Dynamic Programming Examples

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Lecture Outline

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1. Weighted Interval Scheduling
2. Knapsack Problem

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1. Weighted Interval Scheduling
2. Knapsack Problem
3. String Similarity
4. Common Errors with Dynamic Programming

Algorithmic Paradigms

- Greed. Build up a solution incrementally, myopically optimizing some local criterion.
- Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.
- Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

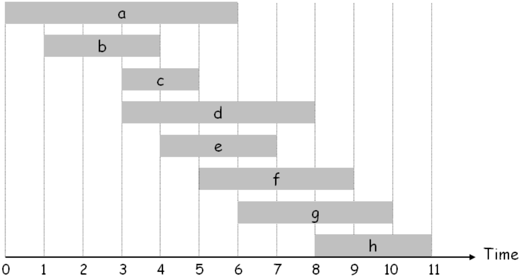
Dynamic Programming Applications

- Areas.
  - Bioinformatics.
  - Control theory.
  - Information theory.
  - Operations research.
  - Computer science: theory, graphics, AI, systems, ...
- Some famous dynamic programming algorithms.
  - Viterbi for hidden Markov models.
  - Unix diff for comparing two files.
  - Smith-Waterman for sequence alignment.
  - Bellman-Ford for shortest path routing in networks.
  - Cocke-Kasami-Younger for parsing context free grammars.

Outline

Weighted Interval Scheduling

- Weighted interval scheduling problem.
  - Job j starts at s_j, finishes at f_j, and has weight or value v_j.
  - Two jobs compatible if they don't overlap.
  - Goal: find maximum weight subset of mutually compatible jobs.

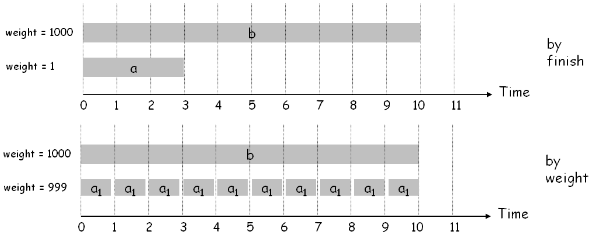


Unweighted Interval Scheduling Review

- Recall. Greedy algorithm works if all weights are 1.
  - Consider jobs in ascending order of finish time.
  - Add job to subset if it is compatible with previously chosen jobs.
- Can Greedy work when there are weights?

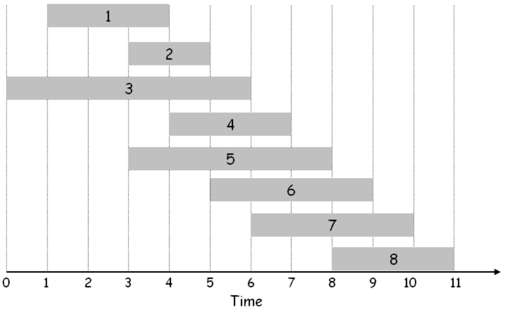
Unweighted Interval Scheduling Review

- Recall. Greedy algorithm works if all weights are 1.
  - Consider jobs in ascending order of finish time.
  - Add job to subset if it is compatible with previously chosen jobs.
- Can Greedy work when there are weights?
- Greedy fails for ordering either by finish time or by weight



Weighted Interval Scheduling

- Notation. Label jobs by finishing time: $f_1, f_2, \ldots f_n$.
- Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.
- Ex: $p(8) = 5, p(7) = 3, p(2) = 0$.



i	$p(i)$
0	-
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

Dynamic Programming: Binary Choice

- Notation. OPT(j) $=$ value of optimal solution to the problem consisting of job requests 1, 2, ..., j.
  - Case 1: OPT selects job j.
    - can't use incompatible jobs $\{p(j)+1, p(j)+2, ..., j-1\}$
    - must include optimal solution to problem consisting of remaining compatible jobs $1, 2, ..., p(j)$
  - Case 2: OPT does not select job j.
    - must include optimal solution to problem consisting of remaining compatible jobs $1, 2, ..., j-1$

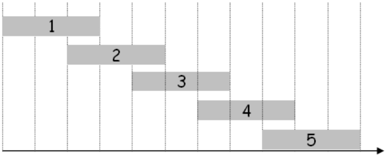
Weighted Interval Scheduling: Brute Force

- Brute force algorithm.

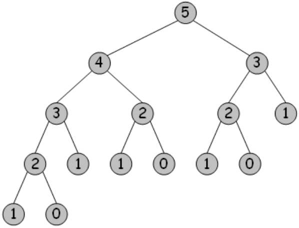
Input $n, s_1, \ldots s_n, f_1, \ldots f_n, v_1, \ldots, v_n$
Sort jobs by finish times so $f_1 \le f_2 \le \ldots \le f_n$
Compute $p(1), p(2), \ldots, p(n)$
procedure $\textsc{Compute-Opt}(j)$
  if $j = 0$ **then return** 0
  else
    return $max(v_j + \textsc{Compute-Opt}(p(j)),\ \textsc{Compute-Opt}(j-1)\ )$
  end if
end procedure

Weighted Interval Scheduling: Brute Force

- Observation. Recursive algorithm fails spectacularly because of redundant sub-problems exponential algorithms.
- Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Weighted Interval Scheduling: Memoization

Input $n, s_1, \dots s_n, f_1, \dots f_n, v_1, \dots, v_n$
Sort jobs by finish times so $f_1 \leq f_2 \leq \dots \leq f_n$
Compute $p(1), p(2), \dots, p(n)$
for $i = 1 \dots n$ **do**
    $M[i] \leftarrow$ empty
end for
$M[0] \leftarrow 0$
procedure $\text{M-OPT}(j)$
    if $M[j]$ is empty **then**
        $M[j] \leftarrow max(v_j + \text{M-OPT}(p(j)), \text{M-OPT}(j-1))$
    end if
    return $M[j]$
end procedure

Weighted Interval Scheduling: Running Time

- Claim. Memoized version of algorithm takes $O(n \log n)$ time.
  - Sort by finish time: $O(n \log n)$.
  - Computing $p()$: $O(n)$ after sorting by start time.
  - $\textsc{M-Opt}(j)$: each invocation takes $O(1)$ time and either
    1. returns an existing value $M[j]$
    2. fills in one new entry $M[j]$ and makes two recursive calls
  - Progress Measure: Θ number of empty cells in M
    - $\Theta \leq n$ always
    - max 2 recursive calls at any level $\Rightarrow \leq 2n$ recursive calls total
  - $\textsc{M-Opt}(n)$ is $O(n)$
  - Overall, $O(n \log n)$, or $O(n)$ if presorted by start & finish times

Weighted Interval Scheduling: Iterative

- Bottom Up Iteration

Input $n, s_1, \ldots s_n, f_1, \ldots f_n, v_1, \ldots, v_n$
Sort jobs by finish times so $f_1 \leq f_2 \leq \ldots \leq f_n$
Compute $p(1), p(2), \ldots, p(n)$
procedure $\textsc{Iter-Opt}(j)$
    $M[0] \leftarrow 0$
    for $i = 1 \ldots n$ **do**
        $M[i] \leftarrow max(v_i + M[p(i)], M[i-1])$
    end for
    return $M[j]$
end procedure

Outline

Knapsack Problem

- Given n objects and a knapsack
- Object i has weight w_i and value v_i.
- Knapsack has maximum weight W
- Goal: fill knapsack to maximize total value
- Example Instance
  - Knapsack max weight $W = 11$.
  - Packing items $\{3, 4\}$ gives total value 40.
- Can we use greedy?
- Greedy by *value*/*weight* ratio is sub-optimal. In the example, it would pack $\{5, 2, 1\}$, which only has value 35.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Subproblems: first try

- Def. $OPT(i)$ = max value subset of items $1, \ldots, i$.
  - Case 1: OPT does not select item i.
    - OPT selects best of $\{1, 2, \ldots, i-1\}$
  - Case 2: OPT selects item i.

Knapsack Subproblems: first try

- Def. $OPT(i)$ = max value subset of items $1, \ldots, i$.
  - Case 1: OPT does not select item i.
    - OPT selects best of $\{1, 2, \ldots, i-1\}$
  - Case 2: OPT selects item i.
    - accepting item i does not immediately imply that we will have to reject other items.
    - without knowing what other items were selected before i, we don't even know if we have enough room for i
- Conclusion. Need more sub-problems!

Knapsack Subproblems: second try

- Def. $OPT(i, S) =$ max value subset of items $1, \ldots, i$, using items in the set S.
- Works, but ...

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- ... 2^n subproblems! we haven't saved any work

Knapsack Subproblems: second try

- Def. $OPT(i, S) =$ max value subset of items $1, \ldots, i$, using items in the set S.
- Works, but ...
- ... 2^n subproblems! we haven't saved any work
- Do we really need to know all of items chosen? Just need to know if we can stick in item i ...

Knapsack Subproblems: third time's a charm

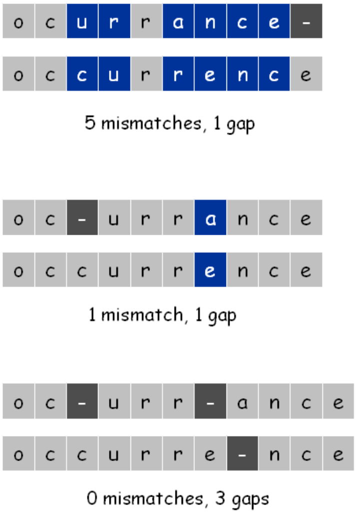
- Only need to know the weight already in the knapsack
- Def. $OPT(i, w)$ = max value subset of items $1, \ldots, i$ weighing no more than w.
  - Case 1: OPT does not select item i.
    - OPT selects best of $\{1, 2, \ldots, i-1\}$ weighing no more than w.
  - Case 2: OPT selects item i.
    - $w' = w - w_i$
    - OPT adds item i to optimal solution from $1, \ldots, i-1$ weighing no more than w', the new weight limit.
- The Reccurence:

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ max(v_i + OPT(i-1, w - w_i), \\ \qquad OPT(i-1, w)) & \end{cases}$$

Outline

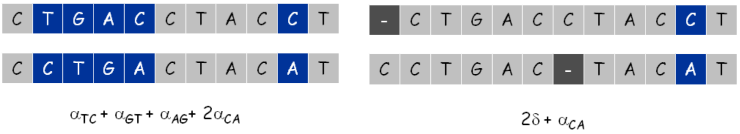
String Similarity

- How similar are two strings?
  1. ocurrance
  2. occurrence



String Edit Distance

- Applications
  - Basis for "diff"
  - Speech Recognition
  - Computational Biology
- Edit Distance
  - Gap Penalty δ; mismatch-penalty α_{pq}
  - Cost $=$ sum of gap and mismatch penalties



Sequence Alignment

- **Goal** Given two strings $X = x_1 x_2 \dots x_m$ and $Y = y_1 y_2 \dots y_n$ find alignment of minimum cost.
- **Def** An alignment M is a set of ordered pairs (x_i, y_j) such that each item occurs in at most one pair and no crossings.
- **Def** The pair (x_i, y_j) and $(x_{i'}, y_{j'})$ cross f $i < i'$ but $j > j'$.

$$cost(M) = \underbrace{\sum_{(x_i,y_j)\in M} \alpha_{x_i,y_j}}_{\text{mismatch}} + \underbrace{\sum_{i:x_i \text{ unmatched}} \delta + \sum_{j:y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Sequence Alignment Subproblems

- **Def** $OPT(i,j) =$ min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.
  - Case 1. OPT matches (x_i, y_j). Pay mismatch for (x_i, y_j) + min cost aligning substrings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$
  - Case 2a. OPT leaves x_i unmatched. Pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$.
  - Case 2b. OPT leaves y_i unmatched. Pay gap for y_i and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$.

Sequence Alignment Subproblems

- **Def** $OPT(i,j)$ = min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.
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  - Case 2a. OPT leaves x_i unmatched. Pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$.
  - Case 2b. OPT leaves y_i unmatched. Pay gap for y_i and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$.

$$OPT(i,j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ min \begin{cases} \alpha_{x_i, y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \end{cases}$$

Sequence Alignment Runtime

- Runtime: $\Theta(mn)$
- Space: $\Theta(mn)$
- English words: $m, n \leq 10$
- Biology: $m, n \approx 10^5$
  - 10^10 operations OK ...
  - 10 GB array is a problem
  - Can cut space down to $O(m+n)$ (see Section 6.7)

Outline

Dynamic Programming and TSP(1)

- Consider this Dyanmic Programming "solution" to the Travelling Salesman Problem

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Order the points $p_1, \ldots, p_n$ arbitrarily.
for $i = 1, \ldots n$ **do**
    for $j = 1, \ldots i$ **do**
        Take optimal solution for points $p_1, \ldots p_{i-1}$, and put point p_i right after p_j.
    end for
    Keep optimal of all the attempts above.
end for

- The runtime of this algorithm is $\Theta(n^2)$. Is it really this easy?

Dynamic Programming and TSP (2)

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Dynamic Programming and TSP (2)

- The runtime of this algorithm is $\Theta(n^2)$. Is it really this easy?
- NO. We don't have the "principle of optimality".
  - Why should the optimal solution for points $p_1, \ldots, p_i$ be based on the optimal solution for $p_1, \ldots, p_{i-1}$???

Dynamic Programming and TSP (2)

- The runtime of this algorithm is $\Theta(n^2)$. Is it really this easy?

- We have not bothered to prove the optimality for many of the problems we considered, because it is "clear". But be sure to check.

Dynamic Programming and TSP (3)

- What if we changed the previous algorithm to keep track of all ordering of points $p_1, \ldots, p_i$? The optimal solution for $p_1, \ldots, p_{i+1}$ must come from one of those, right?

Dynamic Programming and TSP (3)

- What if we changed the previous algorithm to keep track of all ordering of points $p_1, \ldots, p_i$? The optimal solution for $p_1, \ldots, p_{i+1}$ must come from one of those, right?
- Sure, that would work.

Dynamic Programming and TSP (3)

- What if we changed the previous algorithm to keep track of all ordering of points $p_1, \ldots, p_i$? The optimal solution for $p_1, \ldots, p_{i+1}$ <u>must</u> come from one of those, right?

- But now you're doing $n!$ work.