Comparison Sorting

CSE 373
Data Structures & Algorithms
Ruth Anderson

Today's Outline

- **Admin**:
  - HW #5 – Graphs, due Thurs Nov 29 at 11pm
  - HW #6 – last homework, on sorting, individual project, no Java programming, due Thurs Dec 6.
- **Sorting**
  - Comparison Sorting

Why Sort?

Introduction to sorting

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time
- But often we know we want “all the data items” in some order
  - Anyone can sort, but a computer can sort faster
  - Very common to need data sorted somehow
    - Alphabetical list of people
    - Population list of countries
    - Search engine results by relevance
    - …
- Different algorithms have different asymptotic and constant-factor trade-offs
  - No single ‘best’ sort for all scenarios
  - Knowing one way to sort just isn't enough

More reasons to sort

General technique in computing:
Preprocess (e.g. sort) data to make subsequent operations faster

Example: Sort the data so that you can
- Find the **k**th largest in constant time for any **k**
- Perform binary search to find an element in logarithmic time

Whether the performance of the preprocessing matters depends on
- How often the data will change
- How much data there is

The main problem, stated carefully

For now we will assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:
- An array **A** of data records
- A key value in each data record
- A comparison function (consistent and total)
  - Given keys a & b, what is their relative ordering? <, =, >?
  - Ex: keys that implement Comparable or have a Comparator that can handle them

Effect:
- Reorganize the elements of **A** such that for any **`i`** and **`j`**, if **`i < j`** then **`A[i] ≤ A[j]`**
- Usually unspoken assumption: **A** must have all the same data it started with
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Variations on the basic problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn't do so)
2. Maybe in the case of ties we should preserve the original ordering
   - Sorts that do this naturally are called stable sorts
   - One way to sort twice, Ex: Sort movies by year, then for ties, alphabetically
3. Maybe we must not use more than $O(1)$ "auxiliary space"
   - Sorts meeting this requirement are called 'in-place' sorts
   - Not allowed to allocate extra array (at least not with size O(n)), but can allocate O(1) # of variables
   - All work done by swapping around in the array
4. Maybe we can do more with elements than just compare
   - Comparison sorts assume we work using a binary 'compare' operator
   - In special cases we can sometimes get faster algorithms
5. Maybe we have too much data to fit in memory
   - Use an "external sorting" algorithm

11/19/2012 7

Sorting: The Big Picture

Simple algorithms: $O(n^2)$	**Fancier algorithms: $O(n \log n)$**	**Comparison lower bound: $\Omega(n \log n)$**	**Specialized algorithms: $O(n)$**	**Handling huge data sets**
Insertion sort **Selection sort** **Shell sort** …	**Heap sort** **Merge sort** **Quick sort (avg)** …		**Bucket sort** **Radix sort**	**External sorting**

11/19/2012 8

Insertion Sort

- Idea: At the **k**th step put the **k**th input element in the correct place among the first **k** elements
- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted
- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …
- Time?

  Best-case _____ Worst-case _____ "Average" case ____

11/19/2012 9

Insertion Sort

- Idea: At the **k**th step put the **k**th input element in the correct place among the first **k** elements
- "Loop invariant": when loop index is `i`, first `i` elements are sorted
- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …
- Time?

  Best-case O(n) Worst-case O(n^2) "Average" case O(n^2)

  start sorted start reverse sorted (see text)

Selection sort

- Idea: At the **k**th step, find the smallest element among the not-yet-sorted elements and put it at position k
- "Loop invariant": when loop index is `i`, first `i` elements are the `i` smallest elements in sorted order
- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …
- Time?

  Best-case _____ Worst-case _____ "Average" case ____

Selection sort

- Idea: At the **k**th step, find the smallest element among the not-yet-sorted elements and put it at position k
- "Loop invariant": when loop index is `i`, first `i` elements are the `i` smallest elements in sorted order
- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …
- Time?

  Best-case O(n^2) Worst-case O(n^2) "Average" case O(n^2)

  Always $T(1) = 1$ and $T(n) = n + T(n-1)$

Insertion Sort vs. Selection Sort

- They are different algorithms
- They solve the same problem
- They have the same worst-case and average-case asymptotic complexity
  – Insertion sort has better best-case complexity; preferable when input is "mostly sorted"
- Other algorithms are more efficient *for larger arrays that are not already almost sorted*
  – Small arrays may do well with Insertion sort

Aside: We won't cover Bubble Sort

- It doesn't have good asymptotic complexity: $O(n^2)$
- It's not particularly efficient with respect to common factors
- Basically, almost everything it is good at, some other algorithm is at least as good at
- Some people seem to teach it just because someone taught it to them
- For fun see: "Bubble Sort: An Archaeological Algorithmic Analysis", Owen Astrachan, SIGCSE 2003

Sorting: The Big Picture

Simple algorithms: $O(n^2)$
Insertion sort
Selection sort
Shell sort
…

Fancier algorithms: $O(n \log n)$
Heap sort
Merge sort
Quick sort (avg)
…

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$
Bucket sort
Radix sort

Handling huge data sets
External sorting

Heap sort

- Sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, better yet **`buildHeap`**
  - **`for(i=0; i < arr.length; i++)`**
    `arr[i] = deleteMin();`
- Worst-case running time:
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place…

11/19/2012 16

Heap sort

- Sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, better yet **`buildHeap`**
  - **`for(i=0; i < arr.length; i++)`**
    `arr[i] = deleteMin();`
- Worst-case running time: *O*(*n* **`log`** *n*) why?
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place…

11/19/2012 17

In-place heap sort

But this reverse sorts – how would you fix that?

- Treat the initial array as a heap (via **`buildHeap`**)
- When you delete the **`i`**th element, put it at **`arr[n-i]`**
  - It's not part of the heap anymore!

4	7	5	9	**8**	**6**	**10**	**3**	**2**	**1**

heap part: 4 7 5 9 8 6 10 — sorted part: 3 2 1

`arr[n-i]=`
`deleteMin()`

5	7	6	9	**8**	**10**	**4**	**3**	**2**	**1**

heap part: 5 7 6 9 8 10 — sorted part: 4 3 2 1

11/19/2012 18

"AVL sort"

- How?

"AVL sort"

- We can also use a balanced tree to:
  - **`insert`** each element: total time $O(n \log n)$
  - Repeatedly **`delete`** the min value: total time $O(n \log n)$
  - OR: Do an inorder traversal $O(n)$
- But this cannot be made in-place and has worse constant factors than heap sort
  - heap sort is better
  - both are $O(n \log n)$ in worst, best, and average case
  - neither parallelizes well
- Don't even think about trying to sort with a hash table…

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts
2. Solve the parts independently
   - Think recursion
   - Or potential parallelism
3. Combine solution of parts to produce overall solution

Ex: Sort each half of the array, combine together; to sort each half, split into halves…

Divide-and-conquer sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole

2. Quicksort: Pick a "pivot" element
   Divide elements into less-than pivot
   and greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is [*sorted-less-than* then *pivot* then
   sorted-greater-than]

Mergesort

	0	1	2	3	4	5	6	7
a	8	2	9	4	5	3	1	6

lo **hi**

- To sort array from position `lo` to position `hi`:
  - If range is 1 element long, it's sorted! (Base case)
  - Else, split into two halves:
    - Sort from `lo` to `(hi+lo)/2`
    - Sort from `(hi+lo)/2` to `hi`
    - Merge the two halves together
- Merging takes two sorted parts and sorts everything
  - $O(n)$ but requires auxiliary space…

Example, focus on merging

Start with:

a	8	2	9	4	5	3	1	6

After we return from left and right recursive calls (pretend it works for now)

a	2	4	8	9	1	3	5	6

Merge:
Use 3 "fingers" and 1 more array

aux								

(After merge, copy back to original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1							

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2						

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3					

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4				

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5			

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5	6		

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5	6	8	

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5	6	8	9

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

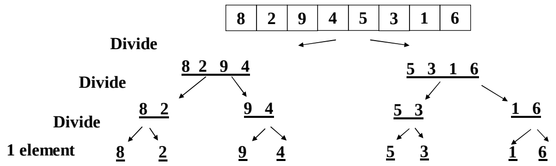
Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5	6	8	9

(After merge,
copy back to
original array)

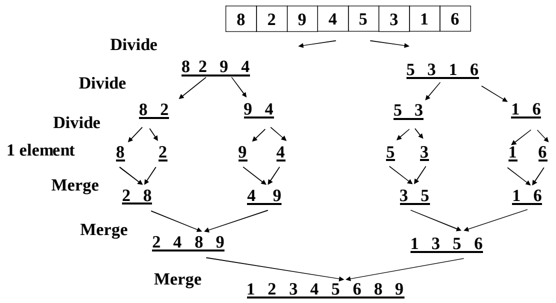
1	2	3	4	5	6	8	9

Mergesort example: Recursively splitting list in half



11/19/2012 34

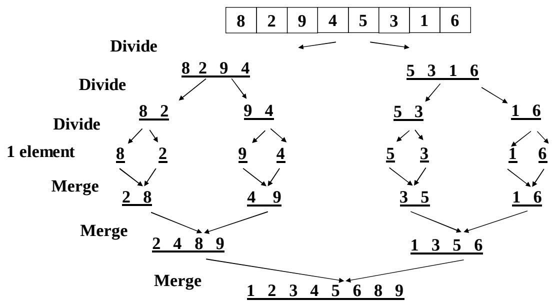
Mergesort example: Merge as we return from recursive calls



When a recursive call ends, it's sub-arrays are each in order; just need to merge them in order together

11/19/2012 35

Mergesort example: Merge as we return from recursive calls

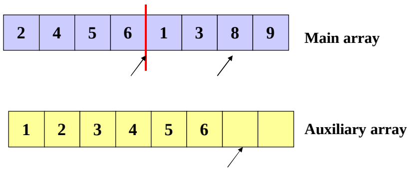


We need another array in which to do each merging step; merge results there, then copy back to original array

11/19/2012 36

Mergesort, some details: saving a little time

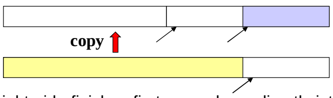
- What if the final steps of our merging looked like the following:



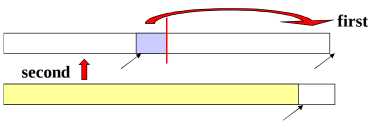
- Seems kind of wasteful to copy 8 & 9 to the auxiliary array just to copy them immediately back…

Mergesort, some details: saving a little time

- Unnecessary to copy 'dregs' over to auxiliary array
  - If left-side finishes first, just stop the merge & copy the auxiliary array:



  - If right-side finishes first, copy dregs directly into right side, then copy auxiliary array



Some details: saving space / copying

Simplest / worst approach:

Use a new auxiliary array of size **`(hi-lo)`** for every merge

Returning from a recursive call? Allocate a new array!

Better:

Reuse same auxiliary array of size **`n`** for every merging stage

Allocate auxiliary array at beginning, use throughout

Best (but a little tricky):

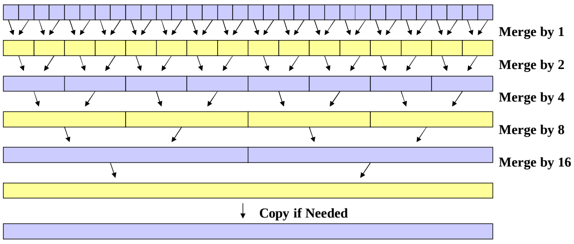
Don't copy back – at 2nd, 4th, 6th, … merging stages, use the original array as the auxiliary array and vice-versa

- Need one copy at end if number of stages is odd

Picture of the "best" from previous slide: Allocate one auxiliary array, switch each step

First recurse down to lists of size 1

As we return from the recursion, switch off arrays



Arguably easier to code up without recursion at all

11/19/2012 40

Linked lists and big data

We defined the sorting problem as over an array, but sometimes you want to sort linked lists

One approach:
- Convert to array: $O(n)$
- Sort: $O(n \log n)$
- Convert back to list: $O(n)$

Or: mergesort works very nicely on linked lists directly
- heapsort and quicksort do not
- insertion sort and selection sort do but they're slower

Mergesort is also the sort of choice for external sorting
- Linear merges minimize disk accesses

11/19/2012 41

Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort n elements, we:
- Return immediately if n=1
- Else do 2 subproblems of size n/2 and then an $O(n)$ merge

Recurrence relation?

11/19/2012 42

Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort n elements, we:

- Return immediately if n=1
- Else do 2 subproblems of size n/2 and then an $O(n)$ merge

Recurrence relation:

$T(1) = c_1$
$T(n) = 2T(n/2) + c_2 n$

11/19/2012 43

MergeSort Recurrence

(For simplicity let constants be 1 – no effect on asymptotic answer)

T(1) = 1
T(n) = 2T(n/2) + n
= 2(2T(n/4) + n/2) + n
= 4T(n/4) + 2n
= 4(2T(n/8) + n/4) + 2n
= 8T(n/8) + 3n
…. (after k expansions)
= 2^kT(n/2^k) + kn

So total is 2^kT(n/2^k) + kn where
n/2^k = 1, i.e., log n = k
That is, $2^{\log n}$ T(1) + n log n
= n + n log n
= O(n log n)

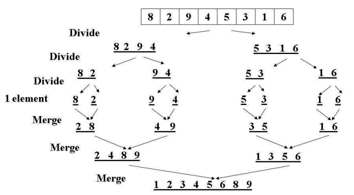
11/19/2012 44

Or more intuitively…

This recurrence comes up often enough you should just "know" it's $O(n \log n)$

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have **log** n height
- At each level we do a *total* amount of merging equal to n



11/19/2012 45

Quicksort

- Also uses divide-and-conquer
  - Recursively chop into two pieces
  - But, instead of doing all the work as we merge together, we'll do all the work as we recursively split into two pieces
  - Also unlike MergeSort, does not need auxiliary space
- $O(n \log n)$ on average ☺, but $O(n^2)$ worst-case ☹
  - MergeSort is always O(nlogn)
  - So why use QuickSort?
- Can be faster than mergesort
  - Often believed to be faster
  - Does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

11/19/2012 46

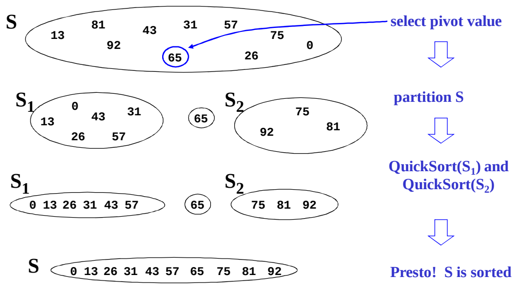
Quicksort overview

1. Pick a pivot element
   - Hopefully an element ~median
   - Good QuickSort performance depends on good choice of pivot; we'll see why later, and talk about good pivot selection later
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C
4. The answer is, "as simple as A, B, C"

(Alas, there are some details lurking in this algorithm)

11/19/2012 47

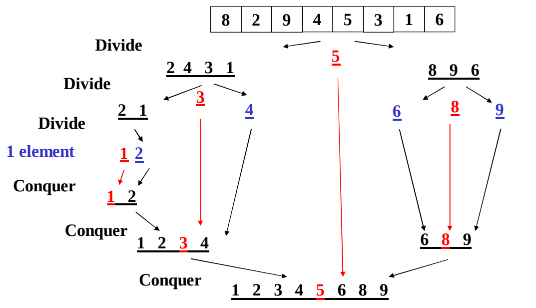
Quicksort: Think in terms of sets

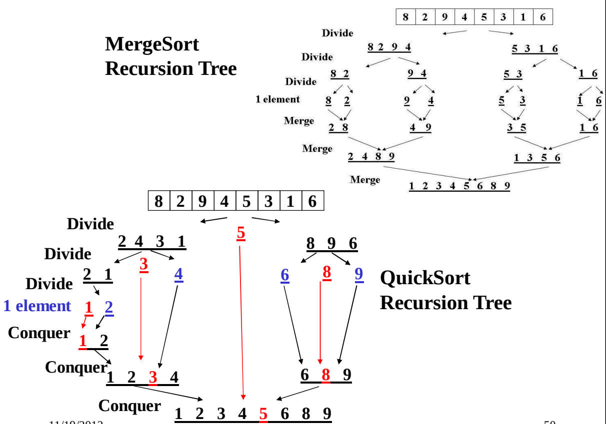


[Weiss]

11/19/2012 48

Quicksort Example, showing recursion



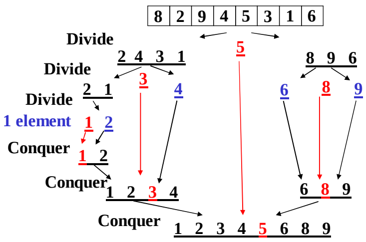


Quicksort Details

We haven't explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size
- How to implement partitioning
  - In linear time
  - In place

Pivots



- Best pivot?
  - Median
  - Halve each time
- Worst pivot?
  - Greatest/least element
  - Reduce to problem of size 1 smaller
  - $O(n^2)$

11/19/2012 52

Quicksort: Potential pivot rules

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive)…

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case is (mostly) sorted input
- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - (Still probably the most elegant approach)
- Median of 3, e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**
  - Common heuristic that tends to work well

11/19/2012 53

Partitioning

- That is, given 8, 4, 2, 9, 3, 5, 7 and pivot 5
  - Getting into left half & right half (based on pivot)
- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition
    - Ideally in linear time
    - Ideally in place
- Ideas?

11/19/2012 54

Partitioning

- One approach (there are slightly fancier ones):
  1. Swap pivot with **`arr[lo]`**; move it 'out of the way'
  2. Use two fingers **`i`** and **`j`**, starting at **`lo+1`** and **`hi-1`** (start & end of range, apart from pivot)
  3. Move from right until we hit something less than the pivot; belongs on left side
     Move from left until we hit something greater than the pivot; belongs on right side
     Swap these two; keep moving inward
     ```
     while (i < j)
         if (arr[j] > pivot) j--
         else if (arr[i] < pivot) i++
         else swap arr[i] with arr[j]
     ```
  4. Put pivot back in middle (Swap with **`arr[i]`**)

Quicksort Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

- Step two: move pivot to the **`lo`** position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Quicksort Example

Often have more than one swap during partition – this is a short example

Now partition in place	6	1	4	9	0	3	5	2	7	8
Move fingers	6	1	4	9	0	3	5	2	7	8
Swap	6	1	4	2	0	3	5	9	7	8
Move fingers	6	1	4	2	0	3	5	9	7	8
Move pivot	5	1	4	2	0	3	6	9	7	8

Quicksort Analysis

- Best-case?
- Worst-case?
- Average-case?

Quicksort Analysis

- Best-case: Pivot is always the median
  T(0)=T(1)=1
  T(n)=2T(n/2) + n -- linear-time partition
  Same recurrence as mergesort: $O(n \log n)$
- Worst-case: Pivot is always smallest or largest element
  T(0)=T(1)=1
  T(n) = 1T(n-1) + n
  Basically same recurrence as selection sort: $O(n^2)$
- Average-case (e.g., with random pivot)
  - O(n **log** n), not responsible for proof (in text)

Quicksort Cutoffs

- For small n, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large n
  - Also, recursive calls add a lot of overhead for small n
- Common engineering technique: switch to a different algorithm for subproblems below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - switch to sequential algorithm
  - None of this affects asymptotic complexity

Quicksort Cutoff skeleton

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls
- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree