CSE373: Data Structures and Algorithms

Lecture 4: Asymptotic Analysis

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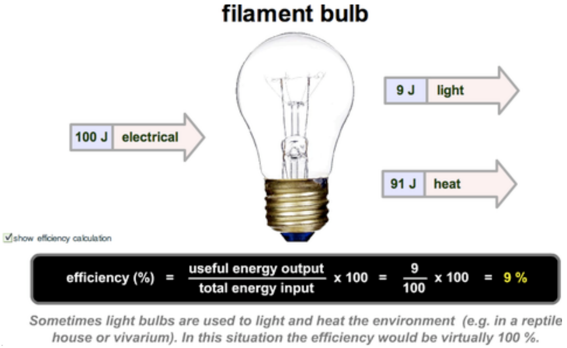
Efficiency



- What does it mean for an algorithm to be *efficient*?
  - We primarily care about *time* (and sometimes *space*)
- Is the following a good definition?
  - "An algorithm is efficient if, when implemented, it runs **quickly** on **real** input instances"
  - What does "quickly" mean?
  - What constitutes "real input?"
  - How does the algorithm *scale* as input size changes?

Gauging efficiency (performance)

- Uh, why not just run the program and time it?
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)
- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

– Various possible answers (clarity, security, …)
– But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

We will focus on large inputs because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast)

– Time difference really shows up as n grows

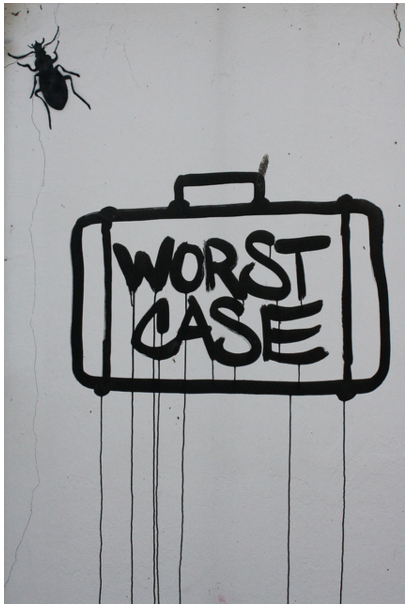
Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!

We usually care about worst-case running times

- Has proven reasonable in practice
  - Provides some guarantees
- Difficult to find a satisfactory alternative
  - What about average case?
  - Difficult to express full range of input
  - Could we use randomly-generated input?
  - May learn more about generator than algorithm



Example

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case?

k is in arr[0]

c1 steps

= $O(1)$

Worst case?

k is not in arr

c2*(arr.length)

= O(arr.length)

Binary search

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

- Can also be done non-recursively but “doesn’t matter” here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
   if(lo==hi)       return false;
   if(arr[mid]==k) return true;
   if(arr[mid]< k) return help(arr,k,mid+1,hi);
   else            return help(arr,k,lo,mid);
}
```

Binary search

Best case: c1 steps = $O(1)$

Worst case: $T(n)$ = c2 steps + $T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

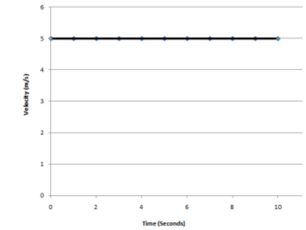
1. Determine the recurrence relation. What is the base case?
   - $T(n) = c2 + T(n/2)$ $T(1) = c1$ first eqn.
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = c2 + c2 + T(n/4)$ 2nd eqn.

     $= c2 + c2 + c2 + T(n/8)$ 3rd eqn.

     $= \ldots$

     $= c2(k) + T(n/(2^k))$ kth eqn.
3. Find a closed-form expression by setting *the number of expansions* to a value (e.g. 1) which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \mathbf{log}_2\, n$
   - So $T(n) = c2\, \mathbf{log}_2\, n + T(1)$
   - So $T(n) = c2\, \mathbf{log}_2\, n + c1$ (get to base case and do it)
   - So $T(n)$ is $O(\mathbf{log}\, n)$

Ignoring constant factors



- So binary search is $O(\log n)$ and linear search is $O(n)$
  - But which is faster?

- Could depend on constant factors
  - How *many* assignments, additions, etc. for each *n*
    - *E.g.* $T(n) = 5{,}000{,}000n$ vs. $T(n) = 5n^2$
  - And could depend on overhead unrelated to *n*
    - *E.g.* $T(n) = 5{,}000{,}000 + \log n$ vs. $T(n) = 10 + n$

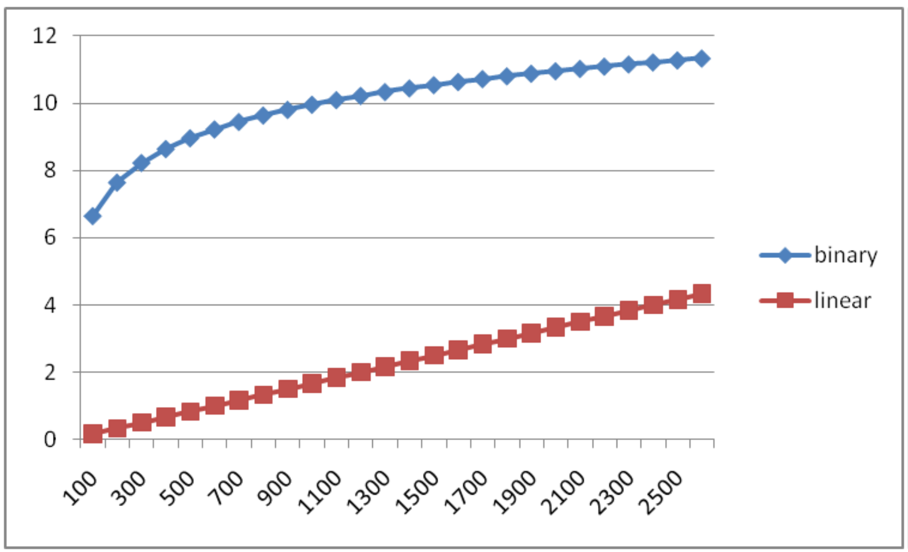
- But there exists some n_0 such that for all $n > n_0$ binary search wins

- Let's play with a couple plots to get some intuition…

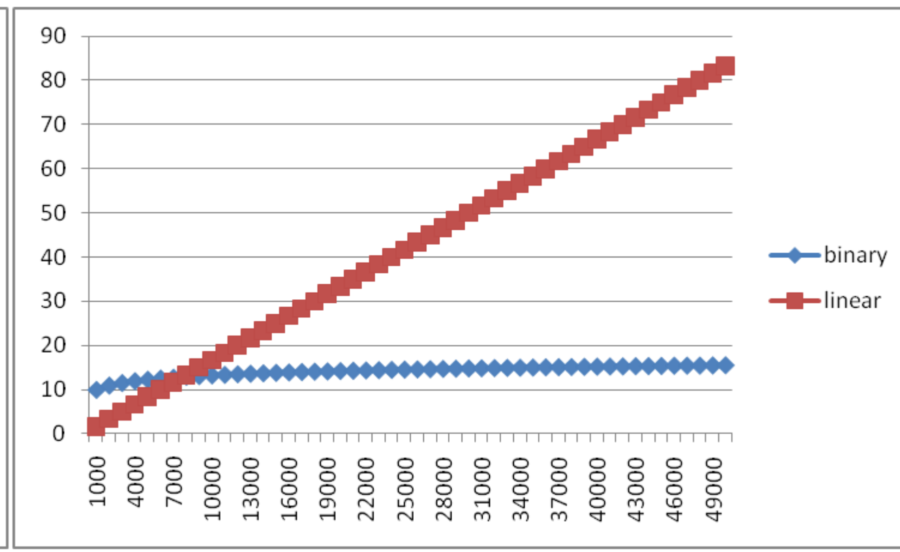
Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2015 model vs. 1994)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search

not enough iterations to show it



enough iterations to show it



Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** $f(n)$

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

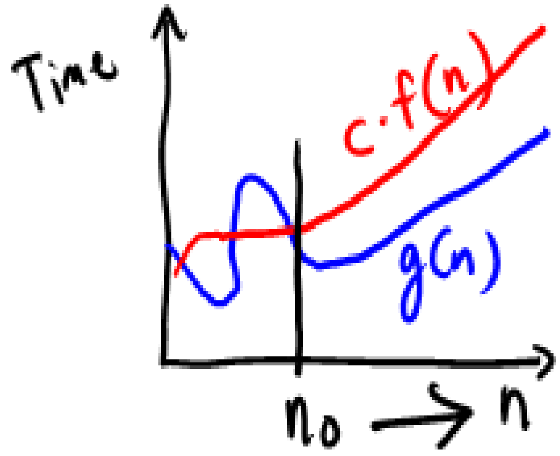
- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

~~But we would never say $O(n^2) = (3n^2+17)$~~

Big-O, formally

Definition: g(n) is in O(f(n)) if there exist positive constants c and n_0 such that

$g(n) \le c\, f(n)$ for all $n \ge n_0$



- To show g(n) is in O(f(n)), pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"
  - Example: Let $g(n) = 3n^2+17$ and $f(n) = n^2$

    c=5 and n_0=10 is more than good enough

    $(3*10^2)+17 \le 5*10^2$ so $3n^2+17$ is $O(n^2)$
- This is "less than or equal to"
  - So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.
    - But usually we're interested in the **tightest** upper bound.

Example 1, using formal definition

- Let $g(n) = 1000n$ and $f(n) = n$
  - To prove $g(n)$ is in $O(f(n))$, find a valid c and n_0
  - We can just let $c = 1000$.
  - That works for any n_0, such as $n_0 = 1$.
  - $g(n) = 1000n \leq c\ f(n) = 1000n$ for all $n \geq 1$.

Definition: $g(n)$ is in $O(\ f(n)\ )$ if there exist positive constants c and n_0 such that

$$g(n) \leq c\ f(n) \qquad \text{for all } n \geq n_0$$

Example 1', using formal definition

- Let g(n) = 1000n and f(n) = n^2
  - To prove g(n) is in O(f(n)), find a valid c and n_0
  - The "cross-over point" is n=1000
    - g(n) = 1000*1000 and f(n) = 1000^2
  - So we can choose n_0=1000 and c=1
  - Then g(n) = 1000n $\leq$ c f(n) = 1n^2 for all n $\geq$ 1000

Definition: g(n) is in O(f(n)) if there exist positive constants c and n_0 such that

$$g(n) \leq c\, f(n) \qquad \text{for all } n \geq n_0$$

Examples 1 and 1'

- Which is it?
- Is $g(n) = 1000n$ called $f(n)$ or $f(n^2)$?

- By definition, it can be either one.
- We prefer to use the smallest one.

Example 2, using formal definition

- Let $g(n) = n^4$ and $f(n) = 2^n$
  - To prove $g(n)$ is in $O(f(n))$, find a valid c and n_0
  - We can choose n_0=20 and c=1
    - $g(n) = 20^4$ vs. $f(n) = 1*2^{20}$
  - $g(n) = n^4 \leq c\ f(n) = 1*2^n$ for all $n \geq 20$

  - If I were doing a complexity analysis, would I pick $O(2^n)$?

Definition: $g(n)$ is in $O(f(n))$ if there exist positive constants c and n_0 such that

$$g(n) \leq c\ f(n) \quad \text{for all } n \geq n_0$$

Comparison

n	n^4	2^n
10	10,000	1,024
20	160,000	1,048,576
30	810,000	1,073,741,824
40	2,560,000	1.0995×10^{12}

What's with the c

- The constant multiplier c is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
- Consider:

  $g(n) = 7n+5$

  $f(n) = n$

– These have the same asymptotic behavior (linear)
  - So $g(n)$ is in $O(f(n))$ even through $g(n)$ is always larger
  - The c allows us to provide a coefficient so that $g(n) \leq c\ f(n)$

– In this example:
  - To prove $g(n)$ is in $O(f(n))$, have $c = 12$, $n_0 = 1$

    $(7*1)+5 \leq 12*1$

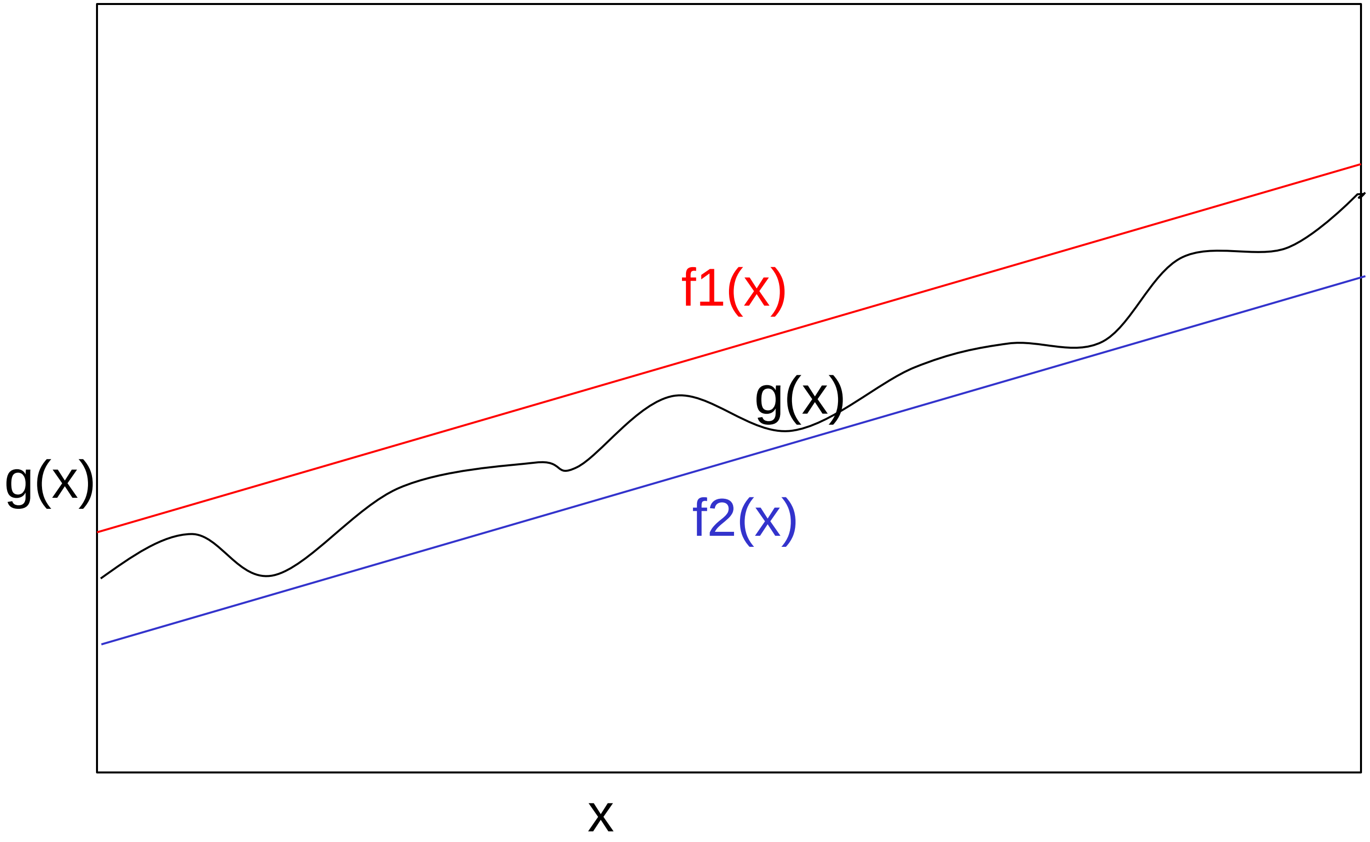
What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations
  - Both are **O($\boldsymbol{n^2}$)**
- Eliminate low-order terms because they have vanishingly small impact as n grows
  - $5n^5 + 40n^4 + 30n^3 + 20n^2 + 10^n + 1$ is ?
  - $O(n^5)$
- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

Upper and Lower Bounds

f1(x) is an upper bound for g(x); f2(x) is a lower bound.

$g(x) \leq f1(x)$ and $g(x) \geq f2(x)$.



More Asymptotic Notation*

*approaching arbitrarily closely

- Upper bound: $O(\ f(n)\ )$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(\ f(n)\ )$ if there exist constants c and n_0 such that
    $g(n) \leq c\ f(n)$ for all $n \geq n_0$
- Lower bound: $\Omega(\ f(n)\ )$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(\ f(n)\ )$ if there exist constants c and n_0 such that
    $g(n) \geq c\ f(n)$ for all $n \geq n_0$
- Tight bound: $\theta(\ f(n)\ )$ is the set of all functions asymptotically equal to $f(n)$
  - $g(n)$ is in $\theta(\ f(n)\ )$ if **both** $g(n)$ is in $O(\ f(n)\ )$ **and**
    $g(n)$ is in $\Omega(\ f(n)\ )$

Correct terms, in theory

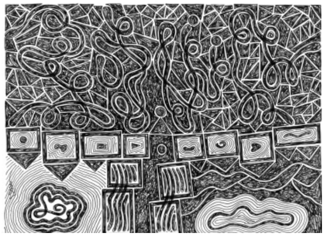
A common error is to say $O(f(n))$ when you mean $\theta(f(n))$

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is $\theta(n)$
- That means that it is not, for example $O(\mathbf{log}\ n)$

Less common notation:

- "little-oh": intersection of "big-Oh" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… ≤
  - Example: array sum is O(n) and $o(n^2)$ but not $o(n)$
- "little-omega": intersection of "big-Omega" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… ≥
  - Example: array sum is O(n) and $\omega(\mathbf{log}\ n)$ but not $\omega(n)$

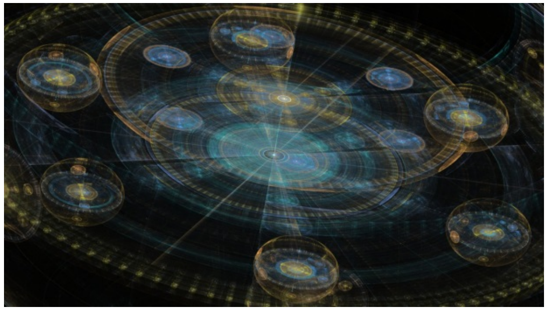
What we are analyzing: Complexity



- The most common thing to do is give an O upper bound to the worst-case running time of an algorithm

- Example: binary-search algorithm
  - Common: $O(\mathtt{log}\ n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\mathtt{log}\ n)$ in the worst-case (lower bound)
    - *No* algorithm can do better
    - A ***problem*** cannot be $O(f(n))$ since you can always make a slower algorithm

Other things to analyze



- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the *probability distribution* of inputs
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting
  - Sometimes an *amortized guarantee*
    - Average time over any sequence of operations

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Addendum: Timing vs. Big-Oh Summary

- Big-oh is an essential part of computer science's mathematical foundation
  - Examine the algorithm itself, not the implementation
  - Reason about (even prove) performance as a function of *n*

- Timing also has its place
  - Compare implementations
  - Focus on data sets you care about (versus worst case)
  - Determine what the constant factors "really are"