# Disjoint Sets and Dynamic Equivalence Relations

CSE 373
Data Structures and Algorithms



# Today's Outline

- **Announcements**
  - **Assignment #3 due Thurs 10/25 at 11pm**.
- **Today's Topics:**
  - **Disjoint Sets & Dynamic Equivalence**



# Motivation

Some kinds of data analysis require keeping track of transitive relations.

Equivalence relations are one family of transitive relations.

Grouping pixels of an image into colored regions is one form of data analysis that uses "dynamic equivalence relations".

Creating mazes without cycles is another application.

Later we'll learn about "minimum spanning trees" for networks, and how the dynamic equivalence relations help out in computing spanning trees.



# Disjoint Sets

- Two sets $S_1$ and $S_2$ are disjoint if and only if they have no elements in common.

- $S_1$ and $S_2$ are disjoint iff $S_1 \cap S_2 = \varnothing$
  (the intersection of the two sets is the empty set)

For example {a, b, c} and {d, e} are disjoint.

But {x, y, z} and {t, u, x} are not disjoint.

## Equivalence Relations

- A binary relation R on a set S is an equivalence relation provided it is reflexive, symmetric, and transitive:
- Reflexive - R(a,a) for all a in S.
- Symmetric - $R(a,b) \rightarrow R(b,a)$
- Transitive - $R(a,b) \wedge R(b,c) \rightarrow R(a,c)$

Is $\leq$ an equivalence relation on integers?

Is "is connected by roads" an equivalence relation on cities?



## Induced Equivalence Relations

- Let S be a set, and let P be a partition of S.

  $P = \{ S_1, S_2, \ldots, S_k \}$

  P being a partition of S means that:

  $i \neq j \rightarrow S_i \cap S_j = \varnothing$ and

  $S_1 \cup S_2 \cup \ldots \cup S_k = S$
- P *induces* an equivalence relation R on S:

  R(a,b) provided a and b are in the same subset (same element of P).

So given any partition P of a set S, there is a corresponding equivalence relation R on S.



## Example

- $S = \{a, b, c, d, e\}$

  $P = \{ S_1, S_2, S_3 \}$

  $S_1 = \{a, b, c\}$, $S_2 = \{d\}$, $S_3 = \{e\}$

  P being a partition of S means that:

  $i \neq j \rightarrow S_i \cap S_j = \varnothing$ and

  $S_1 \cup S_2 \cup \ldots \cup S_k = S$
- P *induces* an equivalence relation R on S:

  R = { (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a),
  (b,c), (c,b),
  (d,d),
  (e,e) }



## Introducing the UNION-FIND ADT

- Also known as the Disjoint Sets ADT or the Dynamic Equivalence ADT.
- There will be a set S of elements that does not change.
- We will start with a partition $P_0$, but we will modify it over time by combining sets.
- The combining operation is called "UNION"
- Determining which set (of the current partition) an element of S belongs to is called the "FIND" operation.

# Example

- Maintain a set of pairwise disjoint* sets.
  - {3,5,7}, {4,2,8}, {9}, {1,6}
- Each set has a unique name: one of its members
  - {3,5,7}, {4,2,8}, {9}, {1,6}

*Pairwise Disjoint: For any two sets you pick, their intersection will be empty)



# Union

- Union(x,y) – take the union of two sets named x and y
  - {3,5,7} , {4,2,8}, {9}, {1,6}
  - Union(5,1)
    {3,5,7,1,6}, {4,2,8}, {9},

To perform the union operation, we replace sets x and y by $(x \cup y)$



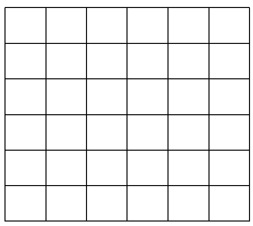
# Find

- Find(x) – return the name of the set containing x.
  - {3,5,7,1,6}, {4,2,8}, {9},
  - Find(1) = 5
  - Find(4) = 8



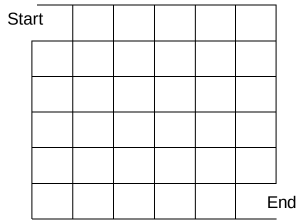
# Application: Building Mazes

- Build a random maze by erasing edges.



## Building Mazes (2)

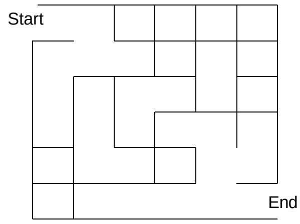
- Pick Start and End






## Building Mazes (3)

- Repeatedly pick random edges to delete.



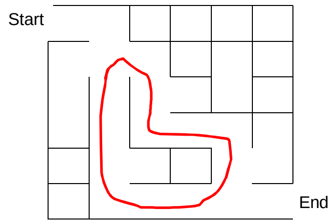



## Desired Properties

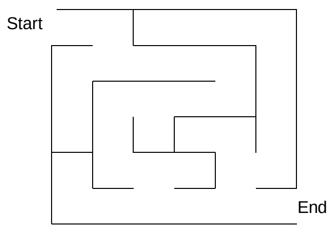
- None of the boundary is deleted
- Every cell is reachable from every other cell.
- Only one path from any one cell to another (There are no cycles – no cell can reach itself by a path unless it retraces some part of the path.)



## A Cycle

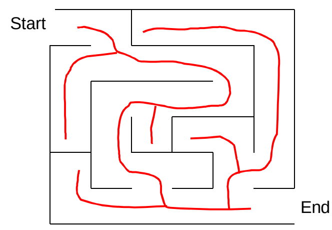


# A Good Solution






# A Hidden Tree






# Number the Cells

We have **disjoint sets** P ={ {1}, {2}, {3}, {4},… {36} } each cell is unto itself.
We have all possible edges E ={ (1,2), (1,7), (2,8), (2,3), … } 60 edges total.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Start | 1 | 2 | 3 | 4 | 5 | 6 | |
| | 7 | 8 | 9 | 10 | 11 | 12 | |
| | 13 | 14 | 15 | 16 | 17 | 18 | |
| | 19 | 20 | 21 | 22 | 23 | 24 | |
| | 25 | 26 | 27 | 28 | 29 | 30 | |
| | 31 | 32 | 33 | 34 | 35 | 36 | End |



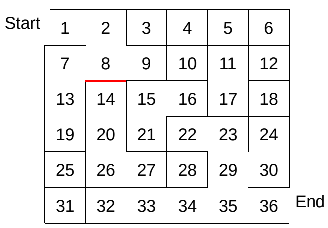
# Basic Algorithm

- P = set of **disjoint sets** of connected cells
- E = set of edges
- Maze = set of maze edges (initially empty)

```
While there is more than one set in P {
  pick a random edge (x,y) and remove from E
  u := Find(x);
  v := Find(y);
  if u ≠ v then    // removing edge (x,y) connects previously non-
                   // connected cells x and y - leave this edge removed!
    Union(u,v)
  else             // cells x and y were already connected, add this
                   // edge to set of edges that will make up final maze.
     add (x,y) to Maze
}
All remaining members of E together with Maze form the maze
```

# Example Step

Pick (8,14)




P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,30,32
33,34,35,36}



# Example

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Find(8) = 7
Find(14) = 20

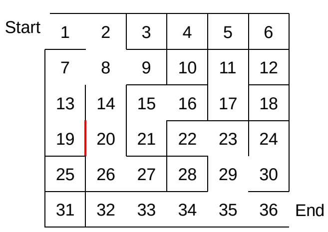
Union(7,20)

P
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}



# Example

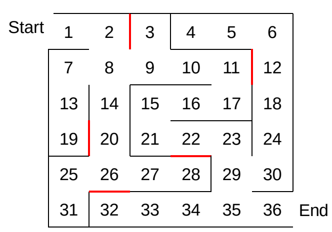
Pick (19,20)




P
{1,2,7,8,9,13,19
14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}



# Example at the End




P
{1,2,3,4,5,6,7,… 36}

E
Maze

## Implementing the Disjoint Sets ADT

- $n$ elements,
  Total Cost of: $m$ finds, $\leq n$-1 unions

- Target complexity: $O(m+n)$
  *i.e.* $O(1)$ amortized

- $O(1)$ worst-case for find as well as union would be great, but…

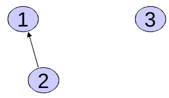
  *Known result*: both find and union *cannot* be done in worst-case $O(1)$ time



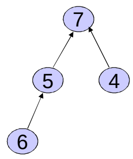
## Up-Tree for Disjoint Union/Find

Initial state:

After several Unions:



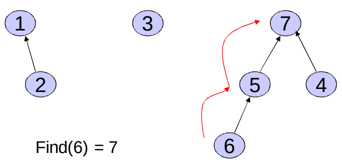
**Roots** are the names of each set.





## Find Operation

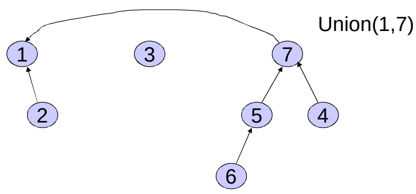
Find(x) - follow x to the root and return the root






## Union Operation

Union(x,y) - assuming x and y are roots, point y to x.

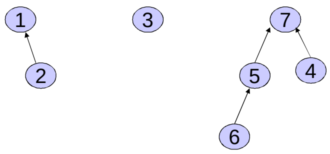


# Simple Implementation

- Array of indices

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 0 | 1 | 0 | 7 | 7 | 5 | 0 |

Up[x] = 0 means
x is a root.






# Implementation

```
int Find(int x) {

  while(up[x] != 0) {
    x = up[x];
  }

  return x;
}
```

```
void Union(int x, int y) {
   up[y] = x;
}
```

*runtime for Union():*

*runtime for Find():*

*runtime for m Finds and n-1 Unions:*



# Find Solutions

Recursive

```
Find(up[] : integer array, x : integer) : integer {
//precondition: x is in the range 1 to size//
if up[x] = 0 then return x
else return Find(up,up[x]);
}
```

Iterative

```
Find(up[] : integer array, x : integer) : integer {
//precondition: x is in the range 1 to size//
while up[x] ≠ 0 do
  x := up[x];
return x;
}
```

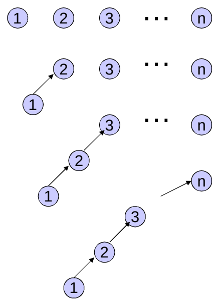


# Now this doesn't look good ☹

Can we do better? *Yes!*

1. Improve union so that *find* only takes $\Theta(\log n)$
   - Union-by-size
   - Reduces complexity to $\Theta(m \log n + n)$
2. Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to <u>almost</u> $\Theta(m + n)$

# A Bad Case



Union(2,1)

Union(3,2)
:
:

Union(n,n-1)

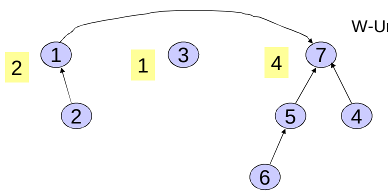
Find(1) n steps!!



# Weighted Union

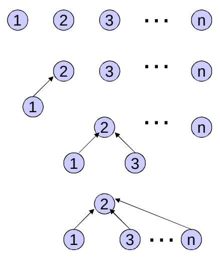
- Weighted Union
  - Always point the *smaller* (total # of nodes) tree to the root of the larger tree

W-Union(1,7)





# Example Again



W-Union(2,1)

W-Union(3,2)
:
:

W-Union(n,2)

Find(1) constant time

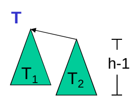


# Analysis of Weighted Union

With weighted union an up-tree of height h has weight *at least* $2^h$.

- Proof by induction
  - **Basis**: h = 0. The up-tree has one node, $2^0 = 1$
  - **Inductive step**: Assume true for all h' < h.

Minimum weight up-tree of height h formed by weighted unions



$W(T_1) \geq W(T_2) \geq 2^{h-1}$

Weighted union    Induction hypothesis

$W(T) \geq 2^{h-1} + 2^{h-1} = 2^h$

# Analysis of Weighted Union (cont)

Let T be an up-tree of weight n formed by weighted union. Let h be its height.

$$n \geq 2^h$$

$$\log_2 n \geq h$$

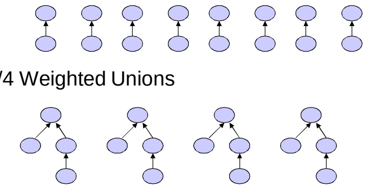
- Find(x) in tree T takes O(log n) time.
  - Can we do better?



# Worst Case for Weighted Union

n/2 Weighted Unions

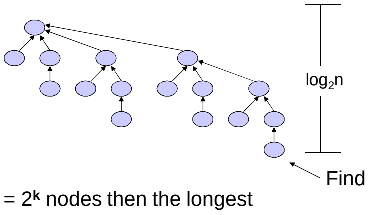
n/4 Weighted Unions





# Example of Worst Case (cont')

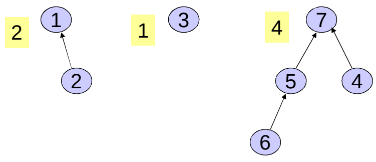
After n/2 + n/4 + …+ 1 Weighted Unions:




If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.



# Array Implementation




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | -1 | 1 | -1 | 7 | 7 | 5 | -1 |
| weight | 2 | | 1 | | | | 4 |

# Weighted Union

```
W-Union(i,j : index){
  //i and j are roots
  wi := weight[i];
  wj := weight[j];
  if wi < wj then
    up[i] := j;
    weight[j] := wi + wj;
  else
    up[j] :=i;
    weight[i] := wi +wj;
}
```

*new runtime for Union():*

*new runtime for Find():*

*runtime for m finds and n-1 unions* =



# Nifty Storage Trick

- Use the same array representation as before
- Instead of storing **`-1`** for the root, simply store **`-size`**

[Read section 8.4, page 299]



# How about Union-by-height?

- Can still guarantee O(log *n*) worst case depth

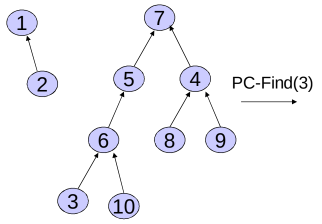
  *Left as an exercise!*

- Problem: Union-by-height doesn't combine very well with the new find optimization technique we'll see next



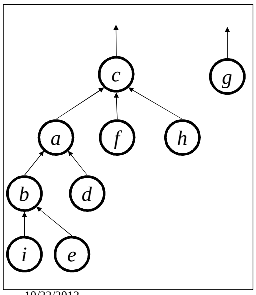
# Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



Student Activity

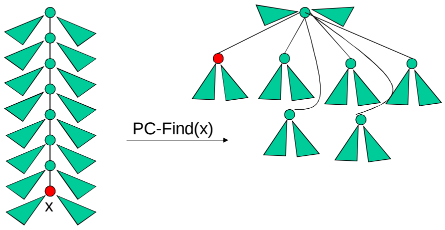
## Draw the result of Find(e):




➡



## Self-Adjustment Works






## Path Compression Find

```
PC-Find(i : index) {
  r := i;
  while up[r] ≠ -1 do //find root
    r := up[r];

  // Assert: r= the root, up[r] = -1
  if i ≠ r then   // if i was not a root

    temp := up[i];
    while temp ≠ r do // compress path
      up[i] := r;
      i := temp;
      temp := up[temp]

  return(r)
}
```

*(New?) runtime for Find:*



## Interlude: A Really Slow Function

**Ackermann's function** is a really big function A(*x*, *y*) with inverse $\alpha(x, y)$ which is really small

How fast does $\alpha(x, y)$ grow?

$\alpha(x, y) = 4$ for *x* **far** larger than the number of atoms in the universe ($2^{300}$)

$\alpha$ shows up in:

- Computation Geometry (surface complexity)
- Combinatorics of sequences

## A More Comprehensible Slow Function

**log* $x$ = number of times you need to compute log to bring value down to at most 1**

E.g. log* 2 = 1

log* 4 = log* $2^2$ = 2

log* 16 = log* $2^{2^2}$ = 3 (log log log 16 = 1)

log* 65536 = log* $2^{2^{2^2}}$ = 4 (log log log log 65536 = 1)

log* $2^{65536}$ = …………… = 5

Take this: $\alpha(m,n)$ grows even slower than log* $n$ !!



## Complex Complexity of Union-by-Size + Path Compression

Tarjan proved that, with these optimizations, $p$ union and find operations on a set of $n$ elements have worst case complexity of $O(p \cdot \alpha(p, n))$

For *all practical purposes* this is amortized constant time: $O(p \cdot 4)$ for $p$ operations!

- Very complex analysis – worse than splay tree analysis etc. that we skipped!



## Disjoint Union / Find with Weighted Union and PC

- Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
- Time complexity for m ≥ n operations on n elements is O(m log* n) where log* n is a very slow growing function.
  - Log * n < 7 for all reasonable n. Essentially constant time per operation!



## Amortized Complexity

- For disjoint union / find with weighted union and path compression.
  - average time per operation is essentially a constant.
  - worst case time for a PC-Find is O(log n).
- An individual operation can be costly, but over time the average cost per operation is not.