CSE 373

Algorithm Analysis and Runtime Complexity

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Evaluating an algorithm

- How to know whether a given algorithm is good, efficient, etc.?

- One idea: *Implement it*, run it, time it / measure it (averaging trials)
  - Pros?
    - Find out how the system effects performance
    - Stress testing – how does it perform in dynamic environment
    - No math!
  - Cons?
    - Need to implement code (takes time)
    - Can be hard to estimate performance
    - When comparing two algorithms, all other factors need to be held constant (e.g., same computer, OS, processor, load)

Range algorithm

How efficient is this algorithm? Can it be improved?

```
// returns the range of values in the given array;
// the difference between elements furthest apart
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int maxDiff = 0;     // look at each pair of values
    for (int i = 0; i < numbers.length; i++) {
        for (int j = 0; j < numbers.length; j++) {
            int diff = Math.abs(numbers[j] – numbers[i]);
            if (diff > maxDiff) {
                maxDiff = diff;
            }
        }
    }
    return diff;
}
```

Range algorithm 2

A slightly better version:

```
// returns the range of values in the given array;
// the difference between elements furthest apart
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int maxDiff = 0;     // look at each pair of values
    for (int i = 0; i < numbers.length; i++) {
        for (int j = i + 1; j < numbers.length; j++) {
            int diff = Math.abs(numbers[j] – numbers[i]);
            if (diff > maxDiff) {
                maxDiff = diff;
            }
        }
    }
    return diff;
}
```

Range algorithm 3

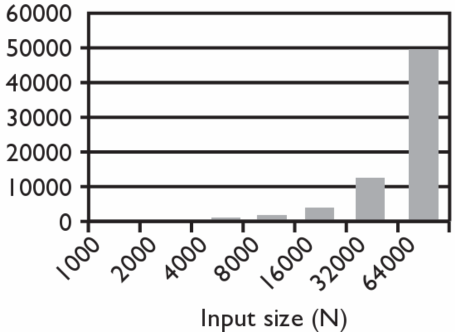
A MUCH faster version. Why is it so much better?

```
// returns the range of values in the given array;
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int max = numbers[0];      // find max/min values
    int min = max;
    for (int i = 1; i < numbers.length; i++) {
        if (numbers[i] < min) {
            min = numbers[i];
        }
        if (numbers[i] > max) {
            max = numbers[i];
        }
    }
    return max - min;
}
```

Runtime of each version

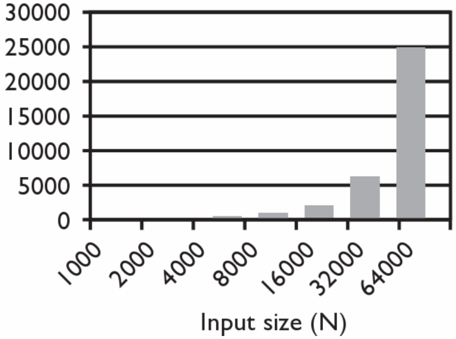
- Version 1:

N	Runtime (ms)
1000	15
2000	47
4000	203
8000	781
16000	3110
32000	12563
64000	49937



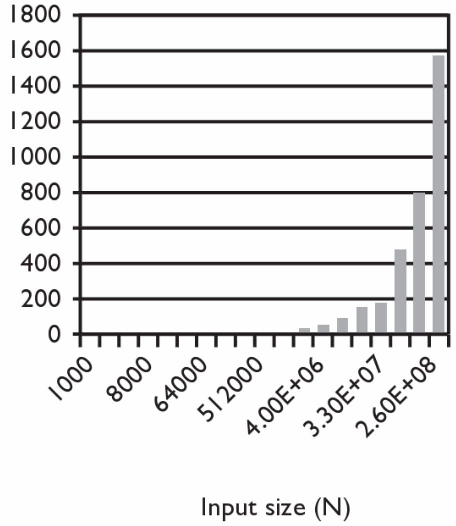
- Version 2:

N	Runtime (ms)
1000	16
2000	16
4000	110
8000	406
16000	1578
32000	6265
64000	25031



Version 3:

N	Runtime (ms)
1000	0
2000	0
4000	0
8000	0
16000	0
32000	0
64000	0
128000	0
256000	0
512000	0
1e6	0
2e6	16
4e6	31
8e6	47
1.67e7	94
3.3e7	188
6.5e7	453
1.3e8	797
2.6e8	1578



Max subsequence sum

- Write a method `maxSum` to find the largest sum of any contiguous subsequence in an array of integers.
  - Easy for all positives: include the whole array.
  - What if there are negatives?

index	0	1	2	3	4	5	6	7	8
value	2	1	-4	10	15	-2	22	-8	5

Largest sum: 10 + 15 + -2 + 22 = 45

  - (Let's define the max to be 0 if the array is entirely negative.)

- Ideas for algorithms?

Algorithm 1 pseudocode

```
maxSum(a):
    max = 0.
    for each starting index i:
        for each ending index j:
            sum = add the elements from a[i] to a[j].
            if sum > max,
                max = sum.

    return max.
```

index	0	1	2	3	4	5	6	7	8
value	2	1	-4	10	15	-2	22	-8	5

Algorithm 1 code

- How efficient is this algorithm?
  - Poor. It takes a few seconds to process 2000 elements.

```
public static int maxSum1(int[] a) {
    int max = 0;
    for (int i = 0; i < a.length; i++) {
        for (int j = i; j < a.length; j++) {
            // sum = add the elements from a[i] to a[j].
            int sum = 0;
            for (int k = i; k <= j; k++) {
                sum += a[k];
            }
            if (sum > max) {
                max = sum;
            }
        }
    }
    return max;
}
```

Flaws in algorithm 1

- Observation: We are redundantly re-computing sums.
  - For example, we compute the sum between indexes 2 and 5:
    a[2] + a[3] + a[4] + a[5]
  - Next we compute the sum between indexes 2 and 6:
    a[2] + a[3] + a[4] + a[5] + a[6]
  - We already had computed the sum of 2-5, but we compute it again as part of the 2-6 computation.
  - Let's write an improved version that avoids this flaw.

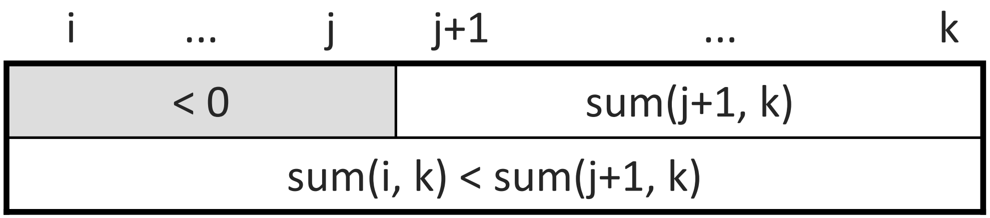
Algorithm 2 code

- How efficient is this algorithm?
    - Mediocre. It can process 10,000s of elements per second.

```
public static int maxSum2(int[] a) {
    int max = 0;
    for (int i = 0; i < a.length; i++) {
        int sum = 0;
        for (int j = i; j < a.length; j++) {
            sum += a[j];
            if (sum > max) {
                max = sum;
            }
        }
    }
    return max;
}
```

A clever solution

- *Claim 1* : The max range cannot start with a negative-sum range.



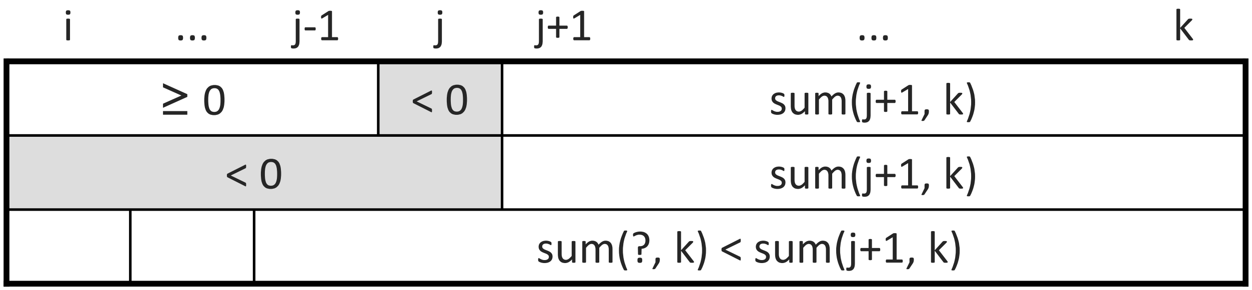
i ... j j+1 ... k

< 0

sum(j+1, k)

sum(i, k) < sum(j+1, k)

- *Claim 2* : If sum(i, j-1) ≥ 0 and sum(i, j) < 0, any max range that ends at j+1 or higher cannot start at any of i through j.



i ... j-1 j j+1 ... k

≥ 0

< 0

sum(j+1, k)

< 0

sum(j+1, k)

sum(?, k) < sum(j+1, k)

  - Together, these observations lead to a very clever algorithm...

Algorithm 3 code

- How efficient is this algorithm?
    - Excellent. It can handle many millions of elements per second!

```
public static int maxSum3(int[] a) {
    int max = 0;
    int sum = 0;
    int i = 0;
    for (int j = 0; j < a.length; j++) {
        if (sum < 0) {    // if sum becomes negative, max range
            i = j;        // cannot start with any of i - j-1,
            sum = 0;      // (Claim 2) so move i up to j
        }
        sum += a[j];
        if (sum > max) {
            max = sum;
        }
    }
    return max;
}
```

Analyzing efficiency

- **efficiency**: A measure of the use of computing resources by code.
  - most commonly refers to run time; but could be memory, etc.

- Rather than writing and timing algorithms, let's *analyze* them. Code is hard to analyze, so let's make the following assumptions:
  - Any *single Java statement* takes a constant amount of time to run.
  - The runtime of a *sequence* of statements is the sum of their runtimes.
  - An *if/else*'s runtime is the runtime of the if test, plus the runtime of whichever branch of code is chosen.
  - A *loop*'s runtime, if the loop repeats *N* times, is *N* times the runtime of the statements in its body.
  - A *method call*'s runtime is measured by the total of the statements inside the method's body.

Runtime example

```
statement1;
statement2;
```

2

```
for (int i = 1; i <= N; i++) {
    statement3;
    statement4;
    statement5;
    statement6;
}
```

$4N$

```
for (int i = 1; i <= N; i++) {
    for (int j = 1; j <= N/2; j++) {
        statement7;
    }
}
```

$\frac{1}{2} N^2$

$\frac{1}{2} N^2 + 4N + 2$

- How many statements will execute if $N = 10$? If $N = 1000$?

Algorithm growth rates

- We measure runtime in proportion to the input data size, *N*.
  - **growth rate**: Change in runtime as *N* changes.

- Say an algorithm runs $\mathbf{0.4N^3 + 25N^2 + 8N + 17}$ statements.
  - Consider the runtime when *N* is *extremely large* .
    (Almost any algorithm is fine if *N* is small.)
  - We ignore constants like 25 because they are tiny next to *N*.
  - The highest-order term (N^3) dominates the overall runtime.
  - We say that this algorithm runs "on the order of" N^3.
  - or $\mathbf{O(N^3)}$ for short ("**Big-Oh** of *N* cubed")

Growth rate example

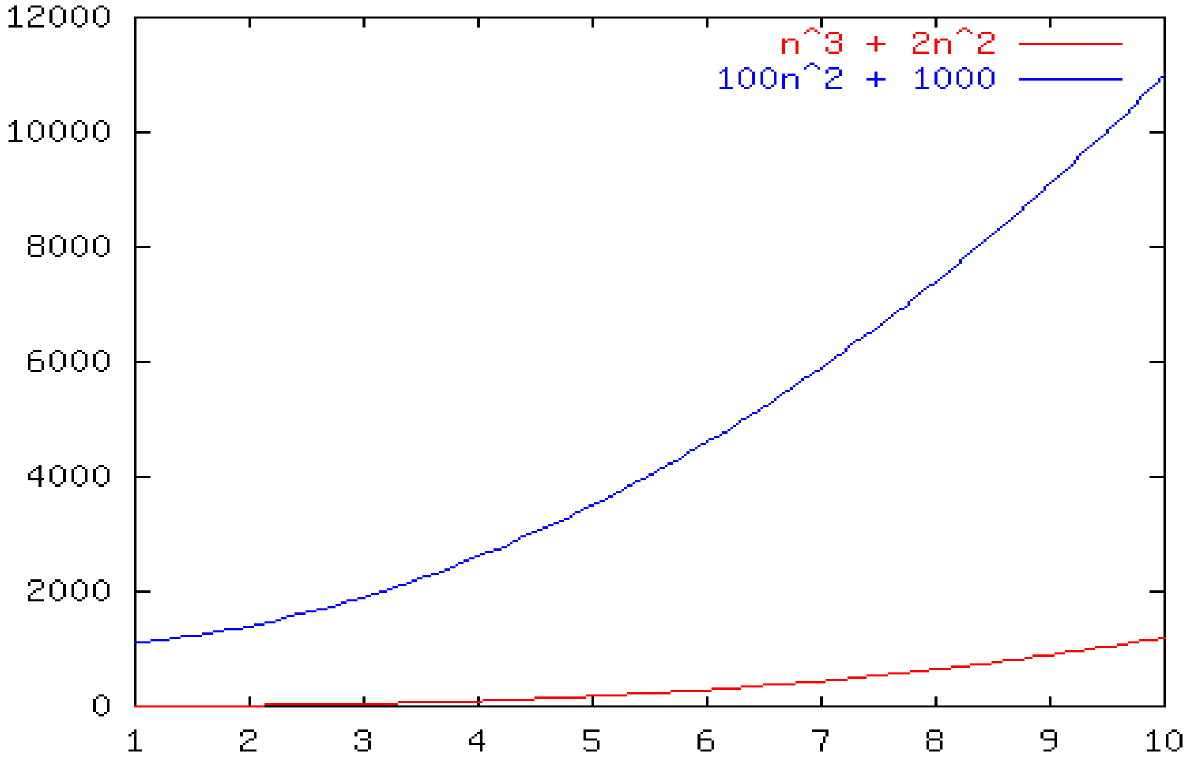
Consider these graphs of functions.

Perhaps each one represents an algorithm:

$N^3 + 2N^2$

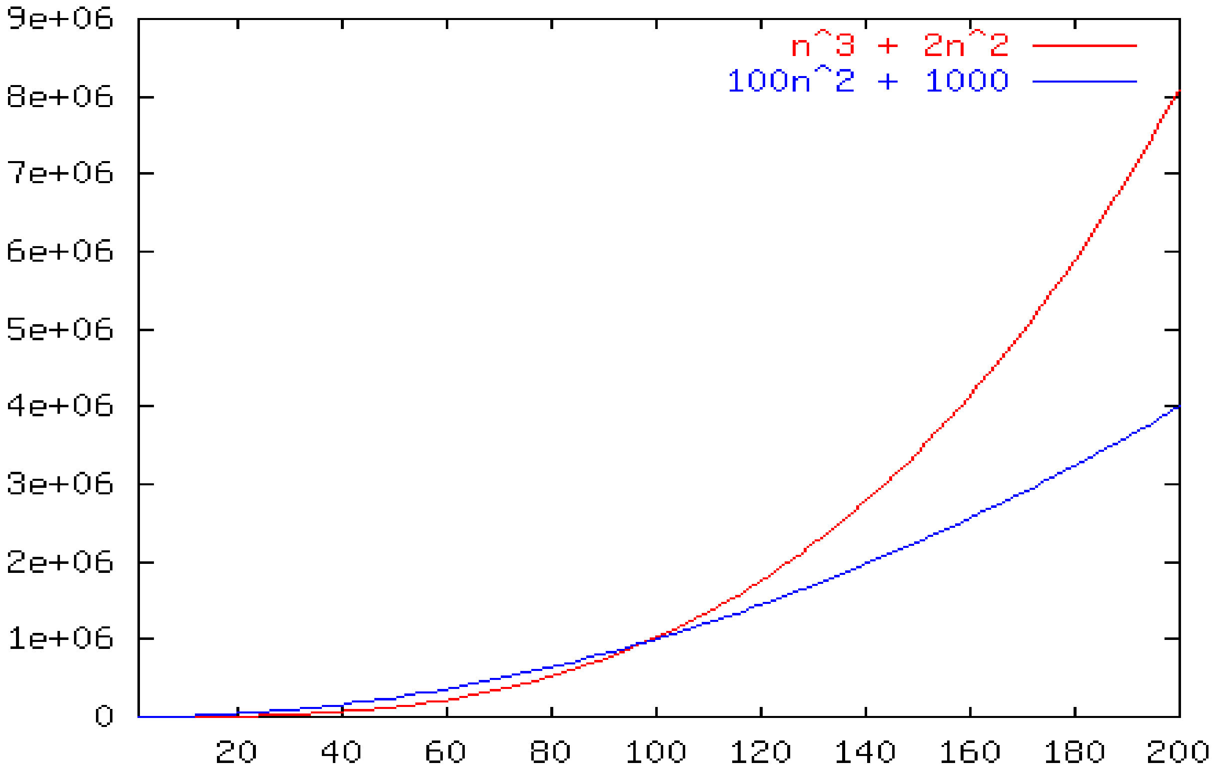
$100N^2 + 1000$

- Which is better?



Growth rate example

- How about now, at large values of *N*?



Complexity classes

- **complexity class**: A category of algorithm efficiency based on the algorithm's relationship to the input size *N*.

Class	Big-Oh	If you double *N*, ...	Example
constant	$O(1)$	unchanged	10ms
logarithmic	$O(\log_2 N)$	increases slightly	175ms
linear	$O(N)$	doubles	3.2 sec
log-linear	$O(N \log_2 N)$	slightly more than doubles	6 sec
quadratic	$O(N^2)$	quadruples	1 min 42 sec
cubic	$O(N^3)$	multiplies by 8	55 min
...	...	...	...
exponential	$O(2^N)$	multiplies drastically	$5 * 10^{61}$ years

Java collection efficiency

Method	Array List	Linked List	Stack	Queue	TreeSet /Map	[Linked] HashSet /Map	Priority Queue
`add` or `put`	$O(1)$	$O(1)$	$O(1)$*	$O(1)$*	$O(\log N)$	$O(1)$	$O(\log N)$*
`add` at index	$O(N)$	$O(N)$	-	-	-	-	-
`contains/` `indexOf`	$O(N)$	$O(N)$	-	-	$O(\log N)$	$O(1)$	-
`get/set`	$O(1)$	$O(N)$	$O(1)$*	$O(1)$*	-	-	$O(1)$*
`remove`	$O(N)$	$O(N)$	$O(1)$*	$O(1)$*	$O(\log N)$	$O(1)$	$O(\log N)$*
`size`	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$

- * = operation can only be applied to certain element(s) / places

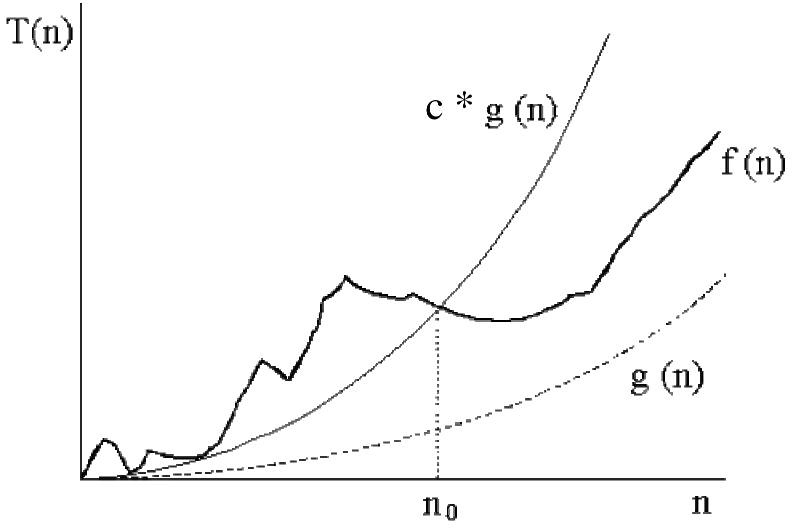
Big-Oh defined

- Big-Oh is about finding an *asymptotic upper bound*.
- Formal definition of Big-Oh:

  $f(N) = O(g(N))$, if there exists positive constants c , N_0 such that

  $f(N) \leq c \cdot g(N)$ for all $N \geq N_0$.

  - We are concerned with how f grows when N is large.
    - not concerned with small N or constant factors
  - Lingo: "$f(N)$ grows no faster than $g(N)$."



Big-Oh questions

- $N + 2 = O(N)$?
  - yes
- $2N = O(N)$?
  - yes
- $N = O(N^2)$?
  - yes
- $N^2 = O(N)$?
  - no
- $100 = O(N)$?
  - yes
- $N = O(1)$?
  - no
- $214N + 34 = O(N^2)$?
  - yes

Preferred Big-Oh usage

- Pick the tightest bound. If $f(N) = 5N$, then:

  $f(N) = O(N^5)$

  $f(N) = O(N^3)$

  $f(N) = O(N \log N)$

  $f(N) = O(N)$ ← preferred

- Ignore constant factors and low order terms:

  $f(N) = O(N)$, *not* $f(N) = O(5N)$

  $f(N) = O(N^3)$, *not* $f(N) = O(N^3 + N^2 + 15)$

  - Wrong: $f(N) \leq O(g(N))$
  - Wrong: $f(N) \geq O(g(N))$
  - Right: $f(N) = O(g(N))$

A basic Big-Oh proof

- *Claim*: $2N + 6 = O(N)$.
- *To prove*: Must find c, N_0 such that for all $N \geq N_0$,

$$2N + 6 \leq c \cdot N$$

- *Proof*: Let c = 3, $N_0 = 6$.

$$2N + 6 \leq 3 \cdot N$$

$$6 \leq N$$

Math background: Exponents

- Exponents:
  - X^Y , or "X to the Y^{th} power";
    X multiplied by itself Y times

- Some useful identities:
  - $X^A \cdot X^B = X^{A+B}$
  - $X^A / X^B = X^{A-B}$
  - $(X^A)^B = X^{AB}$
  - $X^N + X^N = 2X^N$
  - $2^N + 2^N = 2^{N+1}$

Math background: Logarithms

- Logarithms
  - *definition*: $X^A = B$ if and only if $\log_X B = A$
  - *intuition*: $\log_X B$ means:
    "the power X must be raised to, to get B"
  - In this course, a logarithm with no base implies base 2.
    $\log B$ means $\log_2 B$
- Examples
  - $\log_2 16 = 4$ (because $2^4 = 16$)
  - $\log_{10} 1000 = 3$ (because $10^3 = 1000$)

Logarithm bases

- Identities for logs with addition, multiplication, powers:
  - $\log(A \cdot B) = \log A + \log B$
  - $\log(A/B) = \log A - \log B$
  - $\log(A^B) = B \log A$

- Identity for converting bases of a logarithm:

$$\log_A B = \frac{\log_C B}{\log_C A}$$

  - example:
    $\log_4 32 = (\log_2 32) / (\log_2 4)$
    $= 5 / 2$
  - Practically speaking, this means all $\log_c$ are a constant factor away from $\log_2$, so we can think of them as equivalent to $\log_2$ in Big-Oh analysis.

More runtime examples

- What is the exact runtime and complexity class (Big-Oh)?

```
int sum = 0;
for (int i = 1; i <= N; i += c) {
    sum++;
}
```

  - Runtime = $N / c = O(N)$.

```
int sum = 0;
for (int i = 1; i <= N; i *= c) {
    sum++;
}
```

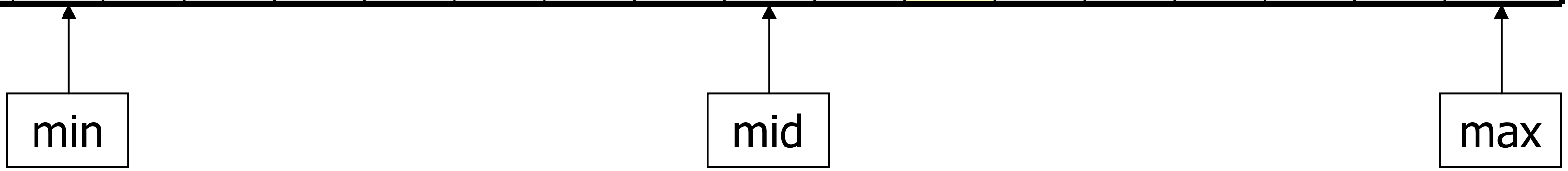
  - Runtime = $\log_c N = O(\log N)$.

Binary search

- **binary search** successively eliminates half of the elements.
  - *Algorithm:* Examine the middle element of the array.
    - If it is too big, eliminate the right half of the array and repeat.
    - If it is too small, eliminate the left half of the array and repeat.
    - Else it is the value we're searching for, so stop.

  - Which indexes does the algorithm examine to find value **42**?
  - What is the runtime complexity class of binary search?

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	42	50	56	68	85	92	103



Binary search runtime

- For an array of size N, it eliminates ½ until 1 element remains.

  $N, N/2, N/4, N/8, \ldots, 4, 2, 1$

  - How many divisions does it take?

- Think of it from the other direction:
  - How many times do I have to multiply by 2 to reach N?

    $1, 2, 4, 8, \ldots, N/4, N/2, N$

  - Call this number of multiplications "x".

    $2^x = N$

    $\boldsymbol{x = \log_2 N}$

- Binary search is in the **logarithmic** (O(log N)) complexity class.

Math: Arithmetic series

- Arithmetic series notation *(useful for analyzing runtime of loops)*:

$$\sum_{i=j}^{k} Expr$$

  - the sum of all values of *Expr* with each value of *i* between *j*--*k*

- Example:

$$\sum_{i=0}^{4} 2i+1$$

= (2(0) + 1) + (2(1) + 1) + (2(2) + 1) + (2(3) + 1) + (2(4) + 1)
= 1 + 3 + 5 + 7 + 9
= 25

Arithmetic series identities

- sum from 1 through *N* inclusive:

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2} = O(N^2)$$

- Intuition:
  - sum = 1 + 2 + 3 + ... + (*N*-2) + (*N*-1) + *N*
  - sum = (1 + *N*) + (2 + *N*-1) + (3 + *N*-2) + ... // rearranged

// *N*/2 pairs total

- sum of squares:

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} = O(N^3)$$

Series runtime examples

- What is the exact runtime and complexity class (Big-Oh)?

```
int sum = 0;
for (int i = 1; i <= N; i++) {
    for (int j = 1; j <= N * 2; j++) {
        sum++;
    }
}
```

  - Runtime = $N \cdot 2N = O(N^2)$.

```
int sum = 0;
for (int i = 1; i <= N; i++) {
    for (int j = 1; j <= i; j++) {
        sum++;
    }
}
```

  - Runtime = $N (N + 1) / 2 = O(N^2)$.