CSE 421
Algorithms

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Lecture 21

Shortest Paths

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- General case – handling negative edges
- If there exists a negative cost cycle, the shortest path is not defined
- Bellman-Ford Algorithm
  - O(mn) time for graphs with negative cost edges

Lemma

- If a graph has no negative cost cycles, then the shortest paths are simple paths
- Shortest paths have at most n-1 edges

Shortest paths with a fixed number of edges

- Find the shortest path from v to w with exactly k edges

Express as a recurrence

- $Opt_k(w) = \min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w) = 0$ if v=w and infinity otherwise

Algorithm, Version 1

```
foreach w
        M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
        foreach w
                M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

Algorithm, Version 2

```
foreach w
        M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
        foreach w
                M[i, w] = min(M[i-1, w], min_x(M[i-1,x] + cost[x,w]))
```

Algorithm, Version 3

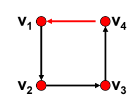
```
foreach w
        M[w] = infinity;
M[v] = 0;
for i = 1 to n-1
        foreach w
                M[w] = min(M[w], min_x(M[x] + cost[x,w]))
```

Correctness Proof for Algorithm 3

- Key lemma – at the end of iteration i, for all w, M[w] <= M[i, w];
- Reconstructing the path:
  - Set P[w] = x, whenever M[w] is updated from vertex x

If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] >= M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let $v_1, v_2, \ldots v_k$ be a cycle in the pointer graph with (v_k, v_1) the last edge added
  - Just before the update
    - $M[v_j] >= M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1,v_2) + cost(v_2,v_3) + \ldots + cost(v_k, v_1)$

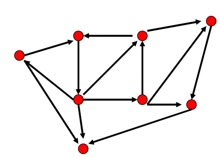


Negative Cycles

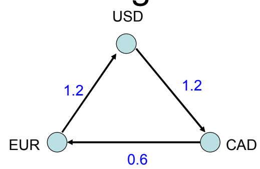
- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

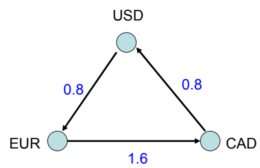
Finding negative cost cycles

- What if you want to find negative cost cycles?



Foreign Exchange Arbitrage





	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----