CSE373: Data Structures & Algorithms

Lecture 10: Disjoint Sets and the Union-Find ADT

Linda Shapiro

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Announcements

- Get started on HW03
  - Keyword search in binary search trees

Where we are

Last lecture:

- Priority queues and binary heaps

Today:

- Disjoint sets
- The union-find ADT for disjoint sets

Disjoint sets

- A set is a collection of elements (no-repeats)

- In computer science, two sets are said to be disjoint if they have no element in common.
  - $\mathbf{S_1 \cap S_2 = \varnothing}$

- For example, {1, 2, 3} and {4, 5, 6} are disjoint sets.
- For example, {x, y, z} and {t, u, x} are not disjoint.

Partitions

A partition *P* of a set *S* is a set of sets {*S1*,*S2*,…,*Sn*} such that every element of *S* is in **exactly one** *Si*

Formally:

– $S_1 \cup S_2 \cup . . . \cup S_k = S$

– $i \neq j$ implies $S_i \cap S_j = \varnothing$ (sets are disjoint with each other)

Example:

– Let *S* be {a,b,c,d,e}

– One partition: {a}, {d,e}, {b,c}

– Another partition: {a,b,c}, ∅, {d}, {e}

– A third: {a,b,c,d,e}

– Not a partition: {a,b,d}, {c,d,e} *…. element d appears twice*

– Not a partition of *S*: {a,b}, {e,c} *…. missing element d*

Binary relations

- $S \times S$ is the set of all pairs of elements of S (Cartesian product)
  - Example: If $S = \{a,b,c\}$
    then $S \times S = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c), (c,a),(c,b),(c,c)\}$
- A binary relation R on a set S is any subset of $S \times S$
  - i.e. a collection of ordered pairs of elements of S.
  - Write $R(x,y)$ to mean (x,y) is “in the relation”
  - (Unary, ternary, quaternary, … relations defined similarly)
- Examples for S = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - First-is-younger-than-second relation

Properties of binary relations

- A relation *R* over set *S* is reflexive means *R*(a,a) for *all* a in *S*
  - e.g. The relation “<=“ on the set of integers {1, 2, 3} is
    {<1, 1>, <1, 2>, <1, 3>, <2, 2>, <2, 3>, <3, 3>}
    It is reflexive because <1, 1>, <2, 2>, <3, 3> are in this relation.

- A relation R on a set S is symmetric if and only if for any a and b in S, whenever <a, b> is in R , <b, a> is in R .
  - e.g. The relation “=“ on the set of integers {1, 2, 3} is
    {<1, 1> , <2, 2> <3, 3> } and it is symmetric.
  - The relation "being acquainted with" on a set of people is symmetric.

- A binary relation *R* over set *S* is transitive means:
    If *R*(a,b) and *R*(b,c) then *R*(a,c) for *all* a,b,c in *S*
  - e.g. The relation “<=“ on the set of integers {1, 2, 3} is transitive, because for <1, 2> and <2, 3> in “<=“, <1, 3> is also in “<=“ (and similarly for the others)

Equivalence relations

- A binary relation *R* is an equivalence relation if *R* is reflexive, symmetric, *and* transitive

- Examples
  - Same gender
  - Connected roads in the world
  - "Is equal to" on the set of real numbers
  - "Has the same birthday as" on the set of all people
  - …

Punch-line

- Equivalence relations give rise to partitions.

- Every partition induces an equivalence relation
- Every equivalence relation induces a partition

- Suppose P={$S1$,$S2$,…,Sn} is a partition
  - Define R(x,y) to mean x and y are in the same Si
    - R is an equivalence relation

- Suppose R is an equivalence relation over S
  - Consider a set of sets S1,S2,…,Sn where
    (1) x and y are in the same Si if and only if R(x,y)
    (2) Every x is in some Si
    - This set of sets is a partition

Example

- Let *S* be {a,b,c,d,e}

- One partition: {a,b,c}, {d}, {e}

- The corresponding equivalence relation:

  (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,d), (e,e)

The Union-Find ADT

- The union-find ADT (or "Disjoint Sets" or "Dynamic Equivalence Relation") keeps track of a set of elements partitioned into a number of disjoint subsets.

- Many uses (which is why an ADT taught in CSE 373):
  - Road/network/graph connectivity (will see this again)
    - “connected components” e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages

- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements

Union-Find Operations

- Given an unchanging set *S*, **`create`** an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*

- Operation **`find`** takes an element of *S* and returns the representative element of the subset it is in

- Operation **`union`** takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent **`find`** operations
  - Choice of representative element up to implementation

Example

- Let *S* = {1,2,3,4,5,6,7,8,9}
- Let initial partition be (will highlight representative elements red)

  {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}
- **`union`**(2,5):

  {1}, {2, 5}, {3}, {4}, {6}, {7}, {8}, {9}
- **`find`**(4) = 4, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(4,6), **`union`**(2,7)

  {1}, {2, 5, 7}, {3}, {4, 6}, {8}, {9}
- **`find`**(4) = 6, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(2,6)

  {1}, {2, 4, 5, 6, 7}, {3}, {8}, {9}

No other operations

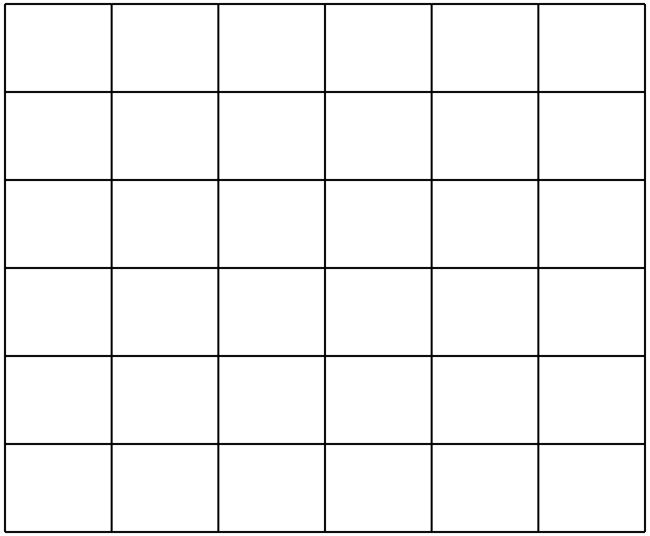
- All that can “happen” is sets get unioned
  - No “un-union” or “create new set” or …

- As always: trade-offs
  - Implementations will exploit this small ADT

- Surprisingly useful ADT
  - But not as common as dictionaries or priority queues

Example application: maze-building

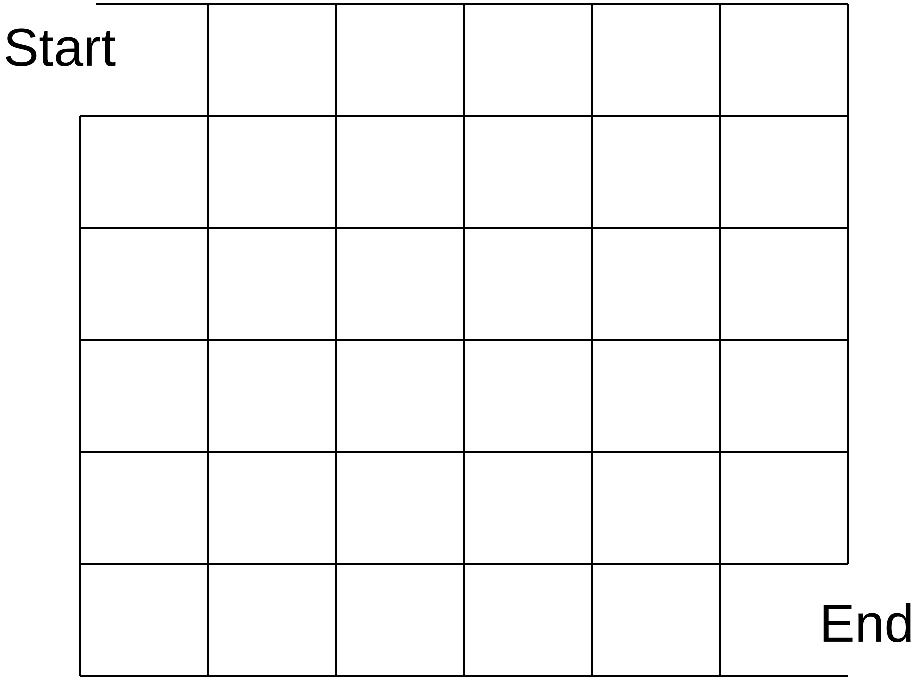
- Build a random maze by erasing edges



  - Possible to get from anywhere to anywhere
    - Including “start” to “finish”
  - No loops possible without backtracking
    - After a “bad turn” have to “undo”

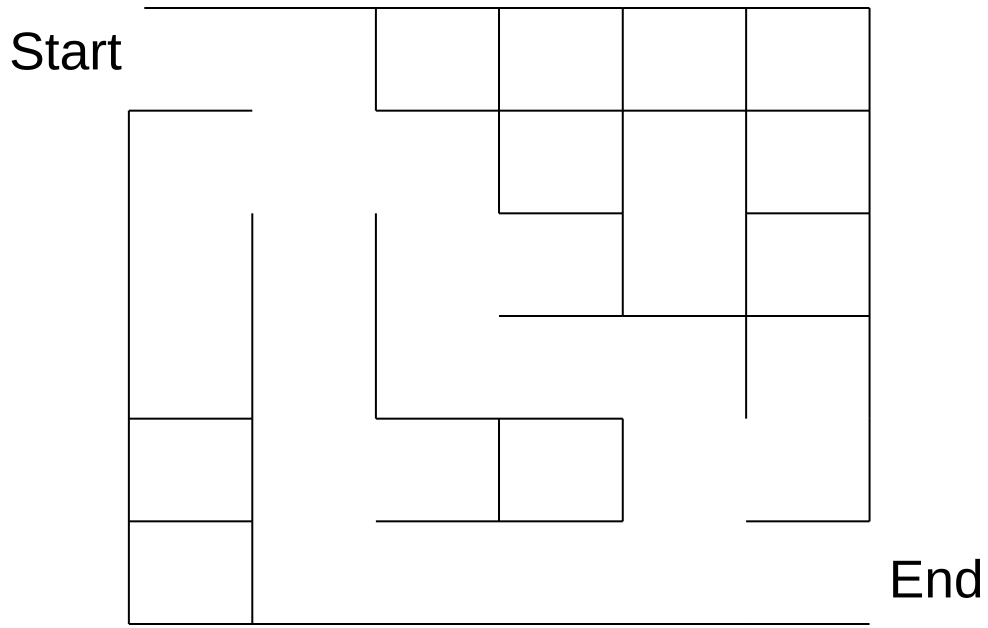
Maze building

Pick start edge and end edge



Repeatedly pick random edges to delete

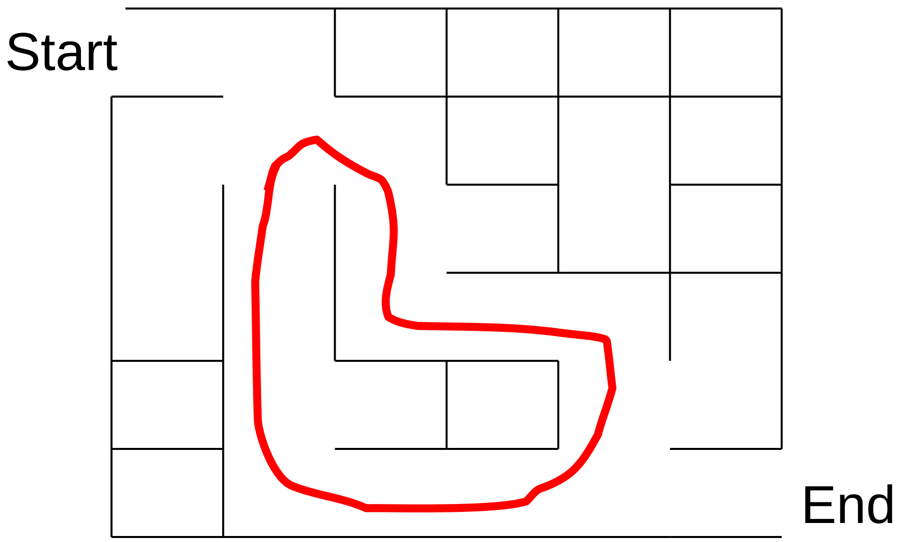
One approach: just keep deleting random edges until you can get from start to finish



Problems with this approach

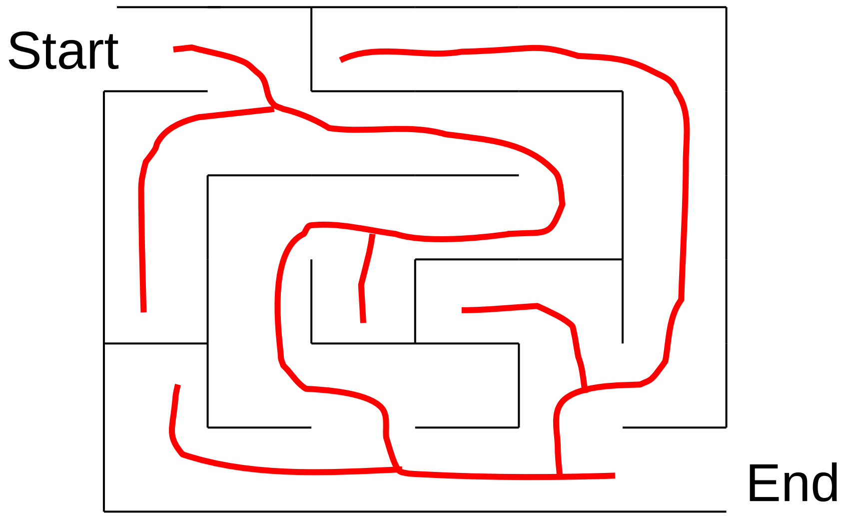
1. How can you tell when there is a path from start to finish?
   – We do not really have an algorithm yet

2. We could have *cycles*, which a “good” maze avoids
   – Want one solution and no cycles



Revised approach

- Consider edges in random order (i.e. pick an edge)
- Only delete an edge if it introduces no cycles (how? TBD)
- When done, we will have a way to get from any place to any other place (including from start to end points)



Cells and edges

- Let's number each cell
  - 36 total for 6 x 6
- An (internal) edge (x,y) is the line between cells x and y
  - 60 total for 6x6: (1,2), (2,3), …, (1,7), (2,8), …

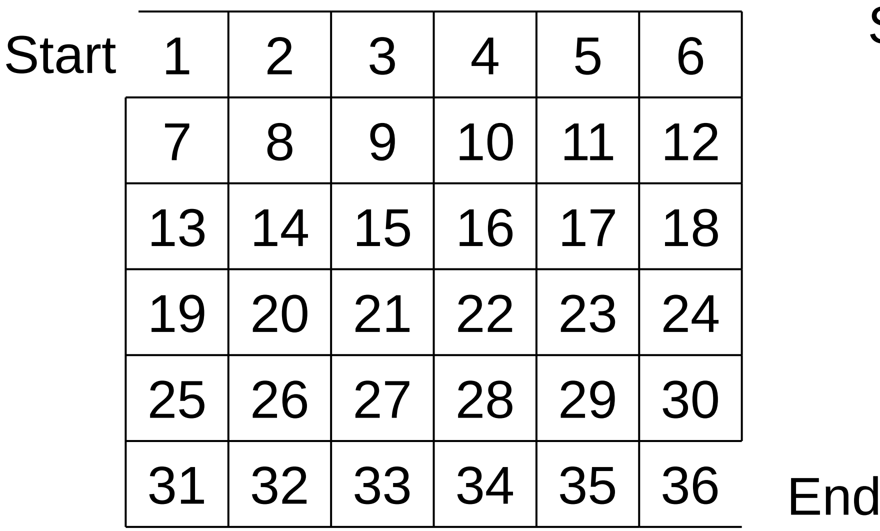
Start

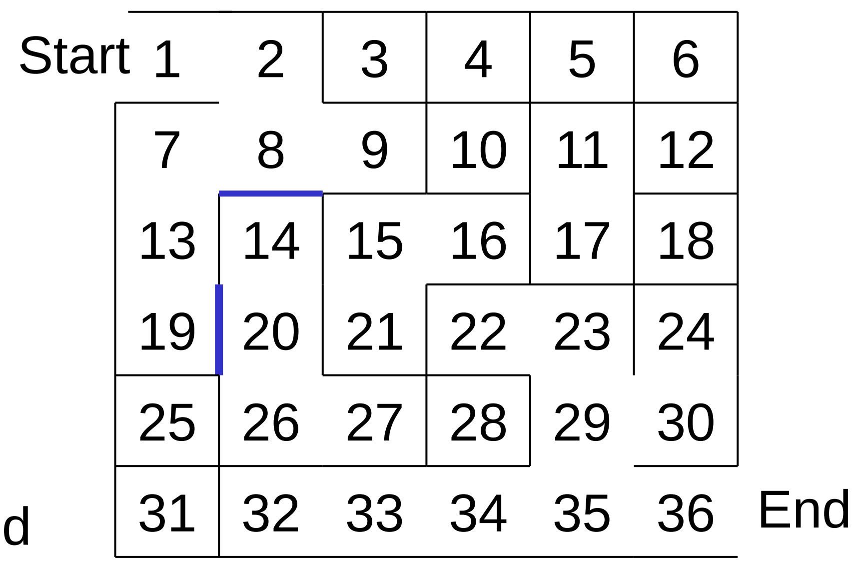
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

The trick

- Partition the cells into disjoint sets
  - Two cells in same set if they are “connected”
  - Initially every cell is in its own subset
- If removing an edge would connect two different subsets:
  - then remove the edge and **union** the subsets
  - else leave the edge because removing it makes a cycle





The algorithm

- P = **disjoint sets** of connected cells

    initially each cell in its own 1-element set
- E = **set** of edges not yet processed, initially all (internal) edges
- M = **set** of edges kept in maze (initially empty)

while P has more than one set {

- Pick a random edge (x,y) to remove from E
- u = **`find`**(x)
- v = **`find`**(y)
- if u==v

    add (x,y) to M // same subset, do not remove edge, do not create cycle

  else

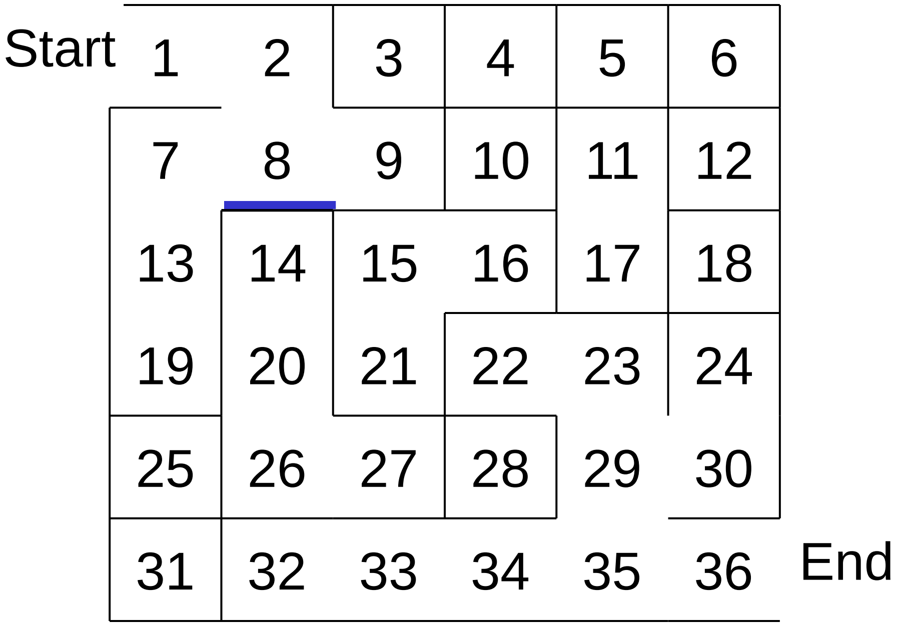
    `union`(u,v) // connect subsets, do not put edge in M

}

Add remaining members of E to M, then output M as the maze

Example at some step

Pick edge (8,14)



P
{1,2,<u>7</u>,8,9,13,19}
{<u>3</u>}
{<u>4</u>}
{<u>5</u>}
{<u>6</u>}
{<u>10</u>}
{11,<u>17</u>}
{<u>12</u>}
{14,<u>20</u>,26,27}
{15,<u>16</u>,21}
{<u>18</u>}
{<u>25</u>}
{<u>28</u>}
{<u>31</u>}
{22,23,24,29,30,32
33,<u>34</u>,35,36}

Example

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

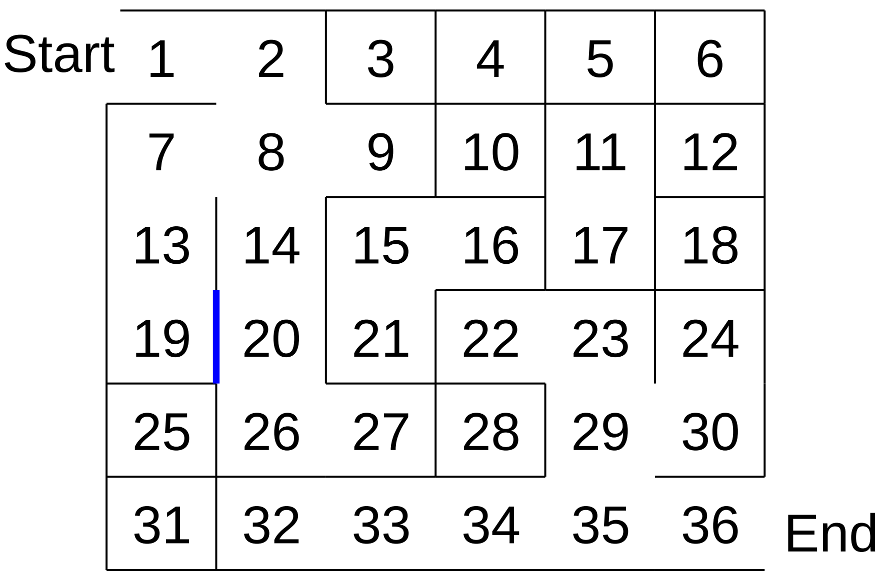
Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

Example: Add edge to M step

Pick edge (19,20)
Find (19) = 7
Find (20) = 7
Add (19,20) to M

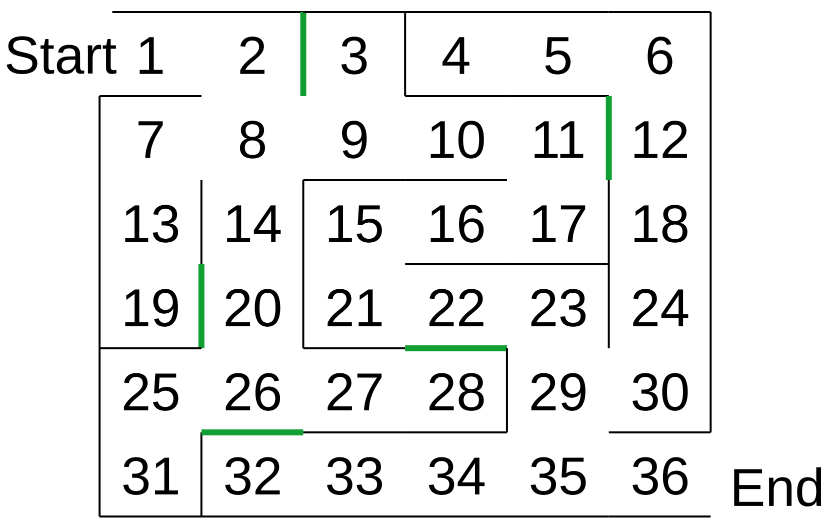


P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

At the end

- Stop when P has one set (i.e. all cells connected)
- Suppose green edges are already in M and black edges were not yet picked
  - Add all black edges to M

P
{1,2,3,4,5,6,<u>7</u>,… 36}



Done! ☺

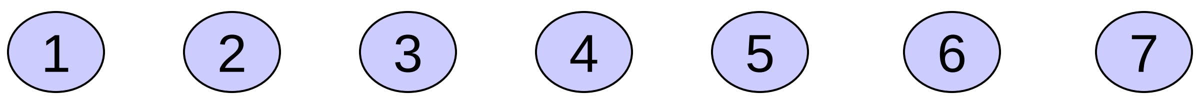
A data structure for the union-find ADT

- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}

- May have any number of **`find`** operations
- May have up to *n*-1 **`union`** operations in any order
  - After *n*-1 **`union`** operations, every **`find`** returns same 1 set

Teaser: the up-tree data structure

- Tree structure with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees



- Possible forest after several unions:
  - Will use roots for set names