Graphs: Topological Sort / Graph Traversals
(Chapter 9)

CSE 373
Data Structures and Algorithms

Today's Outline

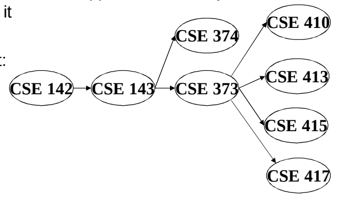
- **Admin**:
  - Homework #4 - due Thurs, Nov 8^{th} at 11pm
  - Midterm 2, Fri Nov 16
- **Graphs**
  - Representations
  - Topological Sort
  - Graph Traversals

Disclaimer: Do not use for official advising purposes!

Topological Sort

Problem: Given a DAG `G=(V,E)`, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:
142, 143, 374, 373, 415, 413, 410, 417

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Topological Sort

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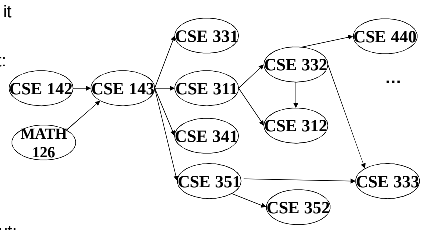
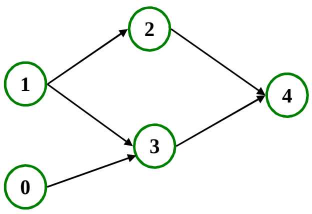


Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352



Valid Topological Sorts:

Questions and comments

- Why do we perform topological sorts only on DAGs?
- Is there always a unique answer?
- What DAGs have exactly 1 answer?
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Questions and comments

- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
- What DAGs have exactly 1 answer?
  - Lists
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Uses

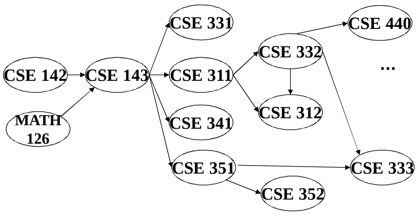
- Figuring out how to graduate
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

A first algorithm for topological sort

1. Label each vertex with its in-degree
   - Labeling also called marking
   - Think “write in a field in the vertex”, though you could also do this with a data structure (e.g., array) on the side
2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and “remove it” (conceptually) from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**

Example

Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1

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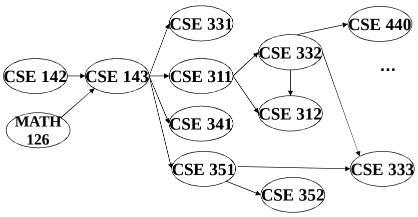
Example

Output: 126



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									

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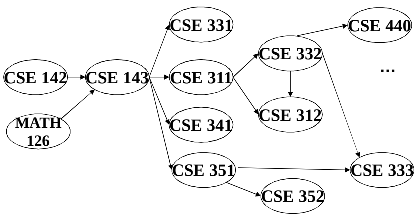
Example

Output: 126
142



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

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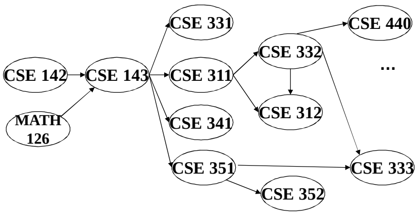
Example

Output: 126
142
143



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x									
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

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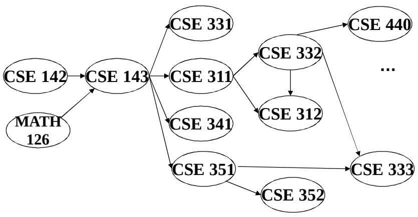
Example

Output: 126
142
143
311



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x								
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

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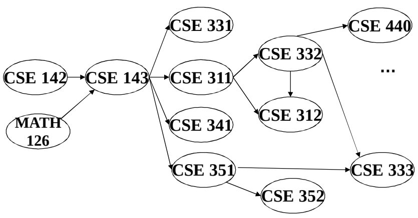
Example

Output: 126
142
143
311
331



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x						
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

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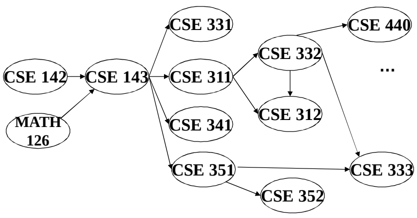
Example



Output: 126
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Removed?	x	x	x	x		x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

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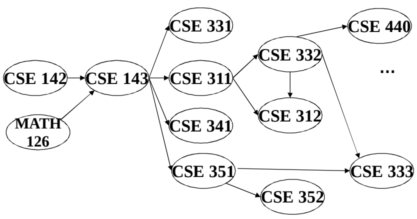
Example



Output: 126
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Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

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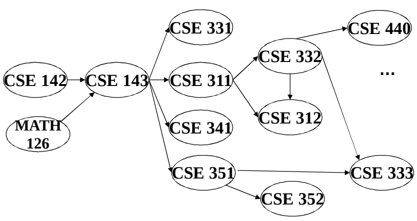
Example



Output: 126
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Removed?	x	x	x	x	x	x	x		x			
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			1	0	1	0	0	1	0	0		0
			0		0							

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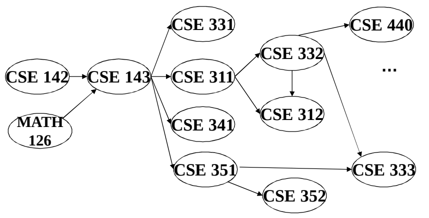
Example



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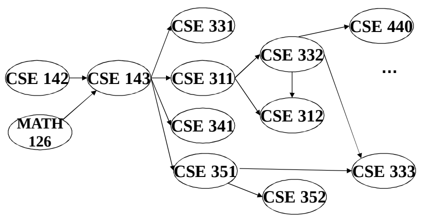
Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x	x		
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



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Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

A couple of things to note

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```


Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
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}
```

- What is the worst-case running time?
  - Initialization $O(|V| + |E|)$
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!
- Keep the "pending" zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Optimized Topological Sort:

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0) enqueue(w);
  }
}
```


Optimized Topological Sort:

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
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  }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V| + |E|)$
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* (i.e., there exists a path) from **v**
- Possibly "do something" for each node (an iterator!)
  - E.g. Print to output, set some field, etc.

Related:
- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:
- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Graph Traversals: Abstract idea

```
traverseGraph(Node start) {
    Set pending = emptySet();
    pending.add(start)
    mark start as visited
    while(pending is not empty) {
      next = pending.remove()
      for each node u adjacent to next
         if(u is not marked) {
           mark u
           pending.add(u)
         }
    }
}
```

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Running time and options

- Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
- The order we traverse depends entirely on add and remove
  - Popular choice: a stack "**depth-first graph search**" (**DFS**)
  - Popular choice: a queue "**breadth-first graph search**" (**BFS**)
- DFS and BFS are "big ideas" in computer science
  - **Depth**: recursively explore one part before going back to the other parts not yet explored
  - **Breadth**: Explore areas closer to the start node first

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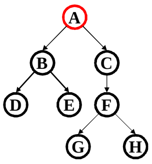
Recursive DFS, Example : trees

- A tree is a graph and DFS and BFS are particularly easy to "see"




```
DFS(Node start) {
  mark and "process"(e.g. print) start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

- Order processed: A, B, D, E, C, F, G, H
- Exactly what we called a "pre-order traversal" for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

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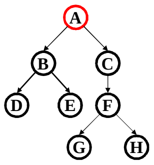
Depth First Search(DFS) with a stack:




```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

- Order processed:

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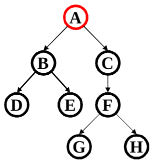
DFS with a stack, Example: trees

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    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

- Order processed: A, C, F, H, G, B, E, D
- A different but perfectly fine traversal

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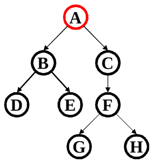
Breadth First Search (BFS) with a queue:




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue()// and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

- Order processed:
- A "level-order" traversal

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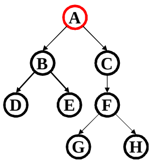
BFS with a queue, Example: trees

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    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

- Order processed: A, B, C, D, E, F, G, H
- A "level-order" traversal

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What if I want to find the "shortest" path?

- **Breadth-first** always finds shortest paths in terms of minimum number of edges from the starting node.

- **An aside: Depth-first** can use less space in finding a path
  - If *longest path* in the graph is **p** and highest out-degree is **d** then DFS stack never has more than **`d*p`** elements
  - But a queue for BFS may hold *O*(|V|) nodes

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Saving the path

- Our graph traversals can answer the "reachability question":
  - "***Is there*** a path from node x to node y?"
- Q: But what if we want to ***output the actual path***?
  - Like getting driving directions rather than just knowing it's possible to get there!
- A: Like this:
  - Instead of just "marking" a node, store the **previous node** along the path (when processing **u** causes us to add **v** to the search, set **`v.path`** field to be **u**)
  - When you reach the goal, follow **`path`** fields backwards to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

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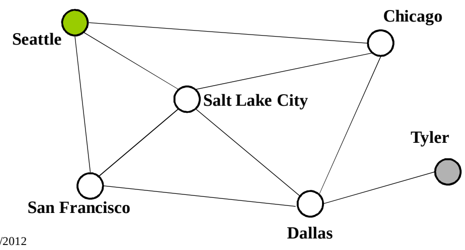
Example using BFS

What is a path from Seattle to Tyler
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



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