CSE373: Data Structures & Algorithms

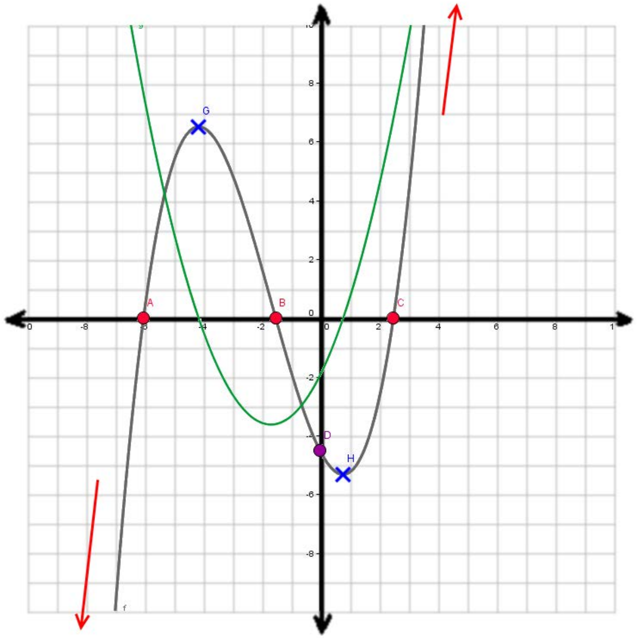
Lecture 16: Introduction to Graphs

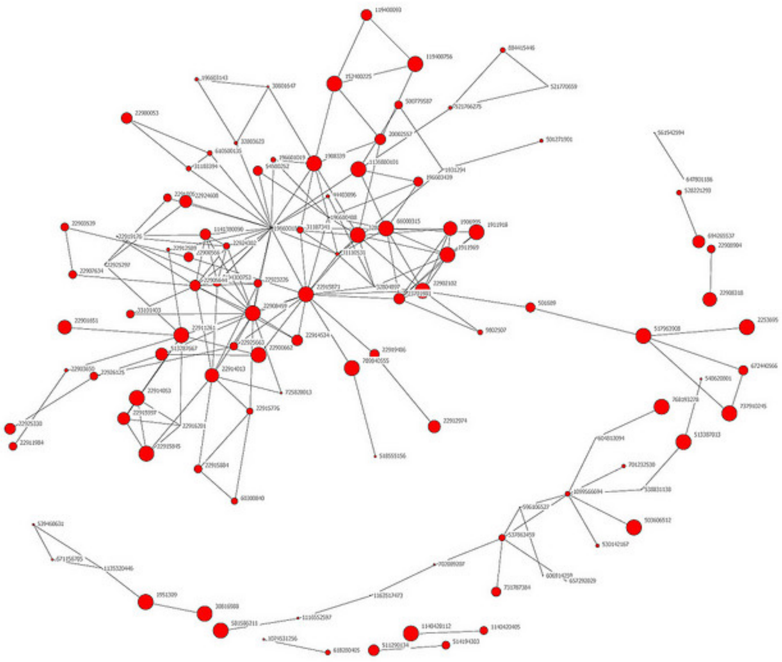
Linda Shapiro

Winter 2015

Announcements

What is a Graph?



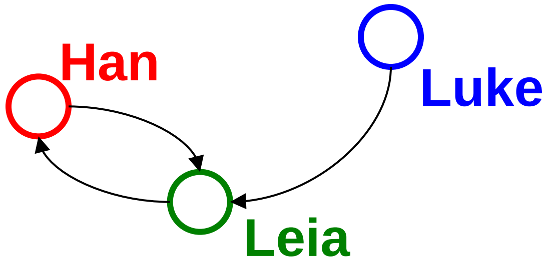


Which kind of graph are we going to study?

Graphs: the mathematical definition

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

- A graph is a pair
  G = (V,E)
  - A set of vertices, also known as nodes
    $\mathbf{V = \{v_1,v_2,\ldots,v_n\}}$
  - A set of edges
    $\mathbf{E = \{e_1,e_2,\ldots,e_m\}}$
    - Each edge $\mathbf{e_i}$ is a pair of vertices $\mathbf{(v_j,v_k)}$
    - An edge "connects" the vertices

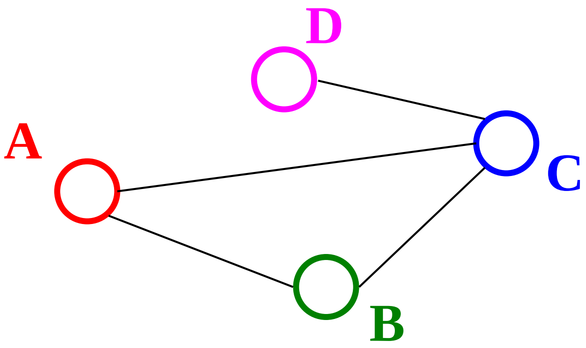


V = {Han,Leia,Luke}
E = {(Luke,Leia),
(Han,Leia),
(Leia,Han)}

- Graphs can be directed or undirected

Undirected Graphs

- In undirected graphs, edges have no specific direction
  - Edges are always "two-way"

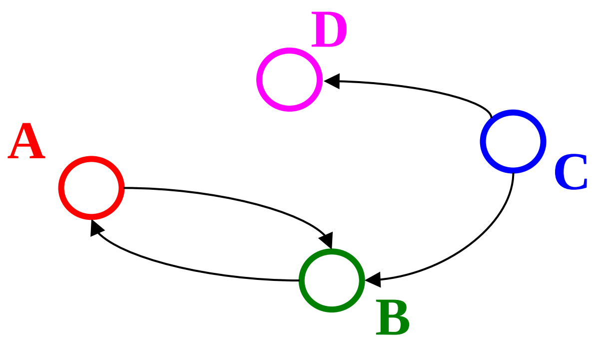


- Thus, **`(u,v)`** $\in$ **`E`** implies **`(v,u)`** $\in$ **`E`** (What do we call this property?)
  - Only one of these edges needs to be in the set
  - The other is implicit, so normalize how you check for it

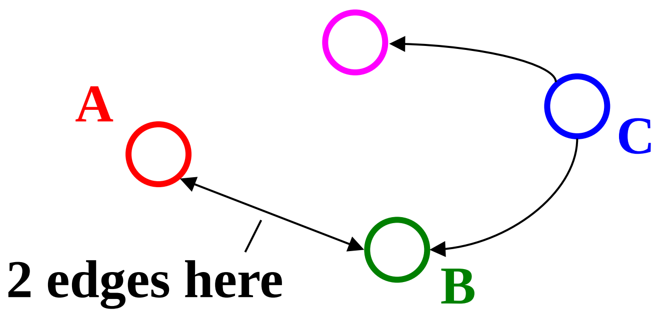
- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices

Directed Graphs

- In directed graphs (sometimes called digraphs), edges have a direction

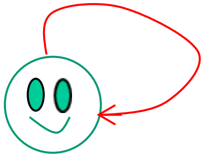


or



- Thus, $(u,v) \in E$ does *not* imply $(v,u) \in E$.
  - Let $(u,v) \in E$ mean $u \rightarrow v$
  - Call **u** the source and **v** the destination
- In-degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-degree of a vertex: number of out-bound edges i.e., edges where the vertex is the source

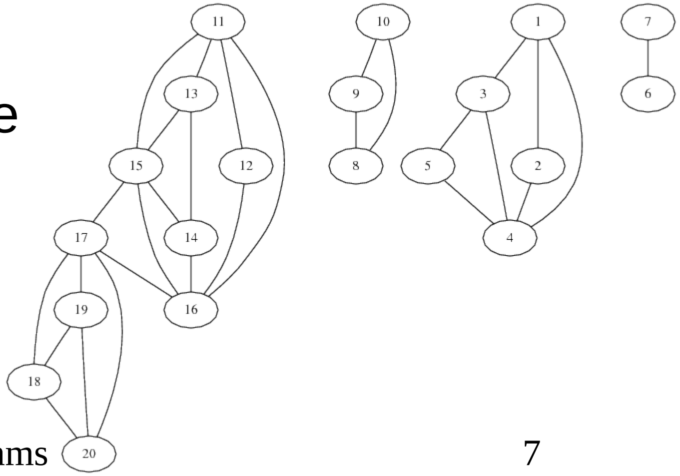
Self-Edges, Connectedness



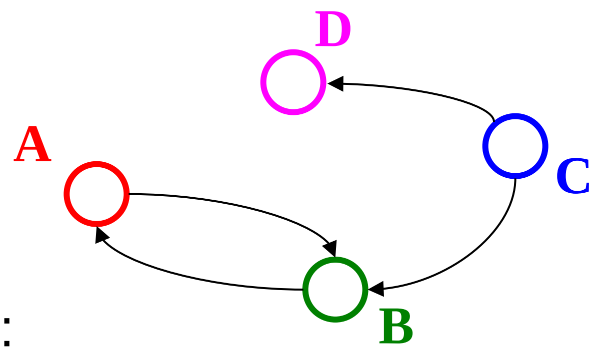
- A self-edge a.k.a. a loop is an edge of the form **`(u,u)`**
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)

- A node can have a degree / in-degree / out-degree of zero

- A graph does not have to be connected
  - Even if every node has non-zero degree



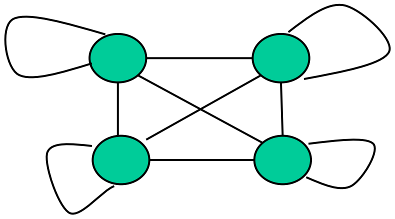
More notation



V = {A, B, C, D}
E = {(C, B), (A, B), (B, A) (C, D)}

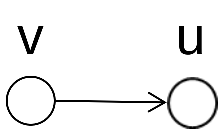
For a graph **G = (V,E)**:

- **|V|** is the number of vertices
- **|E|** is the number of edges
  - Minimum? **0**
  - Maximum for undirected? **|V||V+1|/2 ∈ *O*(|V|²)**
  - Maximum for directed? **|V|² ∈ *O*(|V|²)**



(assuming self-edges allowed, else subtract |V|)

- If **(u,v) ∈ E**
  - Then **v** is a neighbor of **u**, i.e., **v** is adjacent to **u**
  - Order matters for directed edges
    - **u** is not adjacent to **v** unless **(v,u) ∈ E**



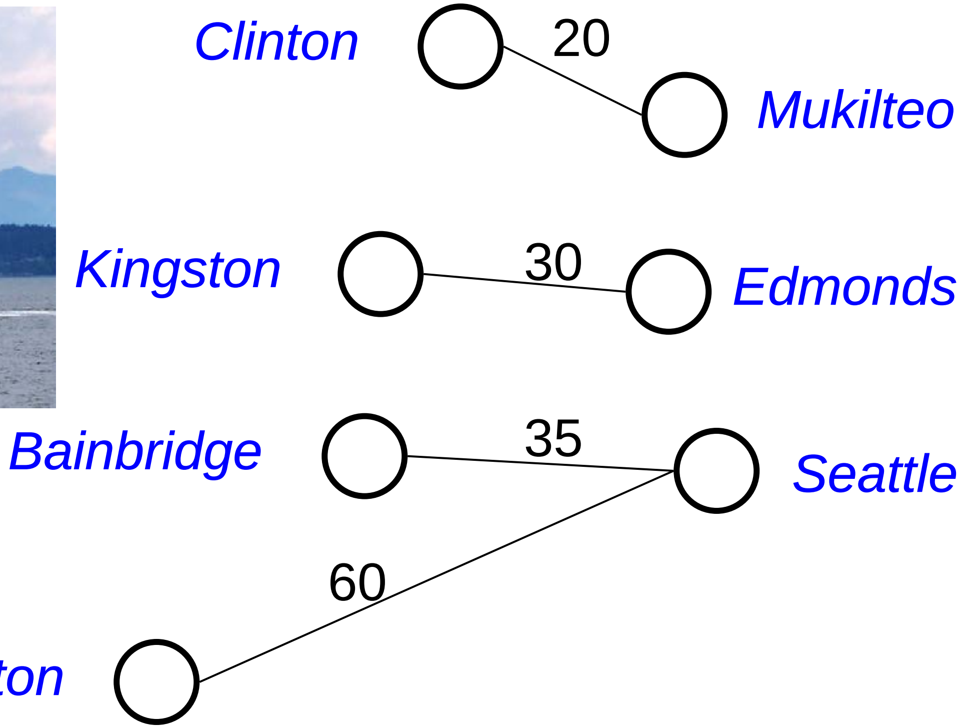
Examples

Which would use directed edges? Which would have self-edges? Which would be connected? Which could have 0-degree nodes?

1. Web pages with links
2. Facebook friends
3. Methods in a program that call each other
4. Road maps (e.g., Google maps)
5. Airline routes
6. Family trees
7. Course pre-requisites

Weighted Graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples use ints)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many do not



Examples

What, if anything, might weights represent for each of these?

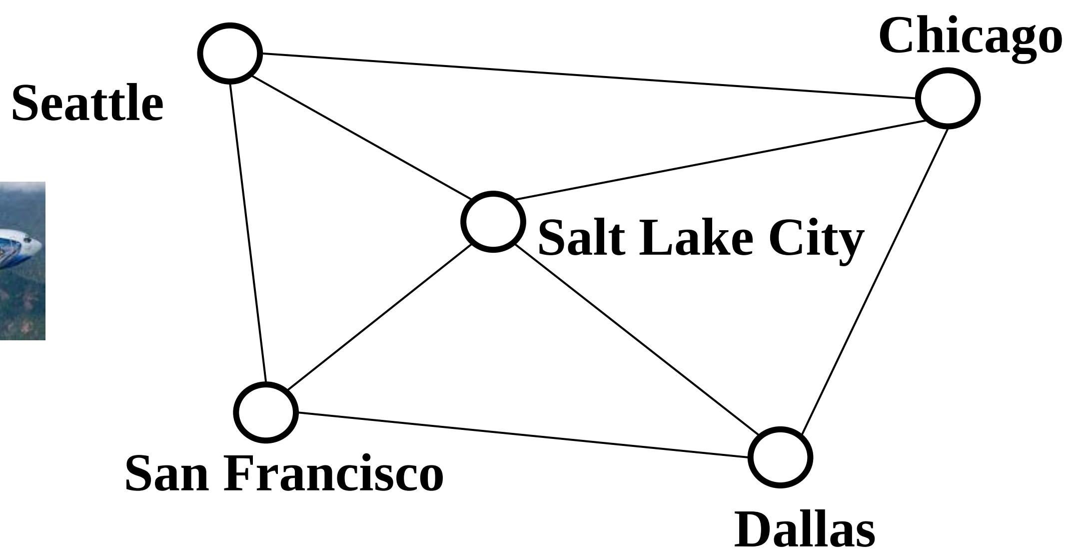
Do negative weights make sense?

- Web pages with links
- Facebook friends
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites

Paths and Cycles

- A path is a list of vertices $[v_0, v_1, \ldots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \le i < n$. Say *"a path from* v_0 *to* v_n*"*

- A cycle is a path that begins and ends at the same node ($v_0 == v_n$)

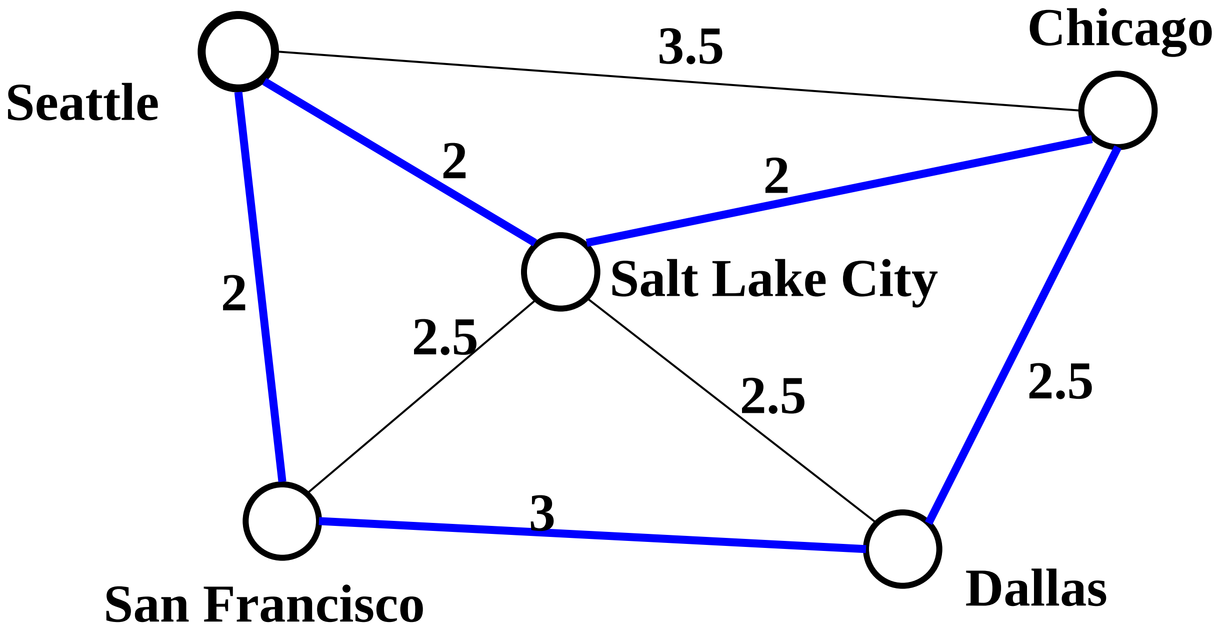


Example: [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

Path Length and Cost

- Path length: Number of *edges* in a path
- Path cost: Sum of *weights* of edges in a path

Example where
P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]



length(P) = 5
cost(P) = 11.5

Simple Paths and Cycles

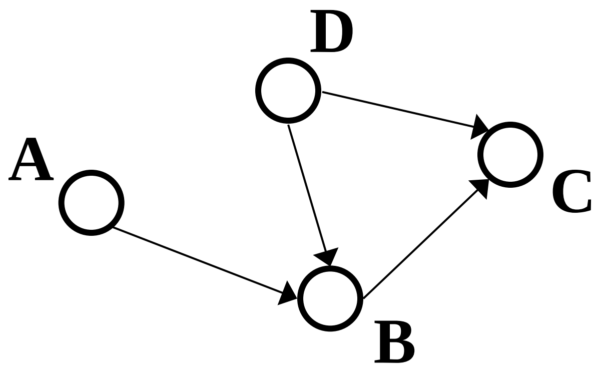
- A simple path repeats no vertices, except the first might be the last
  [Seattle, Salt Lake City, San Francisco, Dallas]
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is both a cycle and a simple path
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

Example:

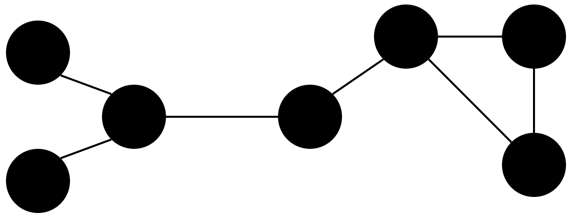


Is there a path from A to D? **No**

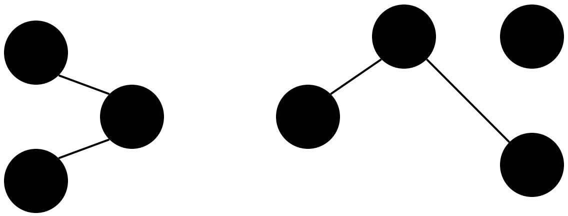
Does the graph contain any cycles? **No**

Undirected-Graph Connectivity

- An undirected graph is connected if for all pairs of vertices **u,v**, there exists a *path* from **u** to **v**



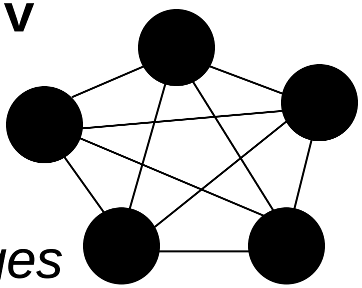
Connected graph



Disconnected graph

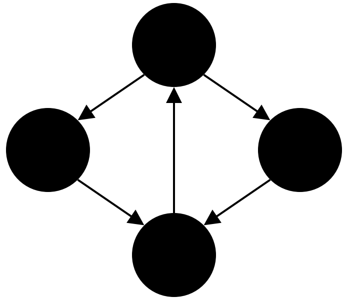
- An undirected graph is complete, a.k.a. fully connected if for *all* pairs of vertices **u,v**, there exists an *edge* from **u** to **v**

plus self edges

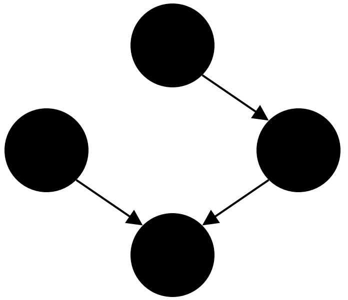


Directed-Graph Connectivity

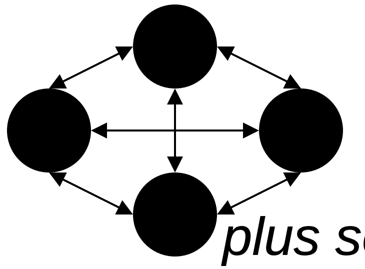
- A directed graph is strongly connected if there is a path from every vertex to every other vertex



- A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



- A complete a.k.a. fully connected directed graph has an edge from every vertex to every other vertex



plus self edges

Trees as Graphs

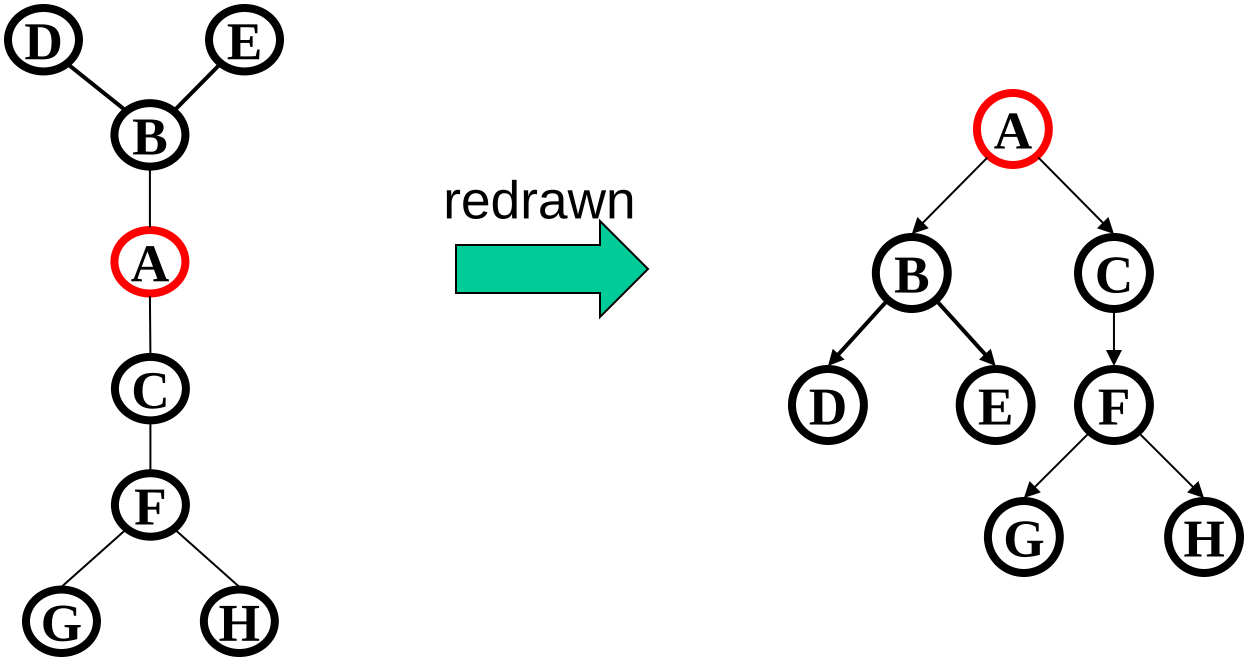
When talking about graphs,
we say a tree is a graph that is:

- Undirected
- Acyclic
- Connected

So all trees are graphs, but not all graphs are trees

Rooted Trees

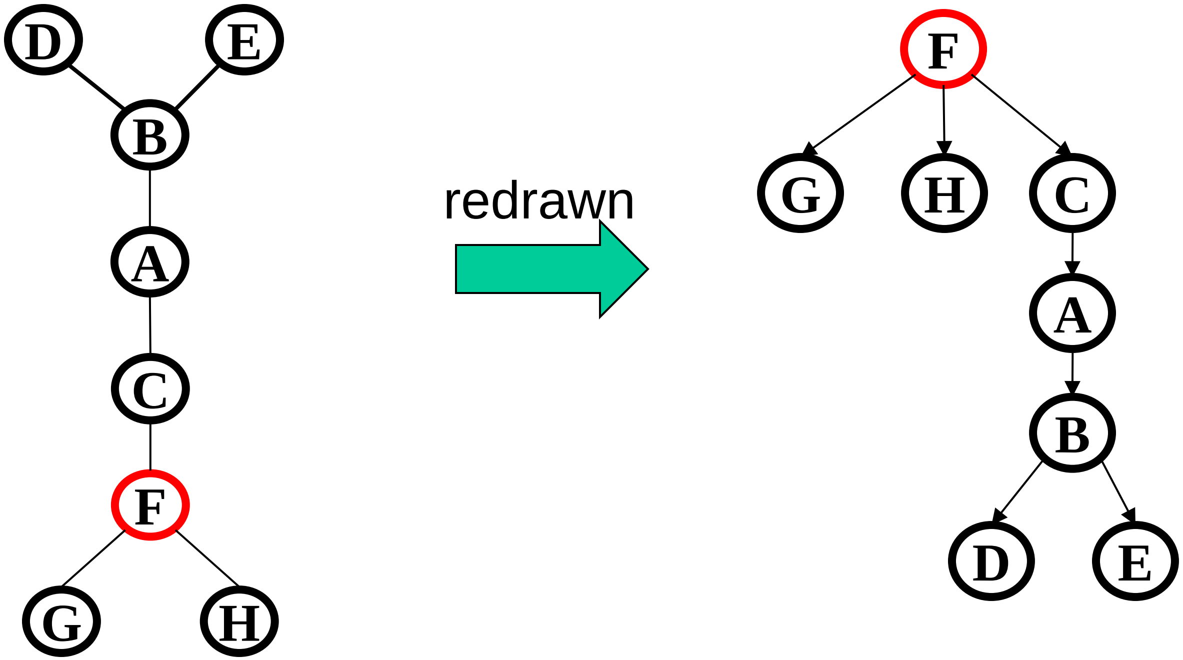
- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
  - The tree is just drawn differently



Rooted Trees

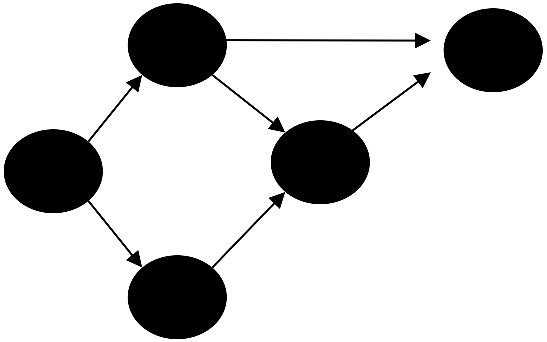
- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children

- Given a tree, picking a root gives a unique rooted tree
  - The tree is just drawn differently

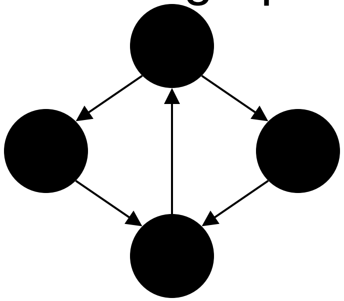


Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree



  - Every DAG is a directed graph
  - But not every directed graph is a DAG



Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites

Density / Sparsity

- Let E be the set of edges and V the set of vertices.
- Then $0 \leq |E| \leq |V|^2$
- And $|E|$ is $O(|V|^2)$

- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means "lots of edges"
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means "most possible edges missing"

What is the Data Structure?

- So graphs are really useful for lots of data and questions
  - For example, "what's the lowest-cost path from x to y"

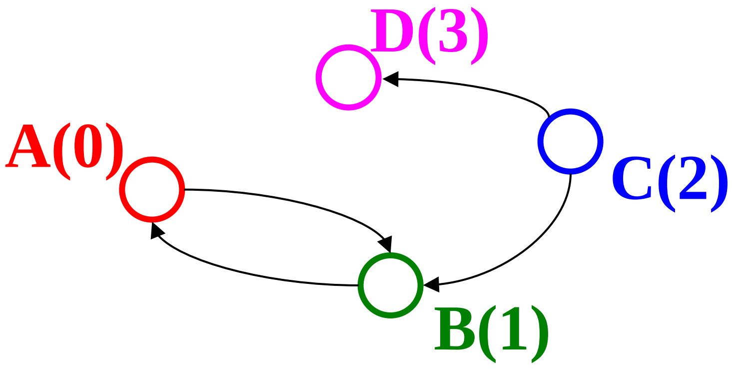
- But we need a data structure that represents graphs

- The "best one" can depend on:
  - Properties of the graph (e.g., dense versus sparse)
  - The common queries (e.g., "is **`(u,v)`** an edge?" versus "what are the neighbors of node **`u`**?")

- So we'll discuss the two standard graph representations
  - Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each node a number from **`0`** to **`|V|-1`**
- A **`|V|`** x **`|V|`** matrix (i.e., 2-D array) of Booleans (or **1 vs. 0**)
  - If **`M`** is the matrix, then **`M[u][v]`** being **`true`** means there is an edge from **`u`** to **`v`**



	0	1	2	3
0	F	T	F	F
1	T	F	F	F
2	F	T	F	T
3	F	F	F	F

Adjacency Matrix Properties

	0	1	2	3
0	F	T	F	F
1	T	F	F	F
2	F	T	F	T
3	F	F	F	F

- Running time to:
  - Get a vertex's out-edges: ***O*(|V|)**
  - Get a vertex's in-edges: ***O*(|V|)**
  - Decide if some edge exists: ***O*(1)**
  - Insert an edge: ***O*(1)**
  - Delete an edge: ***O*(1)**

- Space requirements:
  - $|V|^2$ **bits**

- Best for sparse or dense graphs?
  - Best for dense graphs

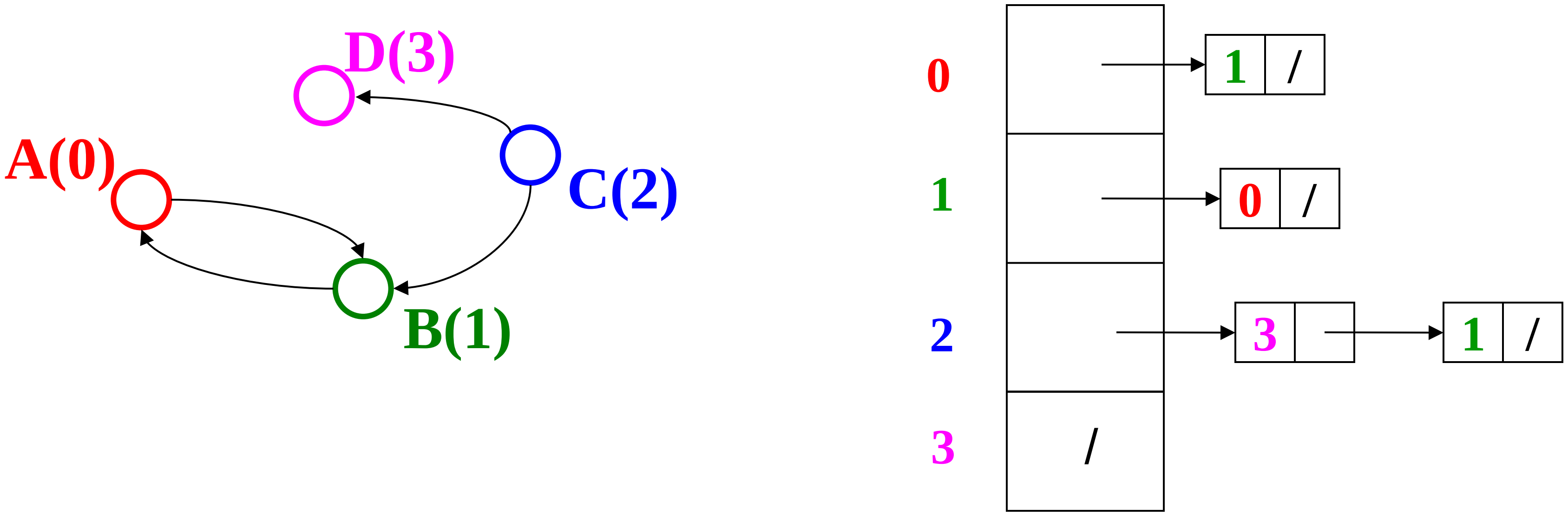
Adjacency Matrix Properties

- How will the adjacency matrix vary for an *undirected graph*?
  - Undirected will be symmetric around the diagonal

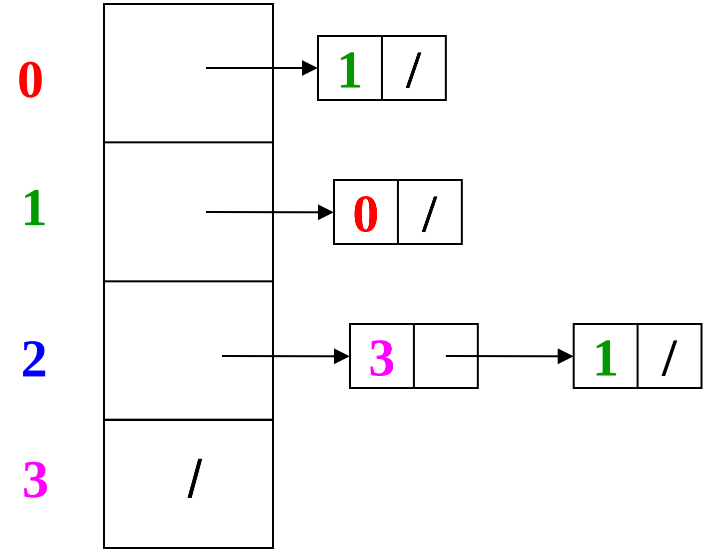
- How can we adapt the representation for *weighted graphs*?
  - Instead of a Boolean, store a number in each cell
  - Need some value to represent ‘not an edge’
    - In *some* situations, 0 or -1 works

Adjacency List

- Assign each node a number from **`0`** to **`|V|-1`**
- An array of length **`|V|`** in which each entry stores a list of all adjacent vertices (e.g., linked list)



Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:
    O(*d*) where *d* is out-degree of vertex
  - Get all of a vertex's in-edges:
    O(|E|) (but could keep a second adjacency list for this!)
  - Decide if some edge exists:
    O(*d*) where *d* is out-degree of source
  - Insert an edge:
    O(1) (unless you need to check if it's there)
  - Delete an edge:
    O(*d*) where *d* is out-degree of source
- Space requirements:
  - *O*(|V|+|E|)
- Good for sparse graphs

Next…

Okay, we can represent graphs

Next lecture we'll implement some useful and non-trivial algorithms

- Topological sort: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors

- Shortest paths: Find the shortest or lowest-cost path from x to y
  - Related: Determine if there even is such a path