CSE373: Data Structures & Algorithms

Lecture 9: Priority Queues and Binary Heaps

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Priority Queue ADT

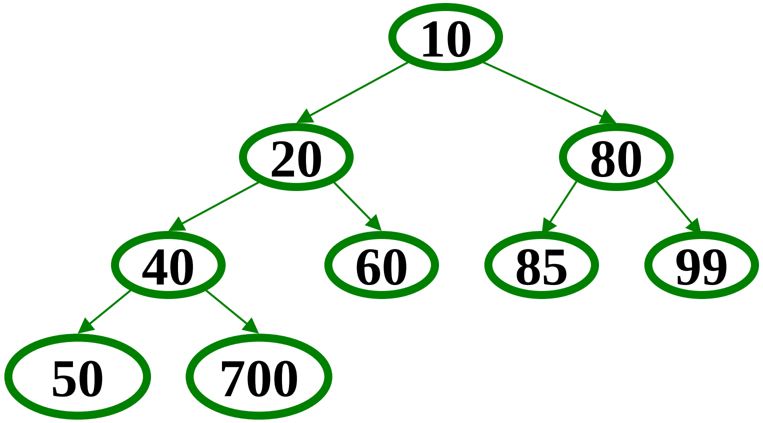
- A **priority queue** holds *compare-able* items
- Each item in the priority queue has a "priority" and "data"
  - In our examples, the *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"

- Operations:
  - **`insert:`** *adds* an element to the priority queue
  - **`deleteMin:`** *returns* and *deletes* the item with greatest priority (min)
  - **`is_empty`**

- Our data structure: A *binary min-heap* (or *binary heap* or *heap*) has:
  - Structure property: A *complete* binary tree
  - Heap property: The priority of every (non-root) node is less important than (>) the priority of its parent ***(Not** a binary search tree)*

Operations: basic idea

- **`deleteMin`**:
  1. Remove root node
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- **`insert:`**
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property

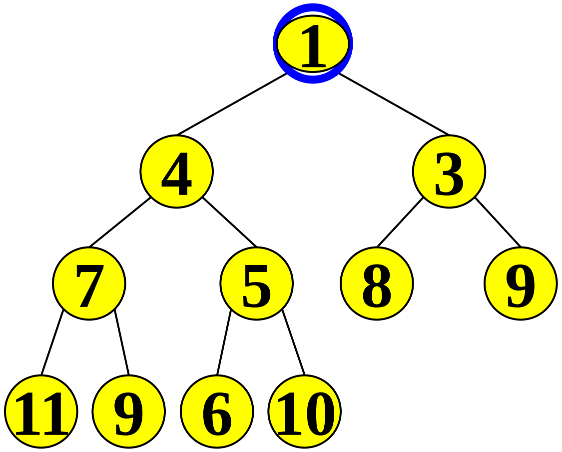


Overall strategy:

- *Preserve structure property*
- *Break and restore heap property*

DeleteMin

Delete (and later return) value at root node

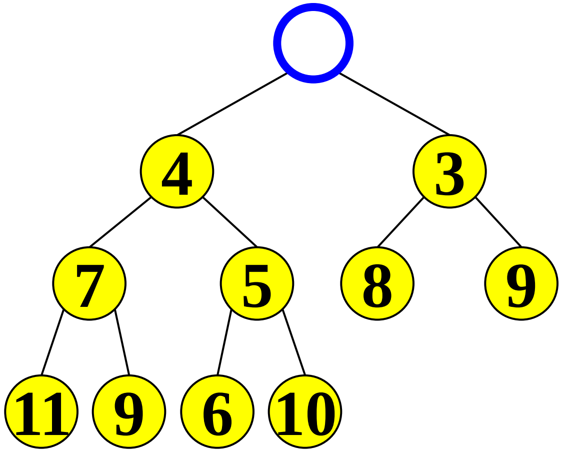


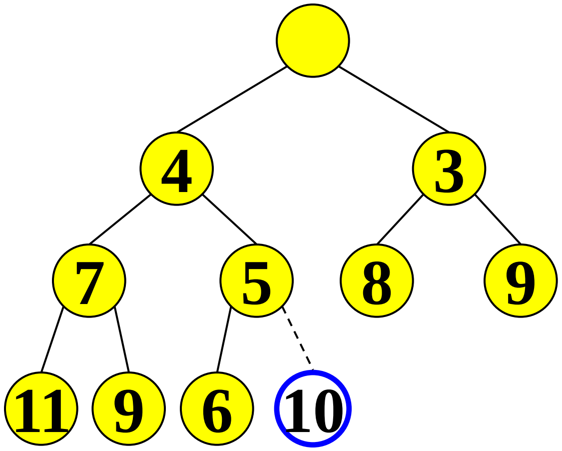
DeleteMin: Keep the Structure Property

- We now have a "hole" at the root
  - Need to fill the hole with another value

- Keep structure property: When we are done, the tree will have one less node and must still be complete

- Pick the last node on the bottom row of the tree and move it to the "hole"

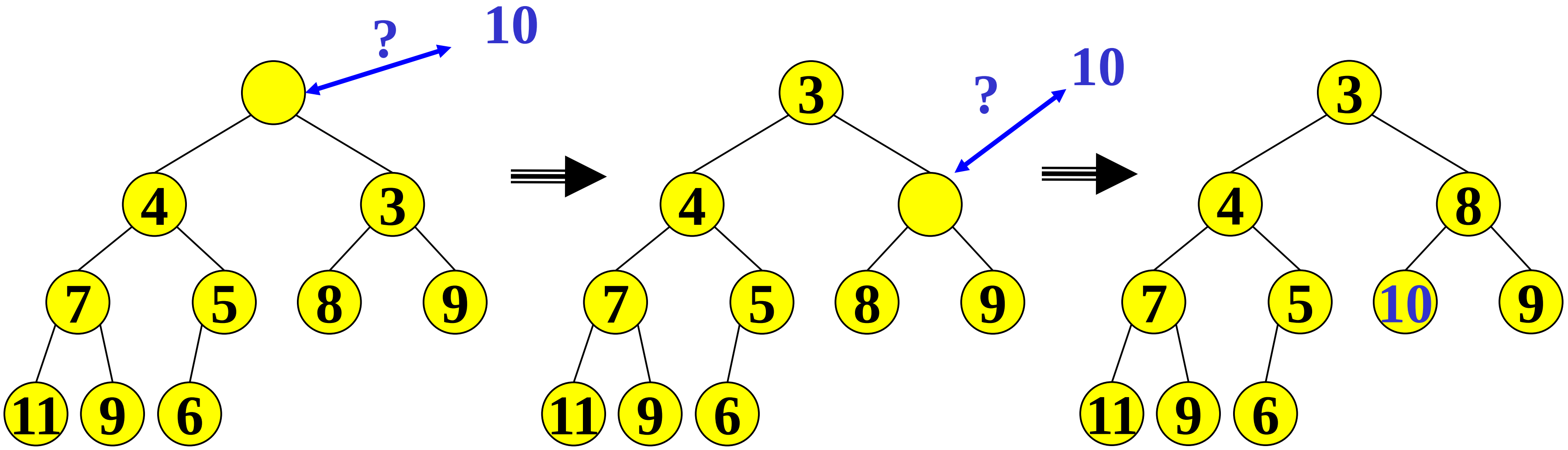




DeleteMin: Restore the Heap Property

Percolate down:

- Keep comparing priority of item with both children
- If priority is less important, swap with the most important child and go down one level
- Done if both children are less important than the item or we've reached a leaf node



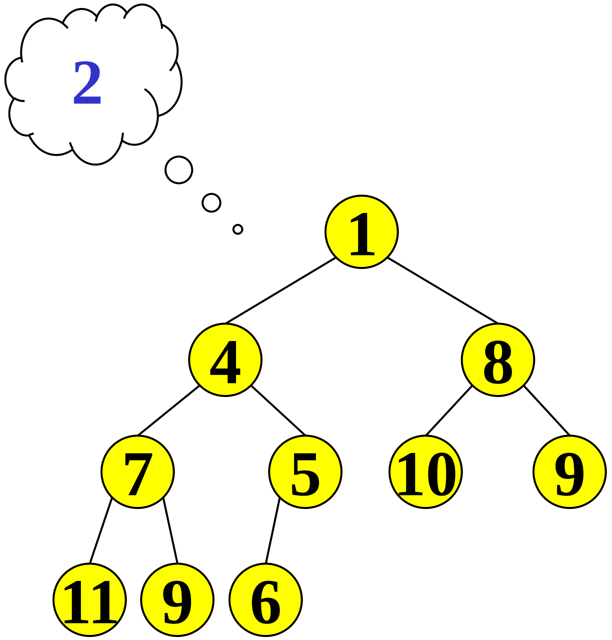
Run time?

Runtime is O(height of heap) $O(\log n)$

Height of a complete binary tree of n nodes = $\lfloor \log_2(n) \rfloor$

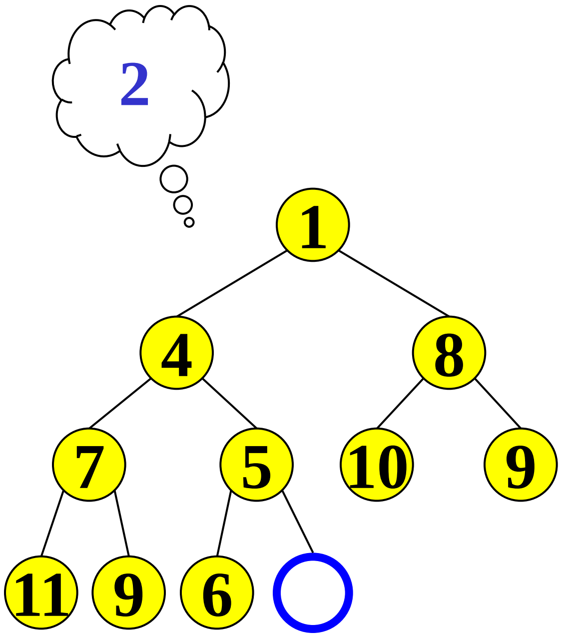
Insert

- Add a value to the tree
- Afterwards, structure and heap properties must still be correct



Insert: Maintain the Structure Property

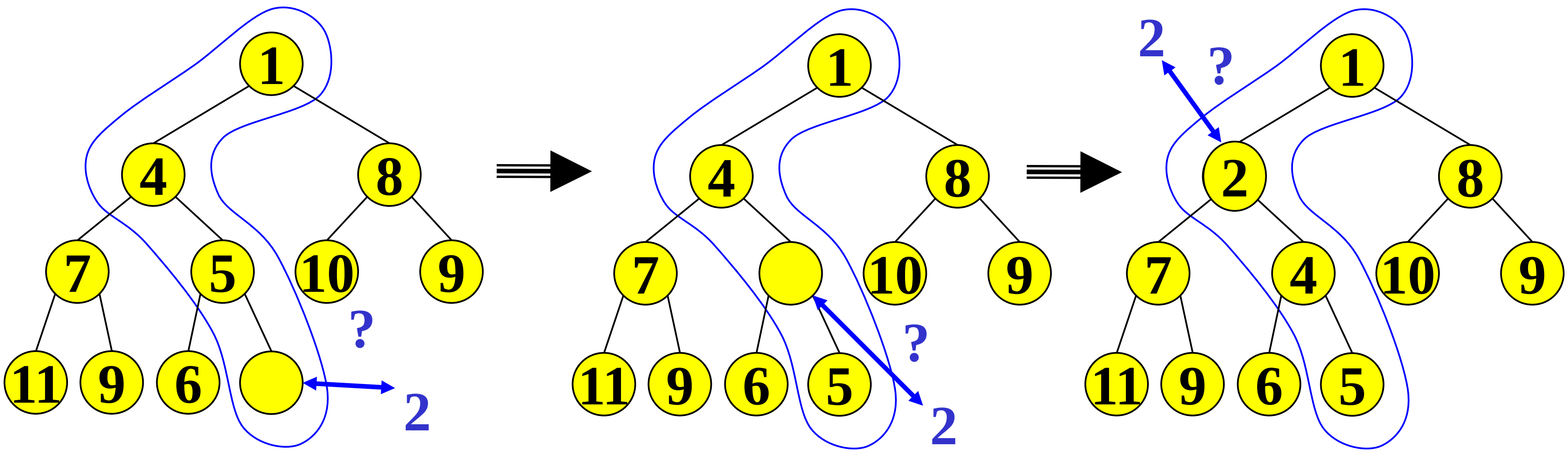
- There is only one valid tree shape after we add one more node
- So put our new data there and then focus on restoring the heap property



Insert: Restore the heap property

Percolate up:

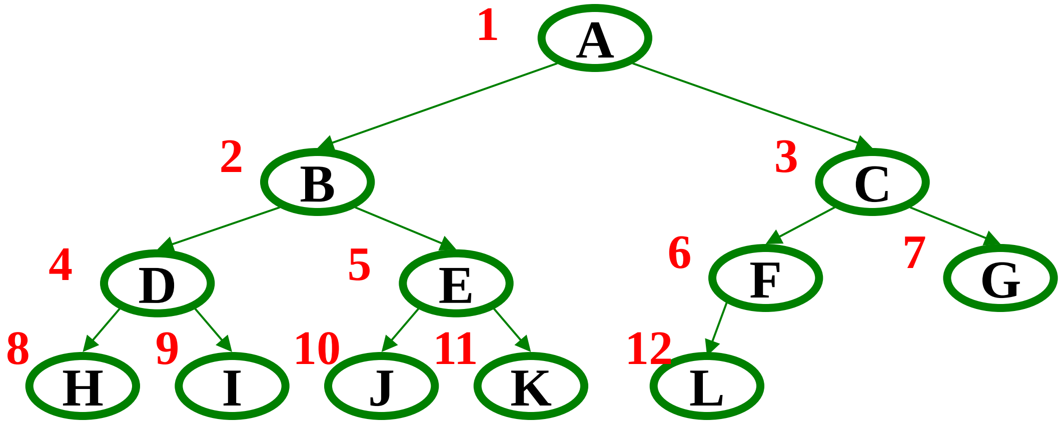
- Put new data in new location
- If parent is less important, swap with parent, and continue
- Done if parent is more important than item or reached root



What is the running time?

Like `deleteMin`, worst-case time proportional to tree height: $O(\log n)$

Array Representation of Binary Trees



From node **`i`**:

left child: **`i*2`**
right child: **`i*2+1`**
parent: **`i/2`**

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:

- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index **`size`**

Minuses:

- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

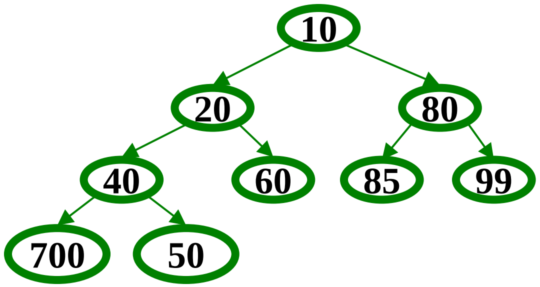
Plusses outweigh minuses: “this is how people do it”

This pseudocode uses ints. In real use, you will have data nodes with priorities.

Pseudocode: insert into binary heap

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

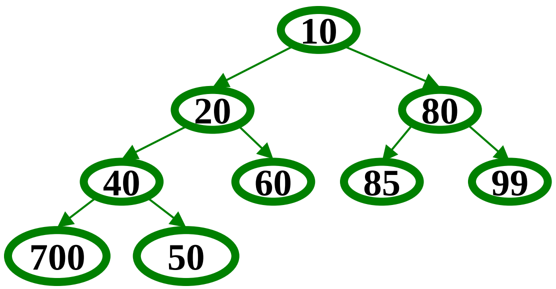
```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



	10	**20**	**80**	**40**	**60**	**85**	**99**	**700**	**50**				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: deleteMin from binary heap

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

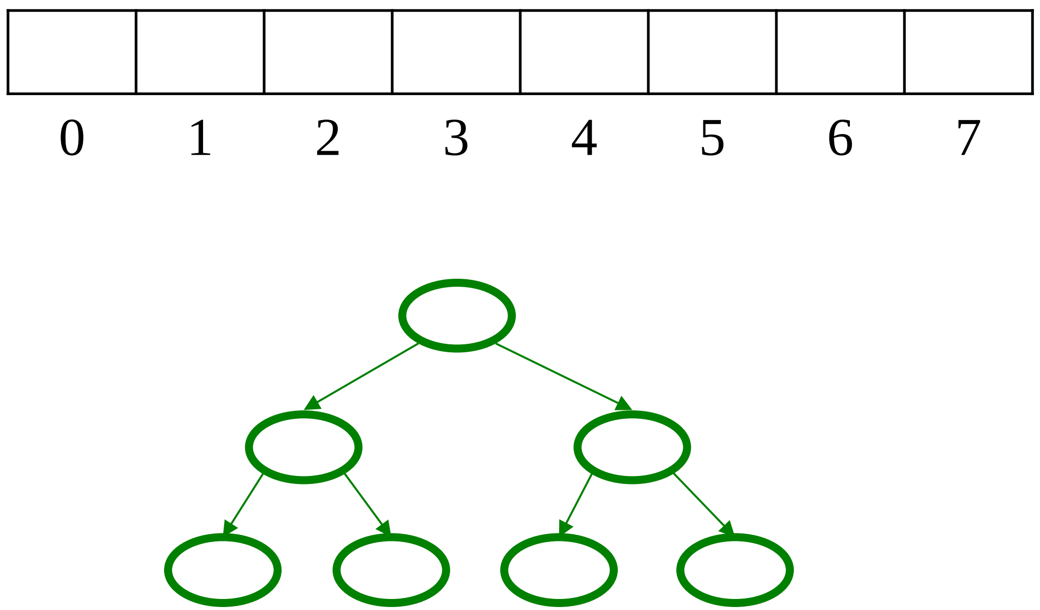



```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(right > size ||
     arr[left] < arr[right])
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```

	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Example

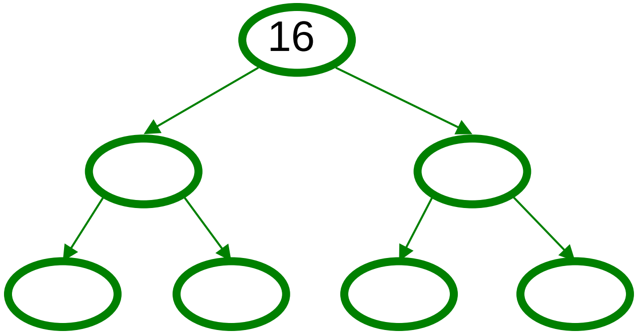
1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

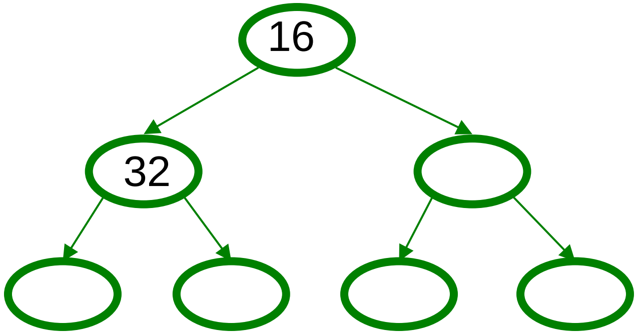
	16						
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

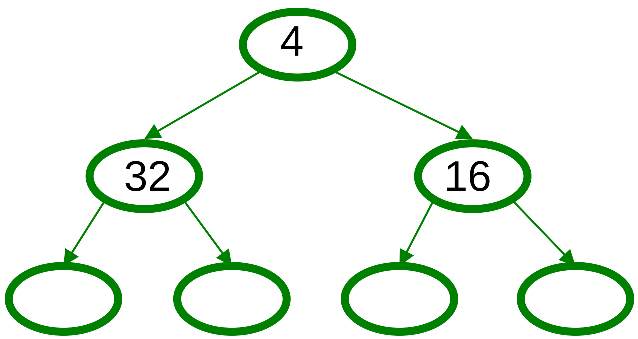
	16	**32**					
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

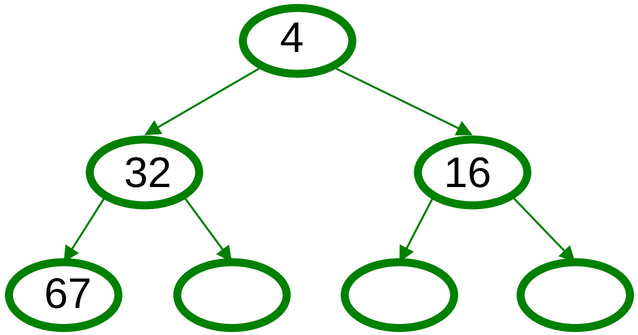
	4	**32**	**16**				
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

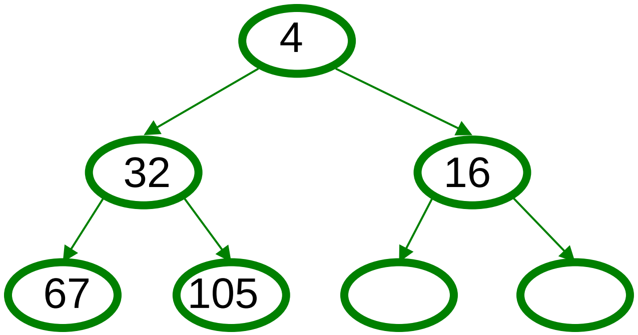
	4	**32**	**16**	**67**			
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

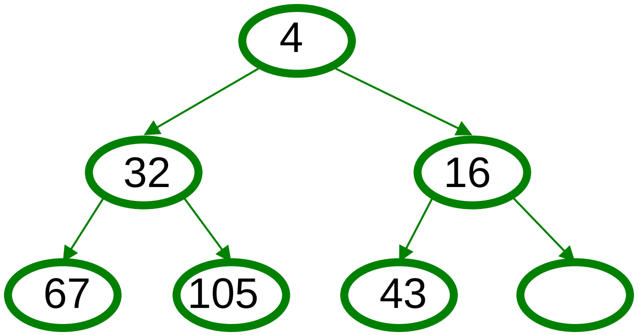
	4	**32**	**16**	**67**	**105**		
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

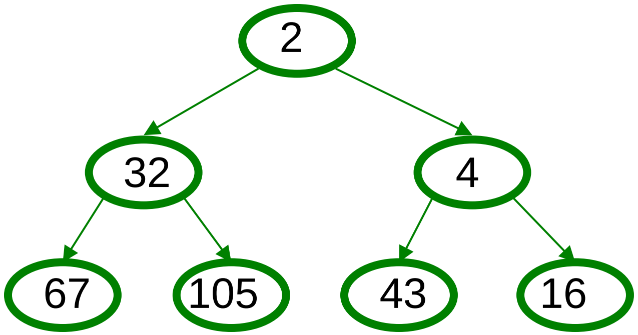
	4	**32**	**16**	**67**	**105**	**43**	
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

	2	**32**	**4**	**67**	**105**	**43**	**16**
0	1	2	3	4	5	6	7



Other operations

- **`decreaseKey`**: given pointer to object in priority queue (e.g., its array index), lower its priority value by p
  - Change priority and percolate up

- **`increaseKey`**: given pointer to object in priority queue (e.g., its array index), raise its priority value by p
  - Change priority and percolate down

- **`remove`**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**

- *n* **`insert`**s works
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - $O(n$ **`log`** $n)$

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an $O(n)$ algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

Floyd's Method

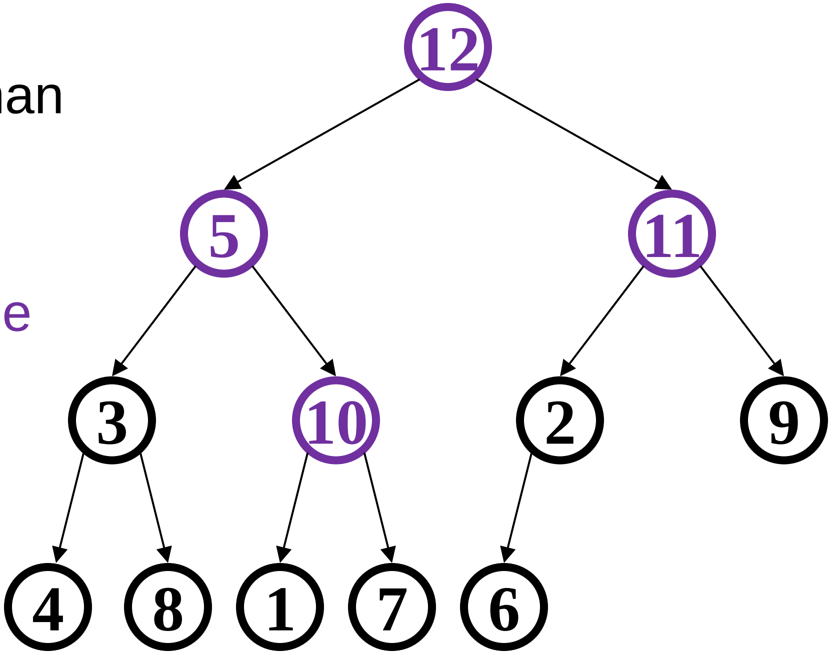
1. Use *n* items to make any complete tree you want
   – That is, put them in array indices 1,…,*n*

2. Treat it as a heap and fix the heap-order property
   – Bottom-up: leaves are already in heap order, work up toward the root one level at a time

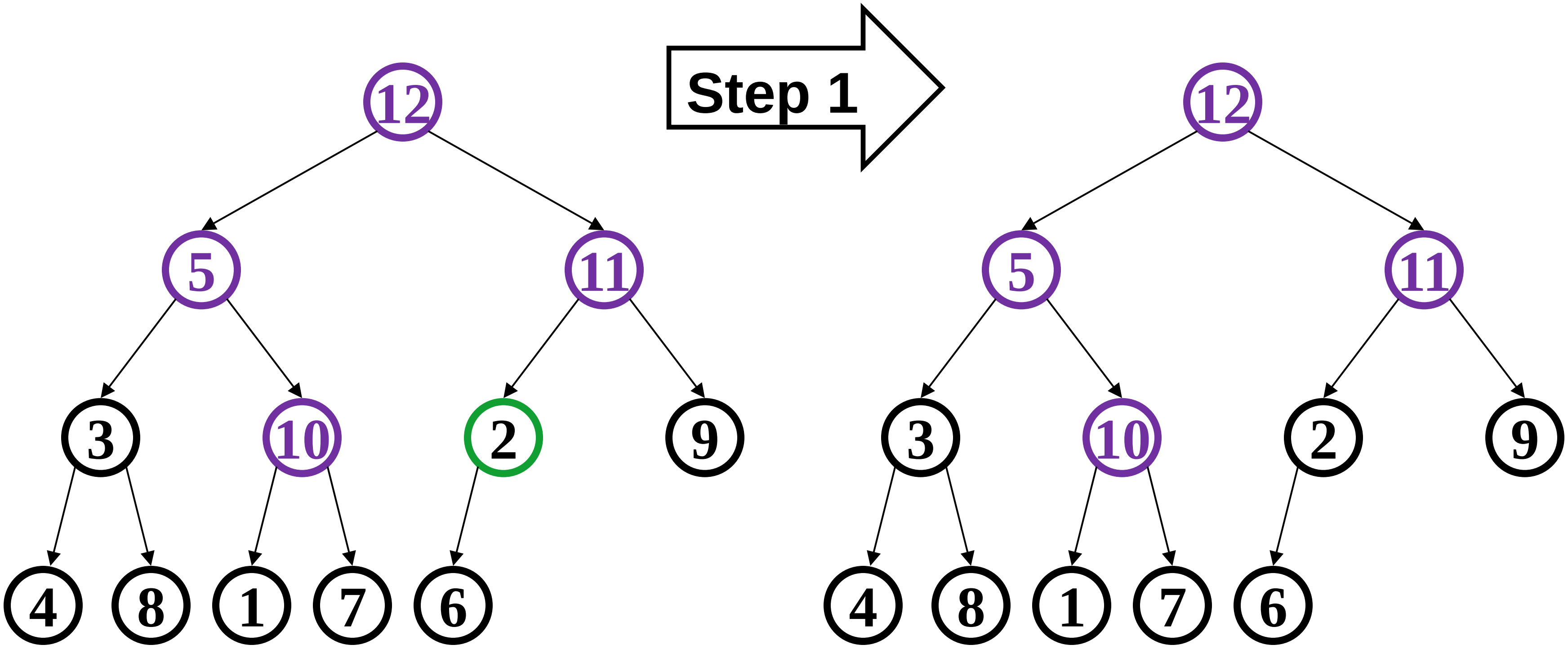
```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Example

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

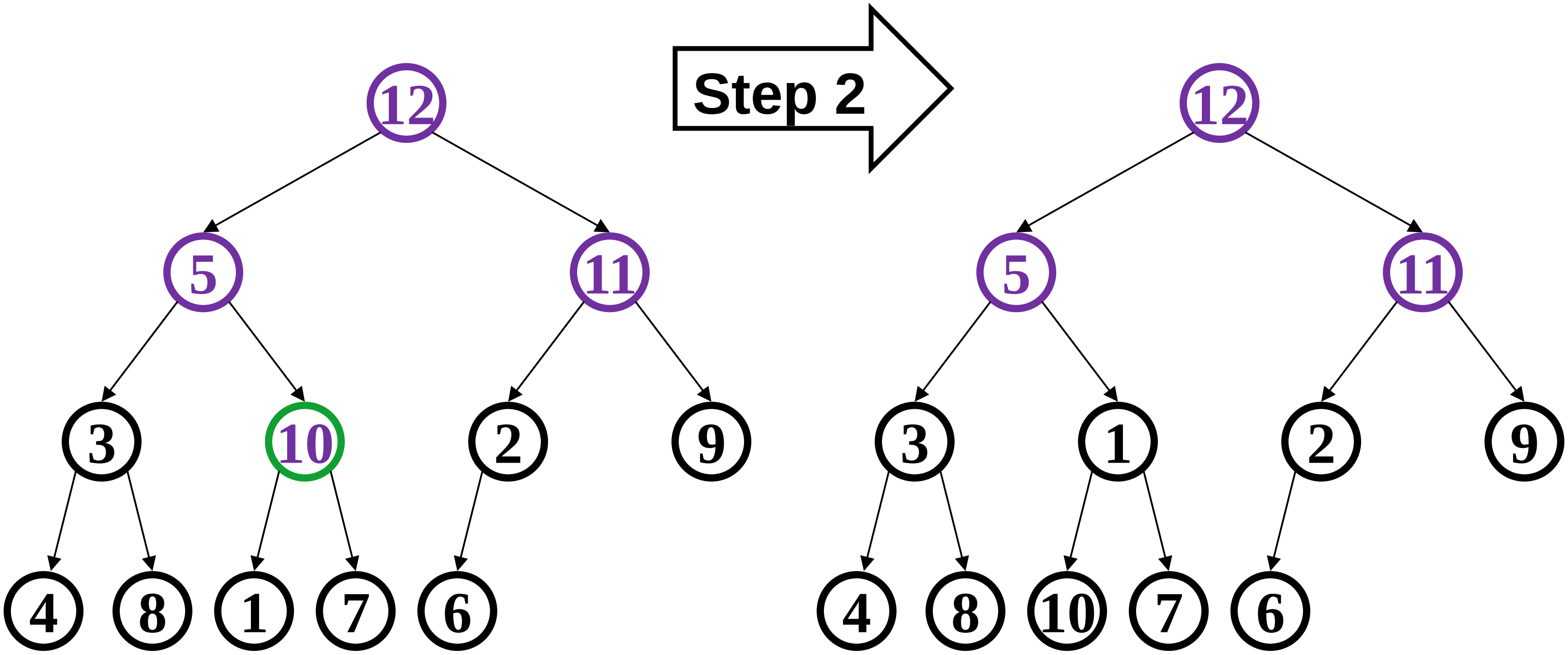


Example



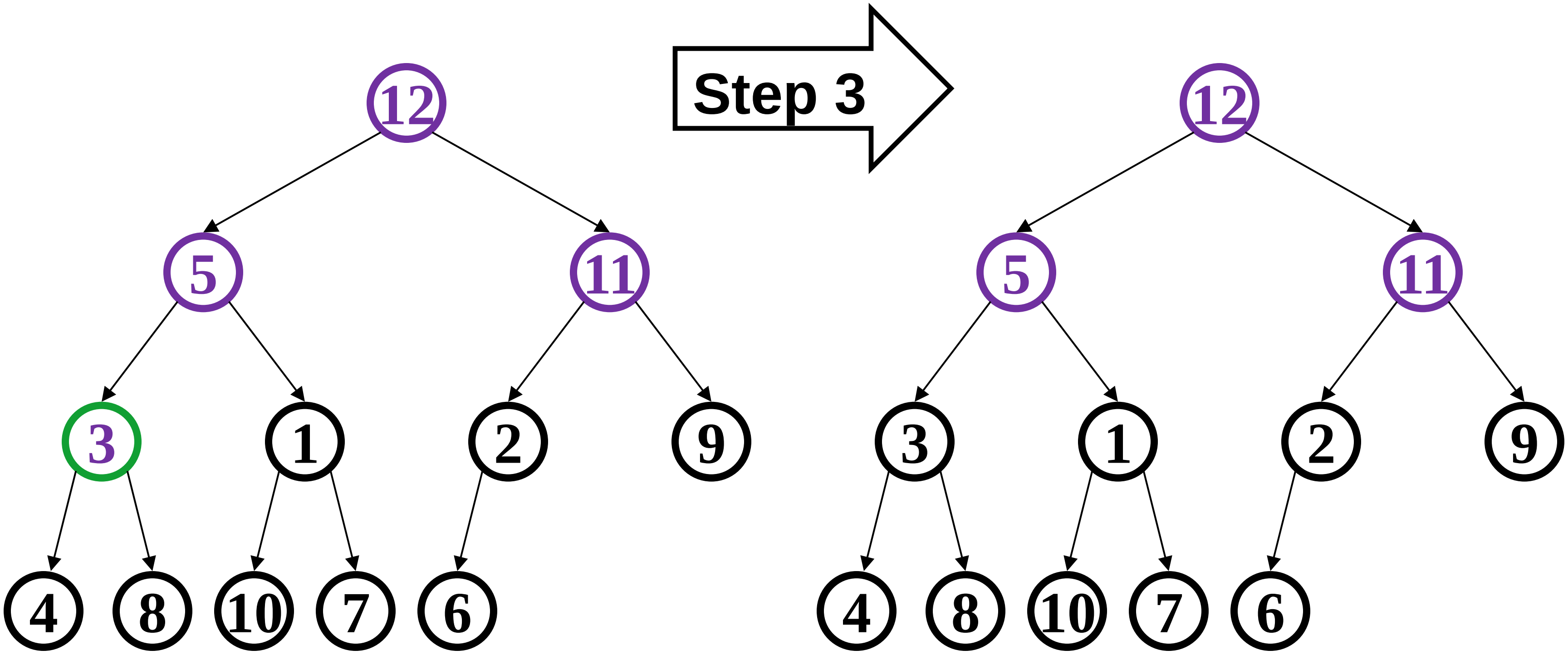
- Happens to already be less than children (er, child)

Example



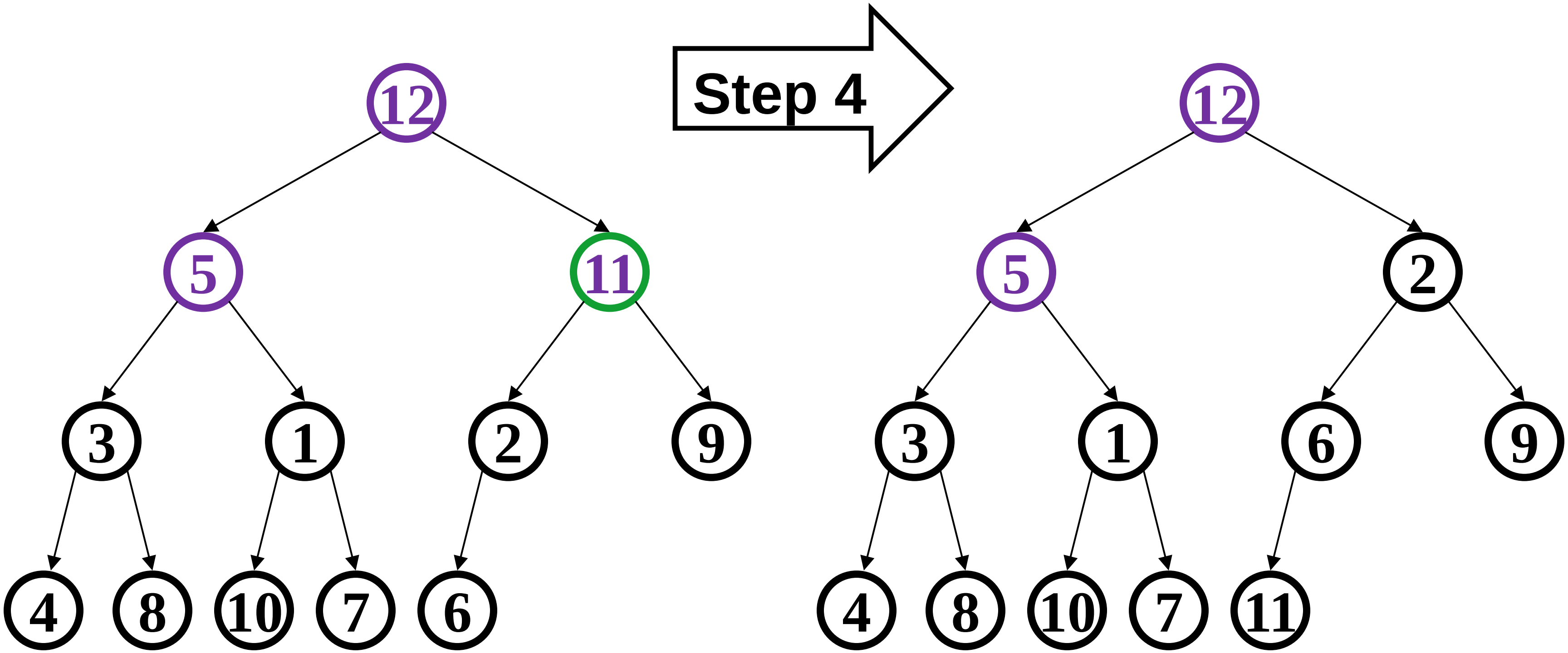
- Percolate down (notice that moves 1 up)

Example



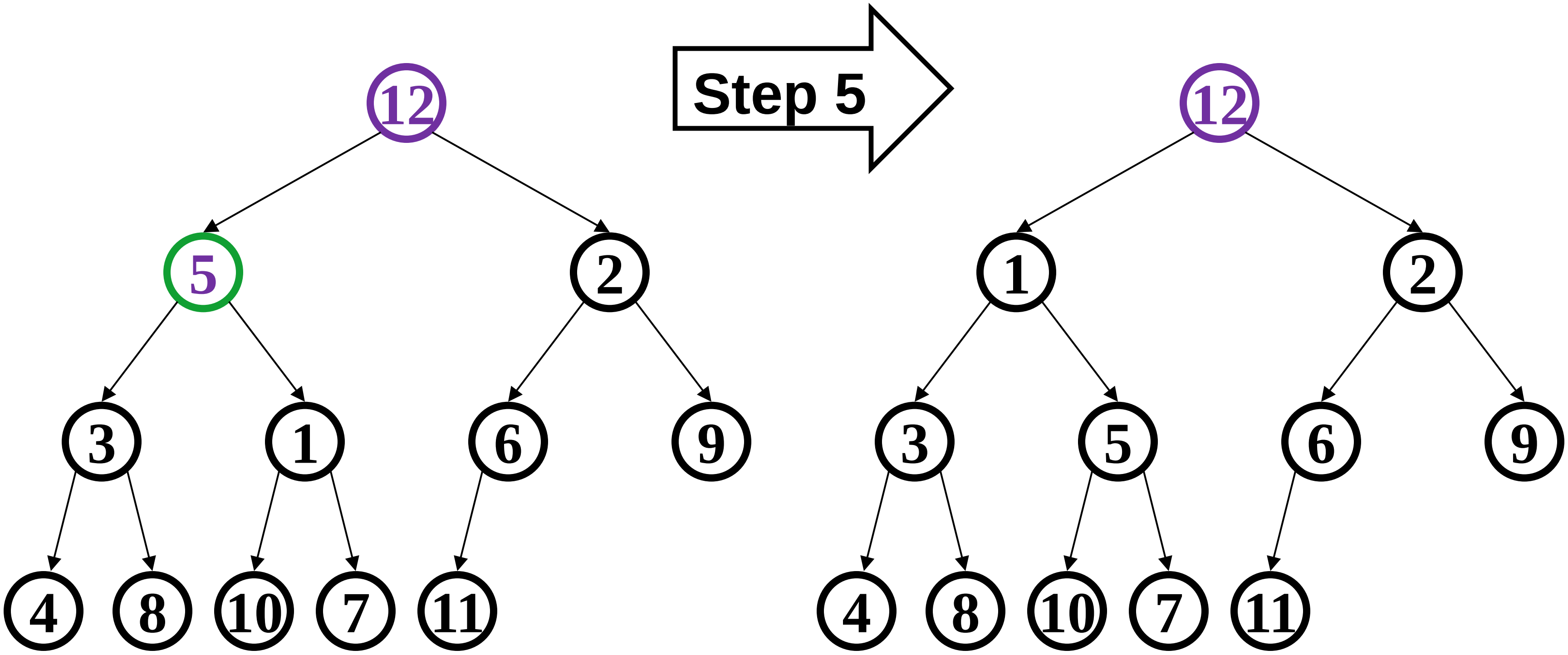
- Another nothing-to-do step

Example

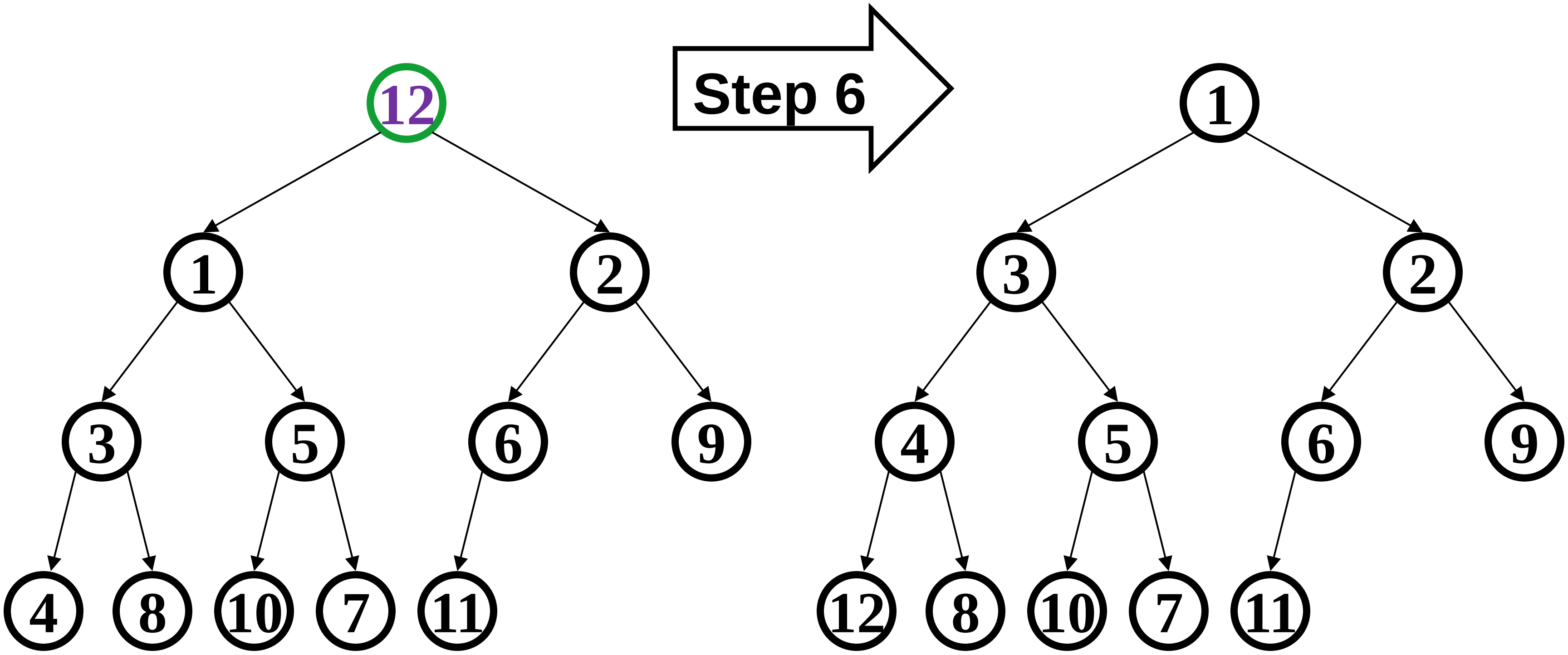


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all **j**>**i**, **arr[j]** is less than its children

- True initially: If **j > size/2**, then **j** is a leaf
  - Otherwise its left child would be at position > **size**
- True after one more iteration: loop body and **percolateDown** make **arr[i]** less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Easy argument: **`buildHeap`** is *O*(*n* **`log`** *n*) where *n* is **`size`**

- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is *O*(**`log`** *n*)

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  - So at most **`2*(size/2)`** *total* percolate steps: $O(n)$

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n$ **`log`** $n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was O(n **`log`** n)
    - Tighter analysis shows same algorithm is $O(n)$