CSE 373 Section

with solutions

Megan Hopp
Rama Gokhale
1/13/15

Today

- Proof by Induction
- Big-Oh
- Algorithm Analysis

Proof by Induction

Base Case:

1.Prove P(0) (sometimes P(1))

Inductive Hypothesis

2.Let k be an arbitrary integer ≥ 0

   3.Assume that P(k) is true

   Inductive Step

   4. …

   5. Prove P(k+1) is true

Examples

Solutions

$$\sum_{i=1}^{n} i = n(n+1)/2 \quad \text{for all } n \geq 1$$

Solution:

1. Base Case: n = 1

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

2. Inductive Hypothesis:

Assume that $\sum_{i=1}^{k} i = k(k+1)/2$ is true for all k ≥ 1.

3. Inductive Step: (k+1)

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{[Inductive hypothesis]}$$

$$= \frac{k^2+k}{2} + k + 1$$

$$= \frac{k^2+k}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)\left((k+1)+1\right)}{2}$$

Examples

Solutions

$$\sum_{i=1}^{N} i^2 = 1 + 2^2 + 3^2 + 4^2 + \cdots = \frac{N(N+1)(2N+1)}{6}$$

- Solution:

1. Base Case: n = 1

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{6}{6} = \frac{1(1+1)(2(1)+1)}{6}$$

2. Inductive Hypothesis:

Assume that

$\sum_{i=1}^{k} k^2 = \frac{k(k+1)(2k+1)}{6}$ for

all k ≥ 1.

3. Inductive Step: (k+1)

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ [Inductive Hypothesis]}$$

$$= \frac{(2k^3 + 2k^2 + k^2 + k)}{6} + (k+1)^2$$

$$= \frac{(2k^3 + 2k^2 + k^2 + k)}{6} + k^2 + 2k + 1$$

$$= \frac{(2k^3 + 2k^2 + k^2 + k)}{6} + \frac{6k^2 + 12k + 6}{6}$$

$$= \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\big((k+1)+1\big)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$ where $n \in Z^+$

Examples

Solutions

Solution:

Base Case: n = 1

$$\sum_{i=1}^{1} 1/i(i+1) = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1}$$

Inductive Hypothesis:

Assume that $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$

for all k ≥ 1.

Inductive Step: (k+1):

We have to prove that

$$\sum_{k=1}^{n+1} \frac{1}{i(i+1)} = \frac{(n+1)}{(n+1)+1}$$

Taking the left hand side…

$$\sum_{k=1}^{n+1} \frac{1}{i(i+1)} = \left(\sum_{k=1}^{n} \frac{1}{i(i+1)}\right) + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n2 + 2n + 1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}$$

$$= \frac{(n+1)}{(n+1)+1}$$

By showing this works for the base case, and then assuming it works for an integer n ϵ Z^+, and proving it works for n+1, we can conclude that it is true for all n ϵ Z^+.

Logarithms

- log means log base of 2
- $\log(N^k) = k \log N$
  - Eg. $\text{Log}(A^2) = \log(A*A) = \log A + \log A = 2\log A$

Big-Oh

- We only look at worst case
- Big input
- Ignore constant factor and lower order terms
  - Why?
- Definition:

> *g(n) is in O(f(n)) if there exist constants*
> *c and n0*
> *such that g(n) □ c f(n) for all n □ n0*

- Also lower bound and tight bound

We use O on a function f(n) (for example n^2) to mean the set of functions with asymptotic behavior less than or equal to f(n)

How to analyze the code?

Consecutive statements

Sum of times

Conditionals Time of test plus slower branch

Loops

Sum of iterations

Calls

Time of call’s body

Recursion Solve recurrence equation

Examples

```
1.int example1(int n) {
        if (n < 10)
                return n - 1;
        else {
                return
example1(n / 2);
        }
}
```

1. O(log N)

```
2.int example2(int n, int sum) {
        for (int k = 0; k < n * n; ++k)
                for (int j = 0; j <
k; j++)

        sum++;
        return sum;
}
```

2. O(n^4)

```
3.int example3(int n, int sum) {
        for (int k = n; k > 0; k--) {
                for (int i = 0; i < k; i++)
                        sum++;
                for (int j = n; j > 0; j--)
                        sum++;
        }
        return sum;
}
```

3. O(n^2)

(From CSE332 12SP Midterm)

Algorithm Analysis

- What is the Big-Oh for the following?
  - Finding the smallest item in an N-item array
    - Sorted? O(1)
    - Unsorted? O(N)
  - Hint: What is the worst case location of the item?