Computer Systems

CSE 410 Autumn 2013

4 – Floating Point

Integer & Floating Point Numbers

- Representation of integers: unsigned and signed
- Unsigned and signed integers in C
- Arithmetic and shifting
- Sign extension

- Background: fractional binary numbers
- IEEE floating-point standard
- Floating-point operations and rounding
- Floating-point in C

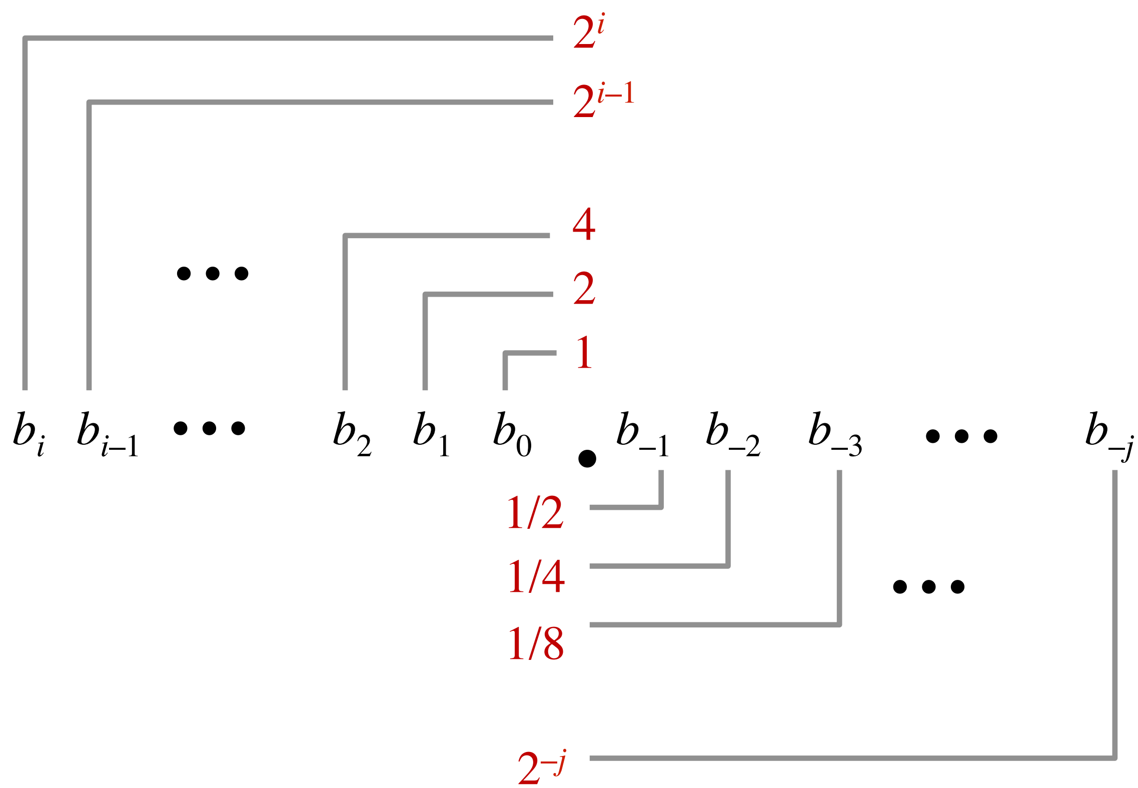
- Reading: Bryant/O'Hallaron sec. 2.4

Fractional Binary Numbers

- **What is 1011.101_2?**

- **How do we interpret fractional *decimal* numbers?**
  - e.g. 107.95_{10}
  - Can we interpret fractional binary numbers in an analogous way?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers: Examples

- **Value** **Representation**
  - 5 and 3/4 101.11_2
  - 2 and 7/8 10.111_2
  - 63/64 0.111111_2

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of the form **0.111111**$\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Shorthand notation for all **1** bits to the right of binary point: $1.0 - \varepsilon$

Representable Values

- **Limitations of fractional binary numbers:**
  - Can only exactly represent numbers that can be written as $x * 2^y$
  - Other rational numbers have repeating bit representations

- **Value** **Representation**
  - 1/3 `0.0101010101[01]`$\ldots_2$
  - 1/5 `0.001100110011[0011]`$\ldots_2$
  - 1/10 `0.0001100110011[0011]`$\ldots_2$

Fixed Point Representation

- **We might try representing fractional binary numbers by picking a fixed place for an implied binary point**
  - "fixed point binary numbers"
- **Let's do that, using 8-bit fixed point numbers as an example**
  - #1: the binary point is between bits 2 and 3
    $b_7\ b_6\ b_5\ b_4\ b_3\ [.]\ b_2\ b_1\ b_0$
  - #2: the binary point is between bits 4 and 5
    $b_7\ b_6\ b_5\ [.]\ b_4\ b_3\ b_2\ b_1\ b_0$
- **The position of the binary point affects the *range* and *precision* of the representation**
  - range: difference between largest and smallest numbers possible
  - precision: smallest possible difference between any two numbers

Fixed Point Pros and Cons

- **Pros**
  - It's simple. The same hardware that does integer arithmetic can do fixed point arithmetic
    - In fact, the programmer can use ints with an implicit fixed point
    - ints are just fixed point numbers with the binary point to the right of b_0
- **Cons**
  - There is no good way to pick where the fixed point should be
    - Sometimes you need range, sometimes you need precision – the more you have of one, the less of the other.

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IEEE Floating Point

- **Analogous to scientific notation**
  - Not 12000000 but 1.2 x 10^7; not 0.0000012 but 1.2 x 10^{-6}
    - (write in C code as: 1.2e7; 1.2e-6)
- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs today
- **Driven by numerical concerns**
  - Standards for handling rounding, overflow, underflow
  - Hard to make fast in hardware but numerically well-behaved
- **1989 Turing Award to William Kahan (UC Berkeley)**

Floating Point Representation

- **Numerical form:**

$$V_{10} = (-1)^{s} * M * 2^{E}$$

  - Sign bit ***s*** determines whether number is negative or positive
  - Significand (mantissa) ***M*** normally a fractional value in range [1.0,2.0)
  - Exponent ***E*** weights value by a (possibly negative) power of two

- **Representation in memory:**
  - MSB **`s`** is sign bit ***s***
  - **`exp`** field encodes ***E*** (but is *not equal* to E)
  - **`frac`** field encodes ***M*** (but is *not equal* to M)

s	exp	frac

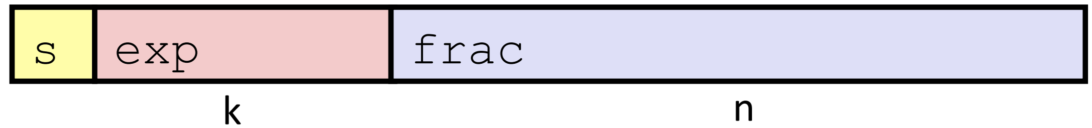
Precisions

- **Single precision: 32 bits**

- **Double precision: 64 bits**

Normalization and Special Values

$V = (-1)^{S} * M * 2^{E}$

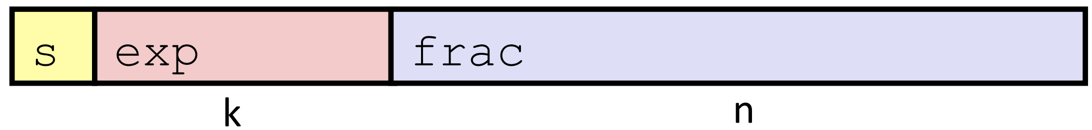


- "Normalized" means the mantissa **M** has the form 1.xxxxx
  - 0.011 x 2^5 and 1.1 x 2^3 represent the same number, but the latter makes better use of the available bits
  - Since we know the mantissa starts with a 1, we don't bother to store it**!**

- **How do we represent 0.0? Or special / undefined values like 1.0/0.0?**

Normalization and Special Values

$V = (-1)^{S} * M * 2^{E}$



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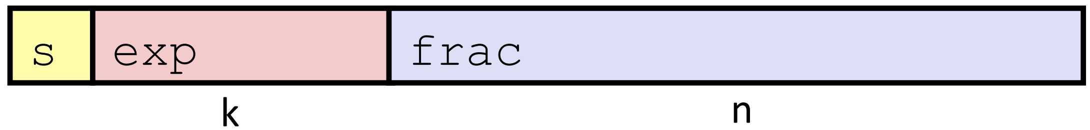
- **Special values:**
  - The bit pattern 00...0 represents zero
  - If **`exp`** == 11...1 and **`frac`** == 00...0, it represents ∞
    - e.g. 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −1.0/0.0 = −∞
  - If **`exp`** == 11...1 and **`frac`** != 00...0, it represents NaN: "Not a Number"
    - Results from operations with undefined result, e.g. sqrt(–1), $\infty - \infty$, $\infty * 0$

Normalized Values

$V = (-1)^{s} * M * 2^{E}$



- **Condition: exp ≠ 000…0 and exp ≠ 111…1**
- **Exponent coded as *biased* value: *E = exp - Bias***
  - **exp** is an *unsigned* value ranging from 1 to 2^{k}-2 (k == # bits in **exp**)
  - *Bias* = 2^{k-1} - 1
    - Single precision: 127 (so *exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (so *exp*: 1…2046, *E*: -1022…1023)
  - These enable negative values for E, for representing very small values

- **Significand coded with implied leading 1: *M* = 1.xxx…x_2**
  - **xxx…x**: the n bits of **frac**
  - Minimum when **000…0** (*M* = 1.0)
  - Maximum when **111…1** (*M* = 2.0 – ε)
  - Get extra leading bit for “free”

Normalized Encoding Example

$V = (-1)^{s} * M * 2^{E}$

s	exp	frac
	k	n

- **Value:** **`float f = 12345.0;`**
  - $12345_{10} = 11000000111001_2$
    $= 1.1000000111001_2 \times 2^{13}$ (normalized form)

- **Significand:**

  M = `1.1000000111001`$_2$

  `frac=` `10000001110010000000000`$_2$

- **Exponent:** E = exp - Bias, so exp = E + Bias

  E = 13

  Bias = 127

  `exp` = 140 = `10001100`$_2$

- **Result:**

0	10001100	10000001110010000000000
s	exp	frac

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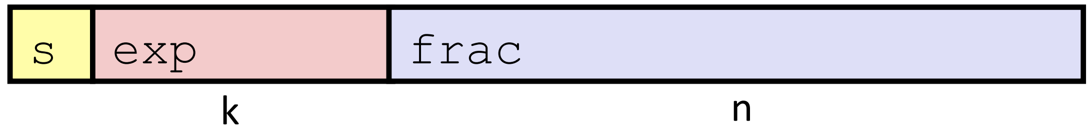
How do we do operations?

- **Unlike the representation for integers, the representation for floating-point numbers is not *exact***

Floating Point Operations: Basic Idea

$$V = (-1)^{s} * M * 2^{E}$$



- **x +$_f$ y = *Round*(x + y)**

- **x *$_f$ y = *Round*(x * y)**

- **Basic idea for floating point operations:**
  - First, compute the exact result
  - Then, *round* the result to make it fit into desired precision:
    - Possibly overflow if exponent too large
    - Possibly drop least-significant bits of significand to fit into **frac**

Rounding modes

- **Possible rounding modes (illustrated with dollar rounding):**

	\$1.40	**\$1.60**	**\$1.50**	**\$2.50**	**–\$1.50**
Round-toward-zero	\$1	\$1	\$1	\$2	–\$1
Round-down (-∞)	\$1	\$1	\$1	\$2	–\$2
Round-up (+∞)	\$2	\$2	\$2	\$3	–\$1
Round-to-nearest	\$1	\$2	??	??	??
Round-to-even	\$1	\$2	\$2	\$2	–\$2

- **What could happen if we're repeatedly rounding the results of our operations?**
  - If we always round in the same direction, we could introduce a statistical bias into our set of values!
- **Round-to-even avoids this bias by rounding up about half the time, and rounding down about half the time**
  - Default rounding mode for IEEE floating-point

Mathematical Properties of FP Operations

- **If overflow of the exponent occurs, result will be ∞ or -∞**
- **Floats with value ∞, -∞, and NaN can be used in operations**
  - Result is usually still ∞, -∞, or NaN; sometimes intuitive, sometimes not

- **Floating point operations are not always associative or distributive, due to rounding!**
  - (3.14 + 1e10) - 1e10 != 3.14 + (1e10 - 1e10)
  - 1e20 * (1e20 - 1e20) != (1e20 * 1e20) - (1e20 * 1e20)

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Floating Point in C

- **C offers two levels of precision**

  `float` single precision (32-bit)

  `double` double precision (64-bit)

- **Default rounding mode is round-to-even**
- **`#include <math.h>` to get `INFINITY` and `NAN` constants**
- **Equality (==) comparisons between floating point numbers are tricky, and often return unexpected results**
  - Just avoid them!
  - Substitute things like: (abs(x-y) / max(abs(x),abs(y))) < epsilon
    - Not guaranteed – depends on what you're doing, the data, etc.
    - When in doubt, find a trained Numerical Analyst or use a carefully-written library

Floating Point in C

- **Conversions between data types:**
  - Casting between **`int`**, **`float`**, and **`double`** changes the bit representation!!
  - `int` → `float`
    - May be rounded; overflow not possible
  - `int` → `double` or `float` → `double`
    - Exact conversion, as long as int has ≤ 53-bit word size
  - `double` or `float` → `int`
    - Truncates fractional part (rounded toward zero)
    - Not defined when out of range or NaN: generally sets to Tmin

Summary

- **Zero**

0	00000000	0000000000000000000000

- **Normalized values**

s	*1 to 2^k-2*	*significand = 1.M*

- **Infinity**

s	11111111	0000000000000000000000

- **NaN**

s	11111111	*non-zero*

- **Denormalized values**

s	00000000	*significand = 0.M*
`s`	`exp`	`frac`

Summary (cont'd)

- **As with integers, floats suffer from the fixed number of bits available to represent them**
  - Can get overflow/underflow, just like ints
  - Some "simple fractions" have no exact representation (e.g., 0.2)
  - Can also lose precision, unlike ints
    - "Every operation gets a slightly wrong result"

- **Mathematically equivalent ways of writing an expression may compute different results**
  - Violates associativity/distributivity

- **Never test floating point values for equality!**