Understanding ADTs

Ways to get your design right

The hard way

- Start hacking
- When something doesn't work, hack some more
  - How do you know it doesn't work?
  - Need to reproduce the errors your users experience
- Apply caffeine liberally

The easier way

- Plan first (specs, system decomposition, tests, ...)
- Less apparent progress upfront
- Faster completion times
- Better delivered product
- Less frustration

Ways to verify your code

The hard way

- Make up some inputs
- If it doesn't crash, ship it
- When it fails in the field, attempt to debug

The easier way

- Reason about possible behaviors and desired outcomes
- Construct simple tests that exercise those behaviors

Another way that can be easy

- Prove that the system does what you want
  - Rep invariants are preserved
  - Implementation satisfies specification
- Proof can be formal or informal (we will be informal)
- Complementary to testing

Uses of reasoning

Goal: correct code

Verify that rep invariant is satisfied

Verify that the implementation satisfies the spec

Verify that client code behaves correctly

Assuming that the implementation is correct

Goal: Demonstrate that rep invariant is satisfied

Exhaustive testing

Create every possible object of the type
Check rep invariant for each object
Problem: impractical

Limited testing

Choose representative objects of the type
Check rep invariant for each object
Problem: did you choose well?

Reasoning

Prove that all objects of the type satisfy the rep invariant
Sometimes easier than testing, sometimes harder
Every good programmer uses it as appropriate

All possible objects (and values) of a type

Make a new object

constructors

producers

Modify an existing object

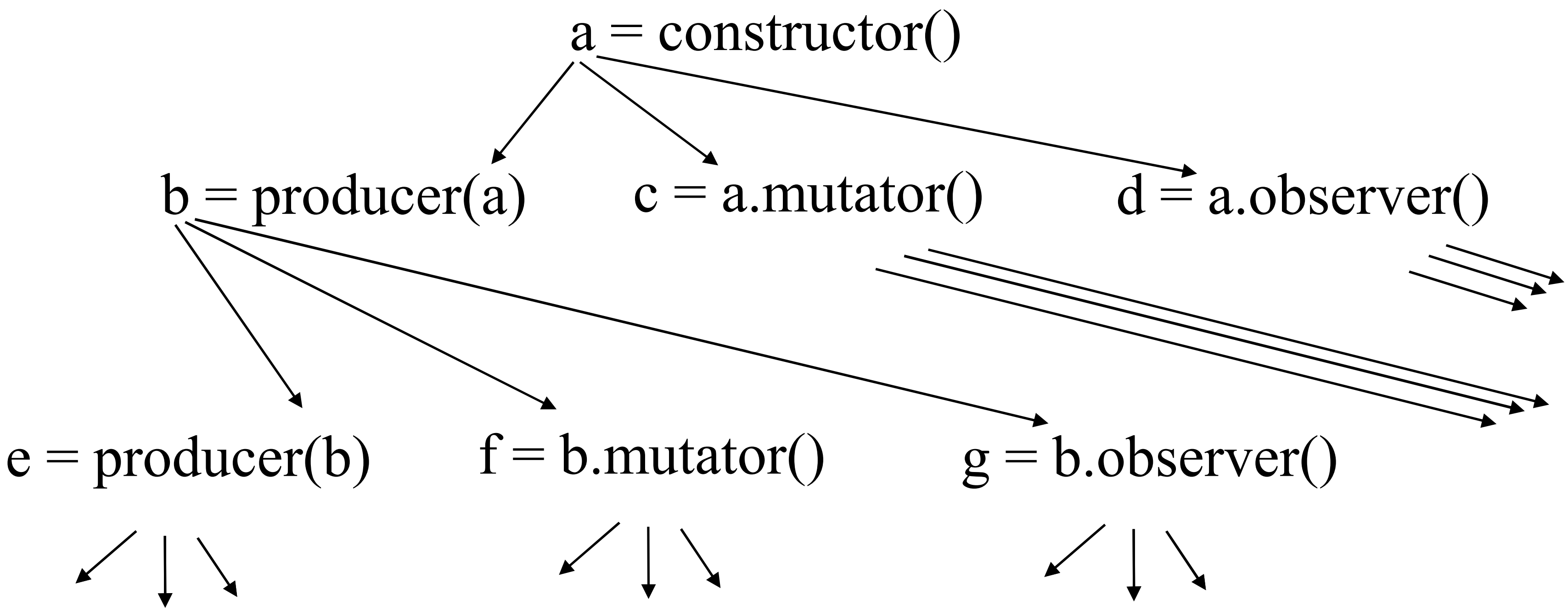
mutators

observers, producers (why?)

Limited number of operations, but infinitely many objects

Maybe infinitely many values as well

Examples of making objects



Infinitely many possibilities

We cannot perform a proof that considers each possibility case-by-case

Solution: induction

Induction: technique for proving infinitely many facts using finitely many proof steps

For constructors ("basis step")

Prove the property holds on exit

For all other methods ("inductive step")

Prove that if the property holds on entry, then it holds on exit

If the basis and industive steps are true:

There is no way to make an object for which the property does not hold

Therefore, the property holds for all objects

Inductive proof that x+1 > x

ADT: the natural numbers (non-negative integers)

constructor: 0 (zero)
producer: succ (successor: $succ(x) = x+1$)
mutators: none
observers: value

Axioms:

1. $succ(0) > 0$
2. $(succ(i) > succ(j)) \Leftrightarrow i > j$

Goal: prove that for all natural numbers x, $succ(x) > x$

Possibilities for x:

1. x is 0

$succ(0) > 0$	axiom #1

2. x is $succ(y)$ for some y

$succ(y) > y$	assumption
$succ(succ(y)) > succ(y)$	axiom #2
$succ(x) > x$	def of $x = succ(y)$

Outline for remainder of lecture

1. **Prove that rep invariant is satisfied**
2. **Prove that client code behaves correctly**
   (Assuming that the implementation is correct)

CharSet Abstraction

```
// Overview: CharSets are finite mutable sets of chars.
// effects: creates a fresh, empty CharSet
public CharSet ( )

// modifies: this
// effects: this_post = this_pre U {c}
public void insert (char c);

// modifies: this
// effects: this_post = this_pre - {c}
public void delete (char c);

// returns: (c ∈ this)
public boolean member (char c);

// returns: cardinality of this
public int size ( );
```

Implementation of CharSet

```
// Rep invariant:  elts has no nulls and no duplicates

public CharSet ( ) {  // constructor
    elts = new ArrayList <Character>( );
}
public void delete (char c) {
    elts.remove (new Character (c));
}
public void insert (char c) {
    if (! member(c))
        elts.add (new Character (c));
}
public boolean member (char c) {
    return elts.contains (new Character (c));
}
```

Proof of CharSet representation invariant

Rep invariant: elts has no nulls and no duplicates

Base case:

Constructor sets elts to the empty ArrayList<Character>

This satisfies the rep invariant

Inductive step:

For each other operation:

Assume rep invariant holds before the operation

Prove rep invariant holds after the operation

Inductive step, `member`

Rep invariant: elts has no nulls and no duplicates

```
public boolean member (char c) {
   return elts.contains (new Character (c));
}
```

`contains` doesn't change `elts`, so neither does `member`.
Conclusion: rep invariant is preserved.

Why do we even need to check member?
After all, the specification says that it does not mutate set.

Reasoning must account for all possible arguments
It's best not to involve the specific values in the proof

Inductive step, `delete`

Rep invariant: elts has no nulls and no duplicates

```
public void delete (char c) {
   elts.remove (new Character (c));
}
```

`remove` either leaves `elts` unchanged or removes element.
Rep invariant can only be made false by adding elements.
Conclusion: rep invariant is preserved.

Inductive step, `insert`

Rep invariant: elts has no nulls and no duplicates

```
public void insert (char c) {
   if (! this.member(c))
      elts.add (new Character (c));
}
```

If c is in $elts_{pre}$:

elts is unchanged

Therefore, rep invariant is preserved.

If c is not in $elts_{pre}$:

new elt is not null or a duplicate

Therefore, rep invariant is preserved.

Reasoning about mutations to the rep

Inductive step must consider all possible changes to the rep

- A possible source of changes: representation exposure
- If the proof does not account for this, then the proof is invalid

An important reason to protect the rep:

- Compiler can help verify that there are no external changes

Induction for reasoning about uses of ADT's

Induction on specification, not on code

Abstract values (e.g., specification fields) may differ from concrete representation

Can ignore observers, since they do not affect abstract state

How do we know that?

Axioms

specs of operations

axioms of types used in overview parts of specifications

Letter sets (case-insensitive character sets)

```
// A LetterSet (case-insensitive char set) is a mutable finite set of characters.
// No LetterSet contains two chars with the same lower-case representation.

// effects: creates an empty LetterSet
public LetterSet ( );

// Insert c if this contains no other char with same lower-case representation.
// modifies: this
// effects: this_post = if (∃c_1 ∈ this_pre s.t.  toLowerCase(c_1) = toLowerCase(c)
//                       then this_pre
//                       else this_pre U {c}
public void insert (char c);

// modifies: this
// effects: this_post = this_pre - {c}
public void delete (char c);

// returns:  (c ∈ this)
public boolean member (char c);

// returns:  |this|
public int size ( );
```

Goal: prove that LetterSet contains two different letters

Prove: $|S| > 1 \Rightarrow (\exists c_1, c_2 \in S\ [toLowerCase(c_1) \neq toLowerCase(c_2)])$

Two possibilities for how S was made: by the constructor, or by insert

Base case: S = { }, (S was made by the constructor):

property holds (vacuously true)

Inductive case (S was made by a call of the form "T.insert(c)"):

Assume: $|T| > 1 \Rightarrow (\exists c_3, c_4 \in T\ [toLowerCase(c_3) \neq toLowerCase(c_4)])$

Show: $|S| > 1 \Rightarrow (\exists c_1, c_2 \in S\ [toLowerCase(c_1) \neq toLowerCase(c_2)])$

where S = T.insert(c)

= "if ($\exists c_5 \in T$ s.t. toLowerCase(c_5) = toLowerCase(c))

then T **else** T U {c}"

The value for S came from the specification of insert, applied to T.insert(c):

public void insert (char c);

modifies: this

effects: $this_{post}$ = ***if (∃**c_1 **∈S s.t. toLowerCase(**c_1***) = toLowerCase(c))***

***then this**$_{pre}$*

else this**$_{pre}$* ***U {c}

(Inductive case is continued on the next slide.)

Goal: prove that LetterSet contains two different letters. Inductive case: S = T.insert(c)

Goal (from previous slide):

Assume: $|T| > 1 \Rightarrow (\exists c_3, c_4 \in T\ [\text{toLowerCase}(c_3) \neq \text{toLowerCase}(c_4)])$

Show: $|S| > 1 \Rightarrow (\exists c_1, c_2 \in S\ [\text{toLowerCase}(c_1) \neq \text{toLowerCase}(c_2)])$

where S = T.insert(c)

= "if ($\exists c_5 \in T$ s.t. toLowerCase(c_5) = toLowerCase(c))

then T **else** T U {c}"

Consider the two possibilities for S (from "if ... then T else T U {c}"):

1. **If S = T, the theorem holds by induction hypothesis**
   The assumption above.

2. **If S = T U {c}, there are three cases to consider:**
   |T| = 0: Vacuous case, since hypothesis of theorem ("|S| > 1") is false
   |T| ≥ 1: We know that T did not contain a char of toLowerCase(h),
   so the theorem holds by the meaning of union
   Bonus: |T| > 1: By inductive assumption, T contains different letters,
   so by the meaning of union, T U {c} also contains different letters

Conclusion

The goal is correct code

A proof is a powerful mechanism for ensuring correctness

Formal reasoning is required if debugging is hard

Inductive proofs are the most effective in computer science

Types of proofs:

Verify that rep invariant is satisfied

Verify that the implementation satisfies the spec

Verify that client code behaves correctly