CSE 373: Data Abstractions

Minimum Spanning Trees

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Autumn 2012

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'=(V, E')** such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

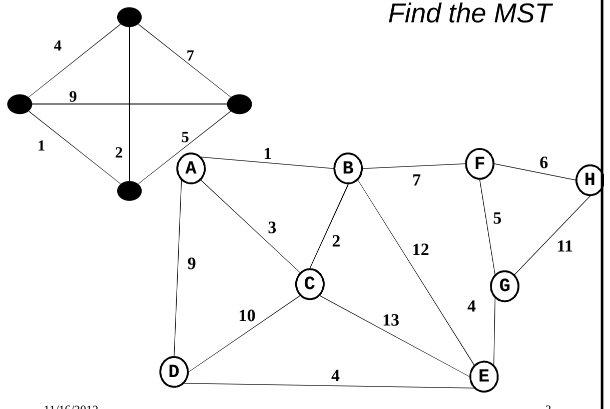
G' is a minimum spanning tree.

Applications:

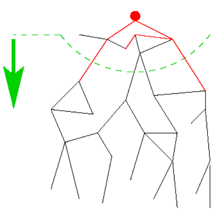
- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

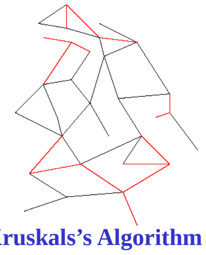
Student Activity

Find the MST



Two Different Approaches





Prim's Algorithm
Almost identical to Dijkstra's

Kruskals's Algorithm
Completely different!

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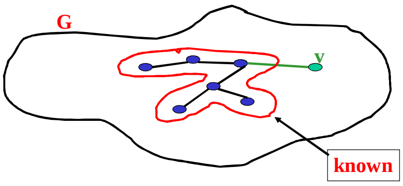
Prim's algorithm

Idea: Grow a tree by picking a vertex from the unknown set that has the smallest cost. Here cost = cost of the edge that connects that vertex to the known set. *Pick the vertex with the smallest cost that connects "known" to "unknown."*

A *node-based* greedy algorithm

Builds MST by greedily adding nodes



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Prim's Algorithm vs. Dijkstra's

Recall:

Dijkstra picked the unknown vertex with smallest cost where cost = ***distance to the source***.

Prim's pick the unknown vertex with smallest cost where cost = ***distance from this vertex to the known set*** (in other words, the cost of the smallest edge connecting this vertex to the known set)

– Otherwise identical

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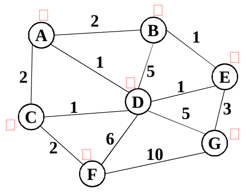
Prim's Algorithm for MST

1. For each node **v**, set **`v.cost = ∞`** and **`v.known = false`**
2. Choose any node **`v.`** (this is like your "start" vertex in Dijkstra)
   a) Mark **v** as known
   b) For each edge **`(v,u)`** with weight **`w`**:
      set **`u.cost=w`** and **`u.prev=v`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known and add **`(v, v.prev)`** to output (the MST)
   c) For each edge **`(v,u)`** with weight **`w`**,

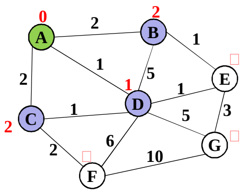
```
if(w < u.cost) {
  u.cost = w;
  u.prev = v;
}
```


Example: Find MST using Prim's



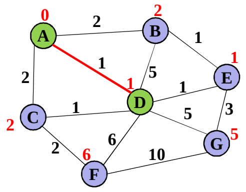
vertex	known?	cost	prev
A		??	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B		2	A
C		2	A
D		1	A
E		??	
F		??	
G		??	

Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B		2	A
C		1	D
D	Y	1	A
E		1	D
F		6	D
G		5	D

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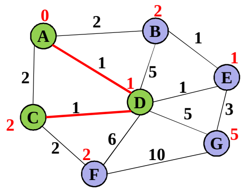
Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B		2	A
C	Y	1	D
D	Y	1	A
E		1	D
F		2	C
G		5	D

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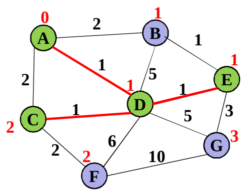
Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B		1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

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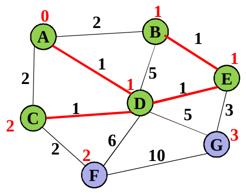
Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

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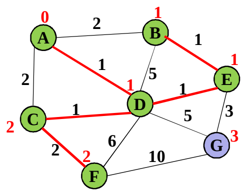
Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G		3	E

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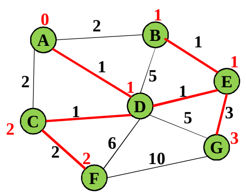
Example: Find MST using Prim's



vertex	known?	cost	prev
A	Y	0	
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G	Y	3	E

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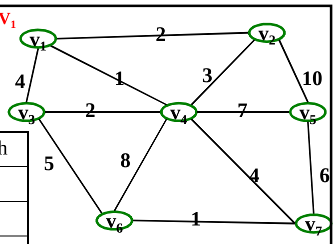
Student Activity

Start with V_1

Find MST using Prim's

V	Kwn	Distance	path
v1			
v2			
v3			
v4			
v5			
v6			
v7			



Order Declared Known:

V_1

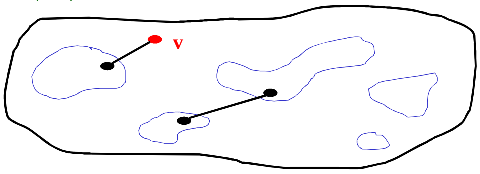
Prim's Analysis

- Correctness ??
  - A bit tricky
  - Intuitively similar to Dijkstra
  - Might return to this time permitting (unlikely)
- Run-time
  - Same as Dijkstra
  - *O*(**|E|log |V|**) using a priority queue

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)



Kruskal's Algorithm for MST

An *edge-based* greedy algorithm
Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge `(u,v)` and mark it
   b. If **u** and **v** are not already connected, add `(u,v)` to the MST and mark **u** and **v** as connected to each other

Kruskal's pseudo code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

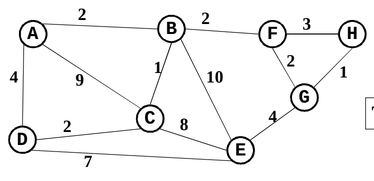
|E| heap ops

2|E| finds

|V| unions

Student Activity

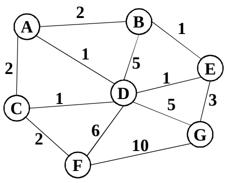
Find MST using Kruskal's



Total Cost:

- **Now find the MST using Prim's method.**
- **Under what conditions will these methods give the same result?**

Example: Find MST using Kruskal's



Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output:

Note: At each step, the union/find sets are the trees in the forest

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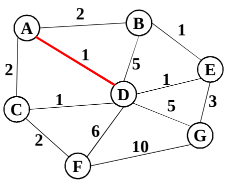
Example: Find MST using Kruskal's



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3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

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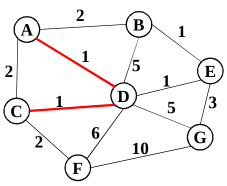
Example: Find MST using Kruskal's



Edges in sorted order:
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3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

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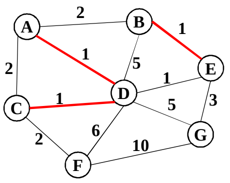
Example: Find MST using Kruskal's

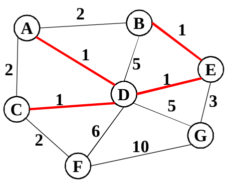


Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

Example: Find MST using Kruskal's



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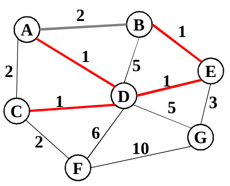

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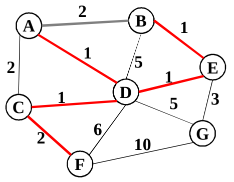

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Output: (A,D), (C,D), (B,E), (D,E), (C,F)

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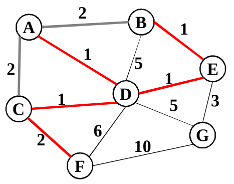

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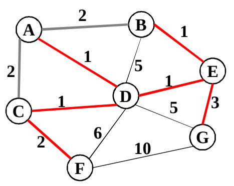

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3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

Correctness

Kruskal's algorithm is clever, simple, and efficient
- But does it generate a minimum spanning tree?
- How can we prove it?

First: it generates a spanning tree
- Intuition: Graph started connected and we added every edge that did not create a cycle
- Proof by contradiction: Suppose **u** and **v** are disconnected in Kruskal's result. Then there's a path from **u** to **v** in the initial graph with an edge we could add without creating a cycle. But Kruskal would have added that edge. Contradiction.

Second: There is no spanning tree with lower total cost…

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The inductive proof set-up

Let **F** (stands for "forest") be the set of edges Kruskal has added at some point during its execution.

Claim: **F** is a subset of *one or more* MSTs for the graph
(Therefore, once **|F|=|V|-1**, we have an MST.)

Proof: By induction on **|F|**

Base case: **|F|=0**: The empty set is a subset of all MSTs

Inductive case: **|F|=k+1**: By induction, before adding the $(k+1)^{th}$ edge (call it **e**), there was some MST **T** such that **F-{e} ⊆ T** …

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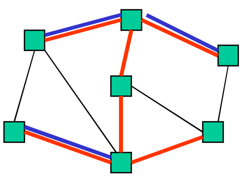
Staying a subset of ***some*** MST

Claim: **F** is a subset of *one or more* MSTs for the graph

So far: **F-{e} ⊆ T**:



Two disjoint cases:
- If **{e} ⊆ T**: Then **F ⊆ T** and we're done
- Else **e** forms a cycle with some simple path (call it **p**) in **T**
  - Must be since **T** is a spanning tree

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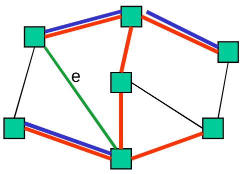
Staying a subset of **some** *MST*

Claim: **F** is a subset of *one or more* MSTs for the graph

So far: **F-{e}** ⊆ **T** and **e** forms a cycle with **p** ⊆ **T**



- There must be an edge **e2** on **p** such that **e2** is not in **F**
  - Else Kruskal would not have added **e**
- Claim: **e2.weight == e.weight**

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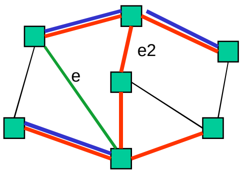
Staying a subset of **some** *MST*

Claim: **F** is a subset of *one or more* MSTs for the graph

So far: **F-{e}** ⊆ **T**
e forms a cycle with **p** ⊆ **T**
e2 on **p** is not in **F**



- Claim: **e2.weight == e.weight**
  - If **e2.weight > e.weight**, then **T** is not an MST because **T-{e2}+{e}** is a spanning tree with lower cost: contradiction
  - If **e2.weight < e.weight**, then Kruskal would have already considered **e2**. It would have added it since **T** has no cycles and **F-{e}** ⊆ **T**. But **e2** is not in **F**: contradiction

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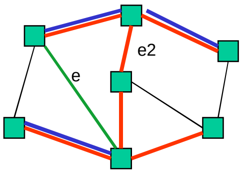
Staying a subset of **some** *MST*

Claim: **F** is a subset of *one or more* MSTs for the graph

So far: **F-{e}** ⊆ **T**
e forms a cycle with **p** ⊆ **T**
e2 on **p** is not in **F**
e2.weight == e.weight



- Claim: **T-{e2}+{e}** is an MST
  - It's a spanning tree because **p-{e2}+{e}** connects the same nodes as **p**
  - It's minimal because its cost equals cost of **T**, an MST
- Since **F** ⊆ **T-{e2}+{e}**, **F** is a subset of one or more MSTs

Done.

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