CSE 421
Algorithms

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Lecture 9

Minimum Spanning Trees

Who was Dijkstra?

- What were his major contributions?

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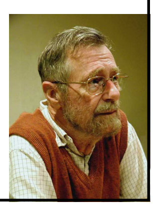
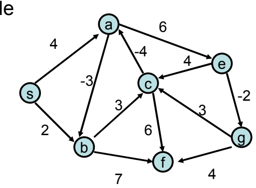
http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments



Shortest Paths

- Negative Cost Edges
  - Dijkstra's algorithm assumes positive cost edges
  - For some applications, negative cost edges make sense
  - Shortest path not well defined if a graph has a negative cost cycle

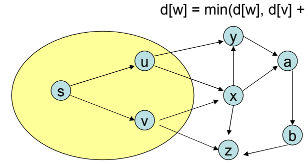


Negative Cost Edge Preview

- Topological Sort can be used for solving the shortest path problem in directed acyclic graphs
- Bellman-Ford algorithm finds shortest paths in a graph with negative cost edges (or reports the existence of a negative cost cycle).

Dijkstra's Algorithm Implementation and Runtime

```
S = {};    d[s] = 0;    d[v] = infinity for v != s
While S != V
        Choose v in V-S with minimum d[v]
        Add v to S
        For each  w in the neighborhood of v
                d[w] = min(d[w], d[v] + c(v, w))
```



HEAP OPERATIONS

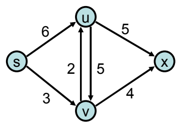
n Extract Mins

m Heap Updates

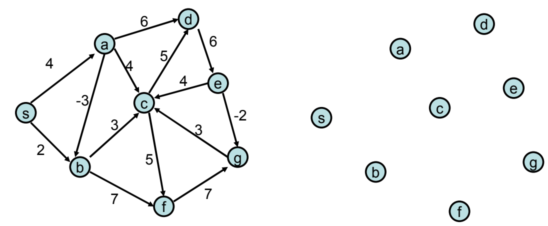
Edge costs are assumed to be non-negative

Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path



Compute the bottleneck shortest paths



Dijkstra's Algorithm for Bottleneck Shortest Paths

S = {}; d[s] = negative infinity; d[v] = infinity for v != s

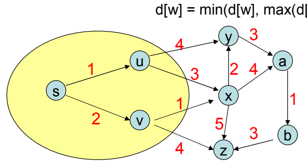
While S != V

Choose v in V-S with minimum d[v]

Add v to S

For each w in the neighborhood of v

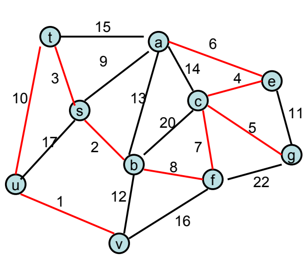
d[w] = min(d[w], max(d[v], c(v, w)))



Minimum Spanning Tree

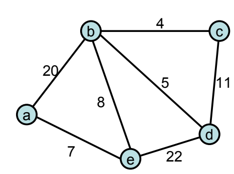
- Introduce Problem
- Demonstrate three different greedy algorithms
- Provide proofs that the algorithms work

Minimum Spanning Tree



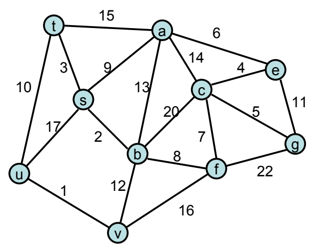
Greedy Algorithms for Minimum Spanning Tree

- Extend a tree by including the cheapest out going edge
- Add the cheapest edge that joins disjoint components
- Delete the most expensive edge that does not disconnect the graph



Greedy Algorithm 1
Prim's Algorithm

- Extend a tree by including the cheapest out going edge

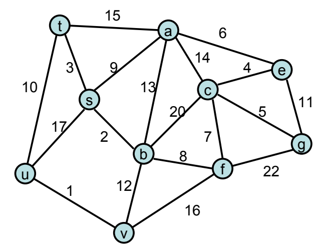


Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion

Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components

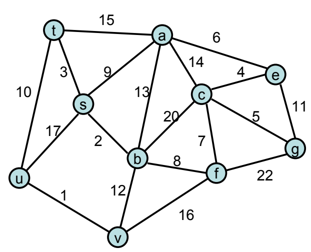


Construct the MST with Kruskal's algorithm

Label the edges in order of insertion

Greedy Algorithm 3
Reverse-Delete Algorithm

- Delete the most expensive edge that does not disconnect the graph



Construct the MST with the reverse-delete algorithm

Label the edges in order of removal

Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S

- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim's Algorithm computes a MST

- Kruskal's Algorithm computes a MST

Reverse-Delete Algorithm

- Lemma: The most expensive edge on a cycle is never in a minimum spanning tree

Dealing with the assumption of no equal weight edges

- Force the edge weights to be distinct
  - Add small quantities to the weights
  - Give a tie breaking rule for equal weight edges

Dijkstra's Algorithm for Minimum Spanning Trees

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