AVL Trees
(4.4 in Weiss)

CSE 373
Data Structures & Algorithms
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Autumn 2012

Today's Outline

- **Announcements**
  - Assignment #2 due Fri, Oct 12 at the BEGINNING of lecture
  - Midterm #1, Fri, Oct 19, 2012.
- **Today's Topics:**
  - **Binary Search Trees (Weiss 4.1-4.3)**
  - **AVL Trees (Weiss 4.4)**

The AVL Balance Condition

Left and right subtrees of *every node*
have equal *heights* **differing by at most 1**

Define: **balance**(x) = height(x.left) – height(x.right)

AVL property: **–1 ≤ balance(x) ≤ 1, for every node x**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $\Theta(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

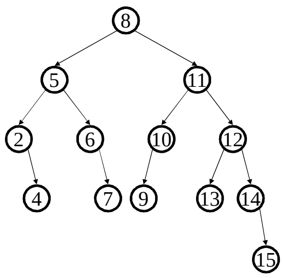
The AVL Tree Data Structure

Structural properties

1. Binary tree property (0,1, or 2 children)
2. Heights of left and right subtrees of *every node* **differ by at most 1**

Result:

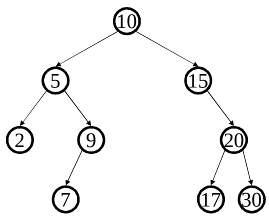
Worst case depth of any node is: O(log n)



Ordering property

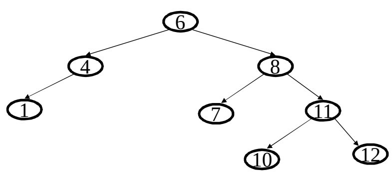
– Same as for BST

Is this an AVL Tree?



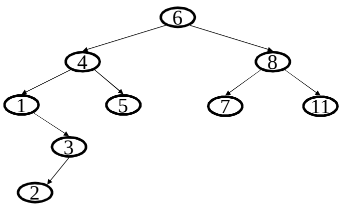
`NULL`s have height **`-1`**

Circle One:



AVL

Not AVL



AVL

Not AVL

Student Activity

If *not* AVL, put a **box** around nodes where AVL property is violated.

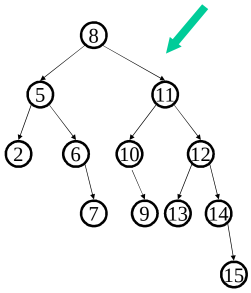
Proving Shallowness Bound

Let **S**(h) be the min # of nodes in an AVL tree of height h

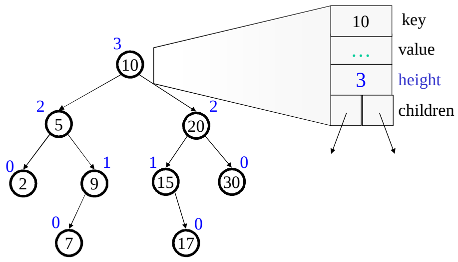
Claim: $\mathbf{S}(h) = \mathbf{S}(h-1) + \mathbf{S}(h-2) + 1$

Solution of recurrence: $\mathbf{S}(h) = \Theta(2^h)$ (like Fibonacci numbers)

AVL tree of height h=4 with the min # of nodes



An AVL Tree



AVL trees: find, insert

- **AVL find**:
  - same as BST find.
- **AVL insert**:
  - same as BST insert, *except* may need to "fix" the AVL tree after inserting new value.

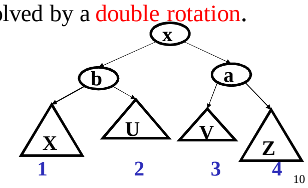
AVL tree insert

Let x be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of x.
2. right subtree of the left child of x.
3. left subtree of the right child of x.
4. right subtree of the right child of x.

Idea: Cases 1 & 4 are solved by a single rotation.
Cases 2 & 3 are solved by a double rotation.



AVL Insert: detect & fix imbalances

1. Insert the new node just as you would in a BST (as a new leaf)
2. For each node on the path from the inserted node up to the root, the insertion may (or may not) have changed the node's height
3. So after recursive insertion in a subtree, check for height imbalance at each of these nodes and perform a *rotation* to restore balance at that node if needed

All the action is in defining the correct rotations to restore balance

Fact that makes it a bit easier:

- There must be a deepest node that is imbalanced after the insert (all descendants still balanced)
- After rebalancing this deepest node, every node is balanced
- So at most one node needs to be rebalanced

Bad Case #1

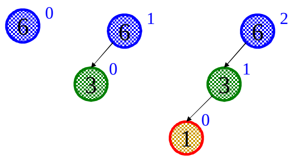
Insert(6)
Insert(3)
Insert(1)

Bad Case #1: Example

Insert(6)
Insert(3)
Insert(1)

Third insertion violates balance property
- happens to be at the root

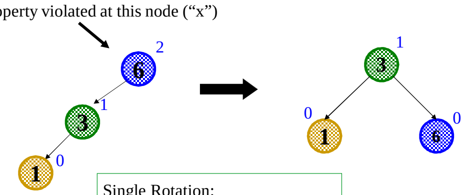
What is the only way to fix this?



Fix: Apply "Single Rotation"

- *Single rotation:* The basic operation we'll use to rebalance
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child (always okay in a BST!)
  - Other subtrees move in only way BST allows (next slide)

AVL Property violated at this node ("x")



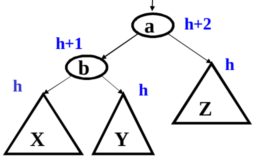
Single Rotation:
1. Rotate between "x" and child

Generalized left-left case

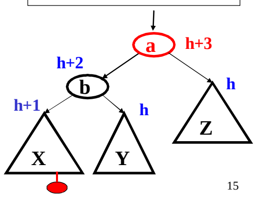
Notational note:
Oval: a node in the tree
Triangle: a subtree

- Node ***a*** imbalanced due to insertion *somewhere* in **left-left grandchild** increasing height of left subtree.
  - 1 of 4 possible imbalance causes (other three coming)
- First we did the insertion, which makes ***a*** imbalanced:

Before insertion – balanced.



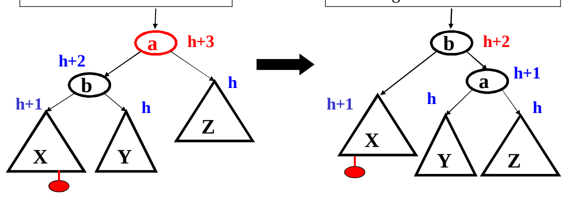
After insertion – unbalanced!



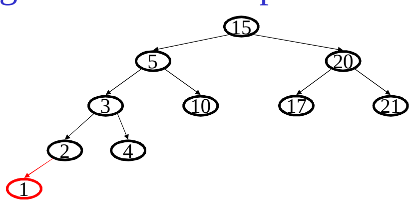
Generalized left-left case (cont.)

- So we rotate at ***a***, using BST facts: X < b < Y < a < Z
- A **single rotation to the right** restores balance at the node
  - To same height as before insertion (so ancestors now balanced)

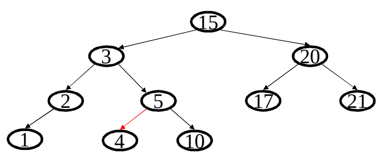
After insertion – unbalanced! After single rotation – balanced!



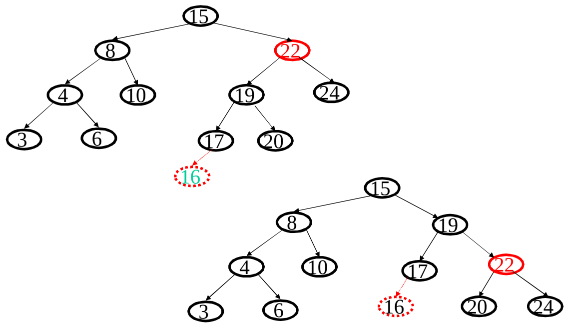
Single rotation example: `insert(1)`



Soln:



Another example: `insert(16)`

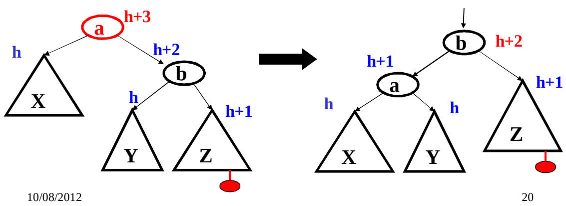


The general right-right case

- Mirror image to left-left case, so you rotate the other way
  - **Single rotation to the left**
  - Exact same concept, but slightly different code

After insertion – unbalanced!

After single rotation – balanced!



Bad Case #3

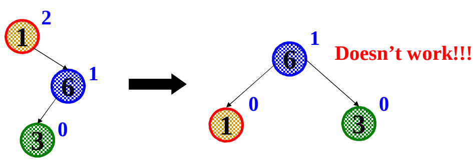
Insert(1)
Insert(6)
Insert(3)

Bad Case #3: Wrong Solution #1

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

Simple example: `insert`(1), `insert`(6), `insert`(3)

- First **wrong** idea: single rotation like we did for left-left

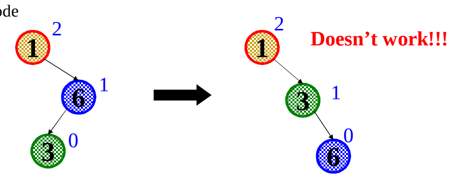


Bad Case #3: Wrong Solution #2

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

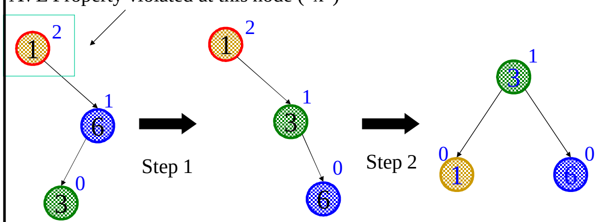
Simple example: `insert`(1), `insert`(6), `insert`(3)

- Second **wrong** idea: single rotation on the child of the unbalanced node



Bad Case #3: Correct Solution: Double Rotation

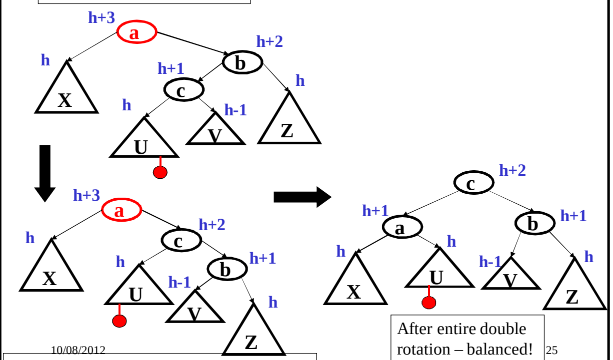
AVL Property violated at this node ("x")



Double Rotation
1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

General right-left case: Double Rotation

After insertion – unbalanced!



After entire double rotation – balanced!

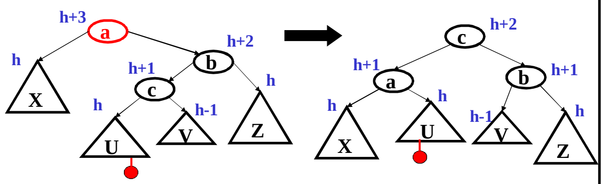
After first single rotation – still unbalanced!

The general right-left case (cont.)

- Like in the left-left and right-right cases, the height of the subtree after rebalancing is the same as before the insert
  - So no ancestor in the tree will need rebalancing
- Does not have to be implemented as two rotations; can just do:

After insertion – unbalanced!

After *entire* double rotation – balanced!



Easier to remember than you may think:

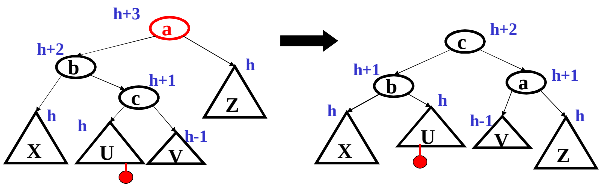
Move c to grandparent's position and then put a, b, X, U, V, and Z in the right places to get a legal BST

The last case: left-right

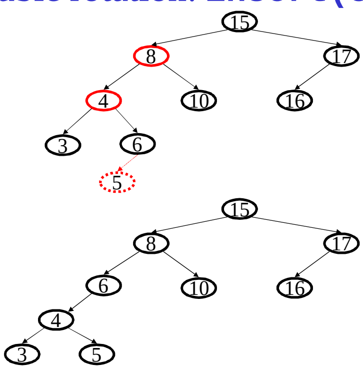
- Mirror image of right-left – double rotation
  - Again, no new concepts, only new code to write

After insertion – unbalanced!

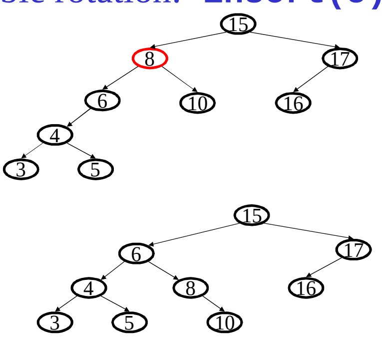
After *entire* double rotation – balanced!



Double rotation: `insert(5)`, step 1



Double rotation: `insert(5)`, step 2



AVL Insert - Summary

- Insert as in a BST
- Check back up path for imbalance, which will be 1 of 4 cases:
  - node's left-left grandchild is too tall
  - node's left-right grandchild is too tall
  - node's right-left grandchild is too tall
  - node's right-right grandchild is too tall
- Only one case occurs because tree was balanced before insert
- After the appropriate single or double rotation, the smallest-unbalanced subtree has the same height as before the insertion
  - So all ancestors are now balanced

Imbalance at node X

Single Rotation
1. Rotate between x and child

Double Rotation
1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Insert into an AVL tree: a b e c d

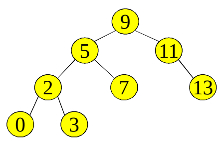
Student Activity

Circle your final answer

Single and Double Rotations:

Inserting what integer values
would cause the tree to need a:

1. single rotation?



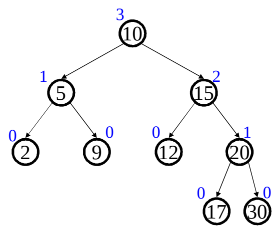
2. double rotation?

3. no rotation?

Student Activity

Insert 3

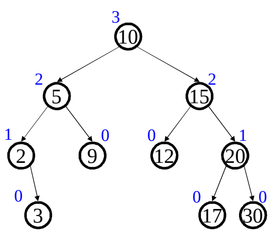
Insert(3)



Unbalanced?

Insert 33

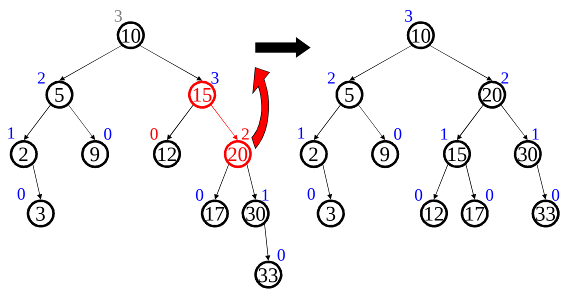
Insert(33)



Unbalanced?

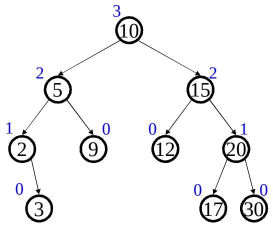
How to fix?

Insert 33: Single Rotation



Insert 18

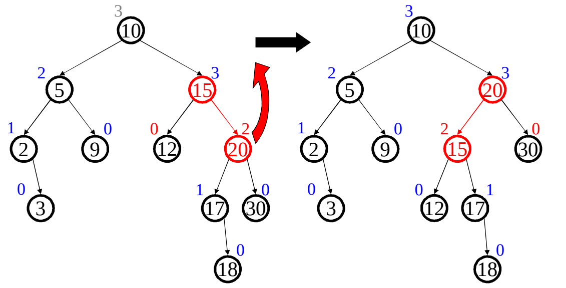
Insert(18)



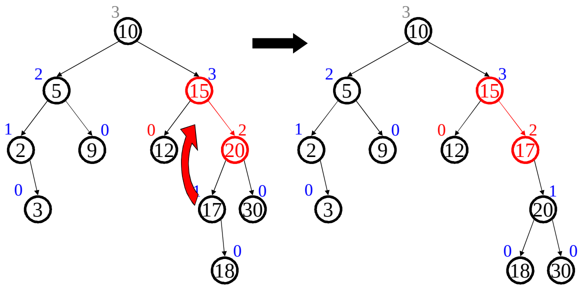
Unbalanced?

How to fix?

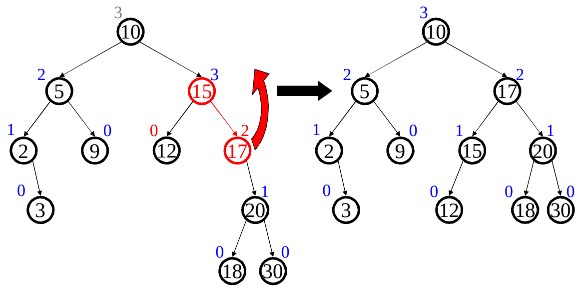
Insert 18: Single Rotation (oops!)



Insert 18: Double Rotation (Step #1)



Insert 18: Double Rotation (Step #2)



AVL Trees Revisited

- Balance condition:

  For every node x, $-1 \le \text{balance}(x) \le 1$
  - Strong enough : Worst case depth is $O(\log n)$
  - Easy to maintain : *one* single or double rotation

- Guaranteed $O(\log n)$ running time for
  - Find ?
  - Insert ?
  - Delete ?
  - buildTree ?

AVL Trees Revisited

- What extra info did we maintain in each node?

- Where were rotations performed?

- How did we locate this node?