CSE 421: Intro Algorithms

Fall 2011
Graphs and Graph Algorithms
Slides by Larry Ruzzo

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Goals

Graphs: defns, examples, utility, terminology
Representation: input, internal
Traversal: Breadth- & Depth-first search
Three Algorithms:
- Connected components
- Bipartiteness
- Topological sort

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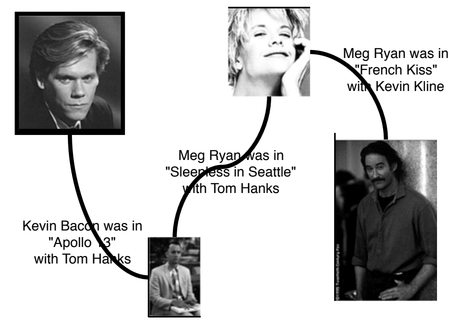

Objects & Relationships

The Kevin Bacon Game:
- Obj: Actors
- Rel: Two are related if they've been in a movie together

Exam Scheduling:
- Obj: Classes
- Rel: Two are related if they have students in common

Traveling Salesperson Problem:
- Obj: Cities
- Rel: Two are related if can travel *directly* between them

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Graphs

An extremely important formalism for representing (binary) relationships

Objects: "vertices," aka "nodes"

Relationships between pairs: "edges," aka "arcs"

Formally, a graph G = (V, E) is a pair of sets, V the vertices and E the edges

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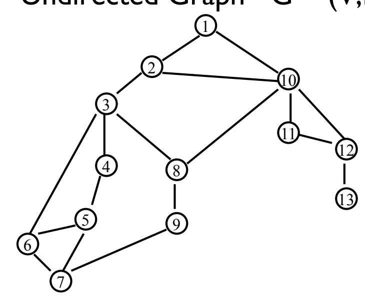
Undirected Graph G = (V,E)



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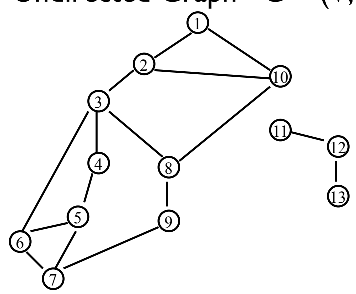

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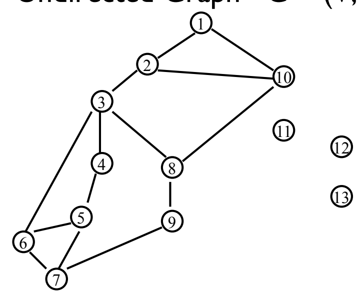
Undirected Graph G = (V,E)



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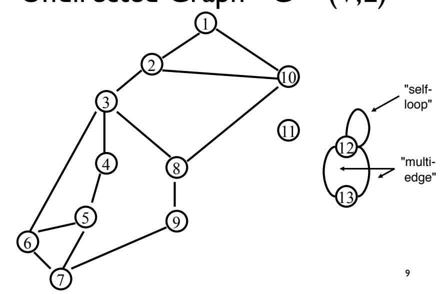
Undirected Graph G = (V,E)



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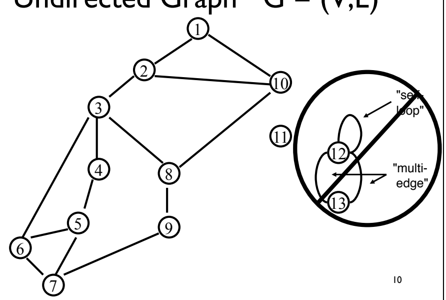
Undirected Graph G = (V,E)



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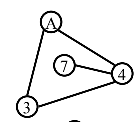
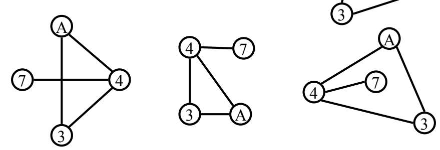
Graphs don't live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.





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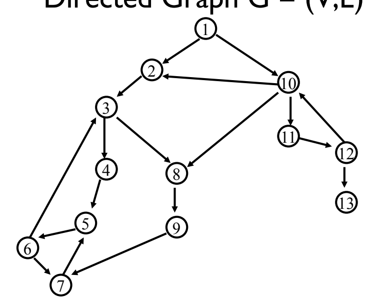
Directed Graph G = (V,E)



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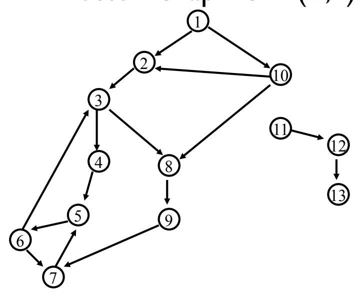
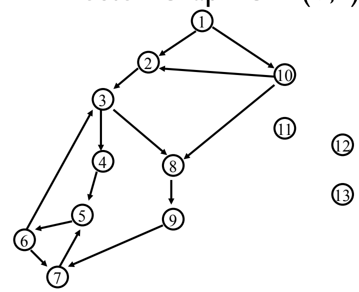
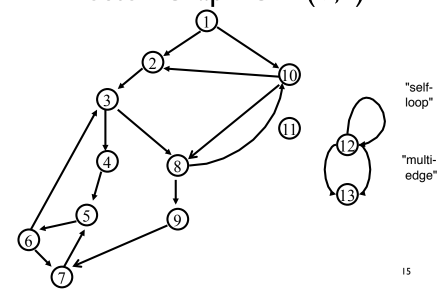
Directed Graph G = (V,E)



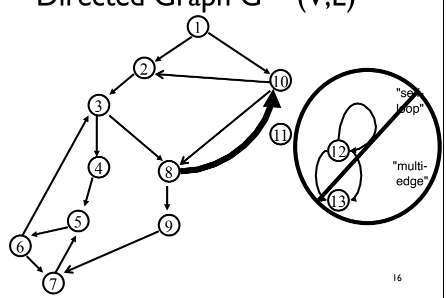
Directed Graph G = (V,E)



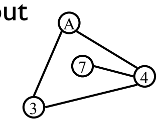
Directed Graph G = (V,E)



Directed Graph G = (V,E)



Specifying undirected graphs as input



What are the vertices?

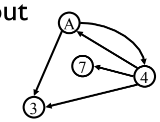
- Explicitly list them: {"A", "7", "3", "4"}

What are the edges?

- Either, set of edges {{A,3}, {7,4}, {4,3}, {4,A}}
- Or, (symmetric) adjacency matrix:

	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Specifying directed graphs as input



What are the vertices?

- Explicitly list them: {"A", "7", "3", "4"}

What are the edges?

- Either, set of directed edges: {(A,4), (4,7), (4,3), (4,A), (A,3)}
- Or, (nonsymmetric) adjacency matrix:

	A	7	3	4
A	0	0	1	1
7	0	0	0	0
3	0	0	0	0
4	1	1	1	0

Vertices vs # Edges

Let G be an undirected graph with *n* vertices and *m* edges. How are *n* and *m* related?

Vertices vs # Edges

Let G be an undirected graph with *n* vertices and *m* edges. How are *n* and *m* related?

Since

- every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges),

it must be true that:

$$\boxed{0 \le m \le n(n-1)/2 = O(n^2)}$$

More Cool Graph Lingo

A graph is called *sparse* if $m \ll n^2$, otherwise it is *dense*

- Boundary is somewhat fuzzy; $O(n)$ edges is certainly sparse, $\Omega(n^2)$ edges is dense.

Sparse graphs are common in practice

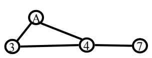
- E.g., all planar graphs are sparse ($m \le 3n-6$, for $n \ge 3$)

Q: which is a better run time, $O(n+m)$ or $O(n^2)$?

A: $O(n+m) = O(n^2)$, but $n+m$ usually way better!

Representing Graph G = (V,E)

internally, indp of input format



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Vertex set $V = \{v_1, \ldots, v_n\}$

Adjacency Matrix A

- $A[i,j] = 1$ iff $(v_i,v_j) \in E$
- Space is n^2 bits

Advantages:

- O(1) test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

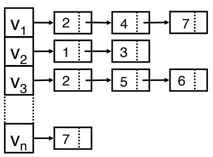
$m \ll n^2$

Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

- O(n+m) words



Advantages:

- Compact for sparse graphs
- Easily see all edges

Disadvantages

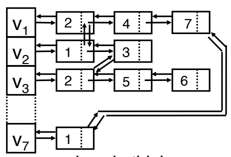
- More complex data structure
- no O(1) edge test

Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

- O(n+m) words



Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, *if needed*, (don't bother if not)

Graph Traversal

Learn the basic structure of a graph

"Walk," *via edges*, from a fixed starting vertex *s* to all vertices reachable from *s*

Being *orderly* helps. Two common ways:

- Breadth-First Search: order the nodes in successive layers based on distance from s
- Depth-First Search: more natural approach for exploring a maze; many efficient algs build on it.

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Breadth-First Search

Completely explore the vertices in order of their distance from *s*

Naturally implemented using a queue

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Graph Traversal: Implementation

Learn the basic structure of a graph

"Walk," via edges, from a fixed starting vertex *s* to all vertices reachable from *s*

Three states of vertices

- *undiscovered*
- *discovered*
- *fully-explored*

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BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

```
BFS(s)
    mark s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully explored
```

Exercise: modify code to number vertices & compute level numbers

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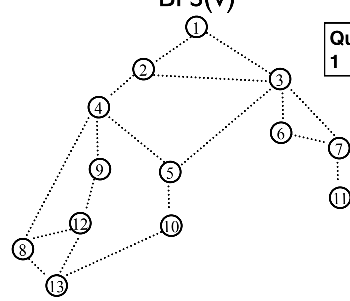
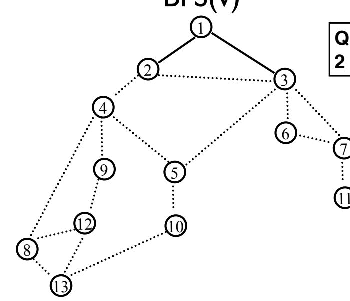
BFS(v)



Queue:
1

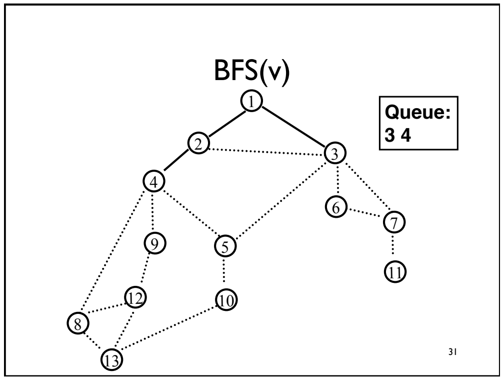
29

BFS(v)



Queue:
2 3

30


BFS(v)
Queue:
3 4
1
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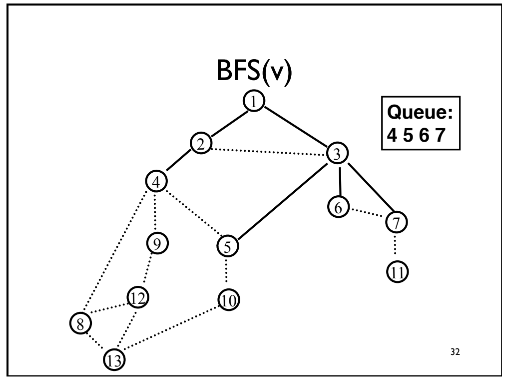
31


BFS(v)
Queue:
4 5 6 7
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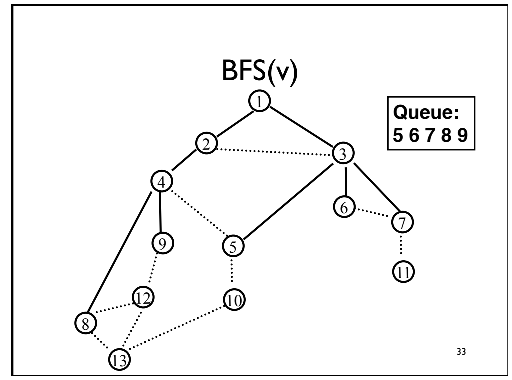
BFS(v)
Queue:
5 6 7 8 9
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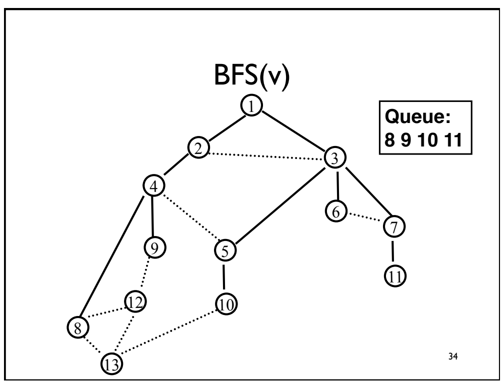
BFS(v)
Queue:
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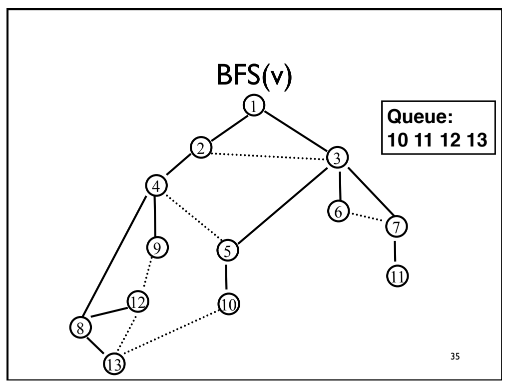
34


BFS(v)
Queue:
10 11 12 13
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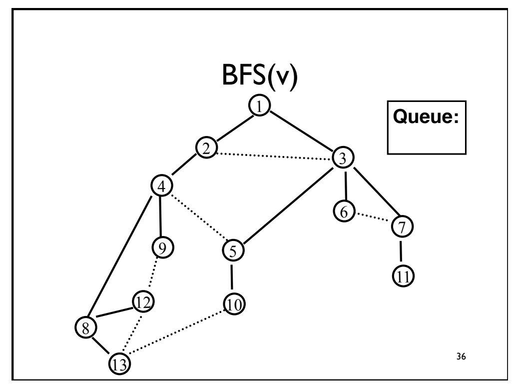
BFS(v)
Queue:
1
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11
12
13
36

BFS: Analysis, I

O(n) Global initialization: mark all vertices "undiscovered"
\+ BFS(s)
O(1) mark s "discovered"
\+ queue = { s }
O(n) while queue not empty
× u = remove_first(queue)
O(n) for each edge {u,x}
if (x is undiscovered)
mark x discovered
append x on queue
= mark u fully explored
$O(n^2)$

Simple analysis: 2 nested loops. Get worst-case number of iterations of each; multiply.

BFS: Analysis, II

Above analysis correct, but pessimistic (can't have Ω(n) edges incident to each of Ω(n) distinct "u" vertices if G is sparse). Alt, more global analysis:

Each edge is explored once from each end-point, so *total* runtime of inner loop is O(m).

Exercise: extend algorithm and analysis to non-connected graphs

Total O(n+m), n = # nodes, m = # edges

Properties of (Undirected) BFS(v)

BFS(v) visits x if and only if there is a path in G from v to x.

Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G

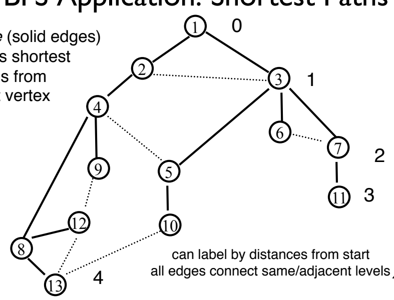
Level i in this tree are exactly those vertices *u* such that the shortest path (in G, not just the tree) from the root v is of length i.

All non-tree edges join vertices on the same or adjacent levels

not true of every spanning tree!

BFS Application: Shortest Paths

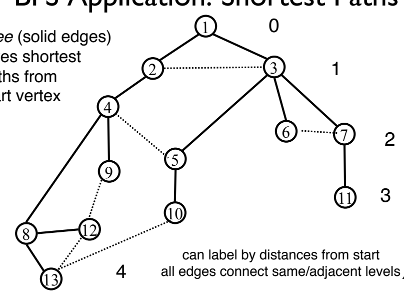
Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex



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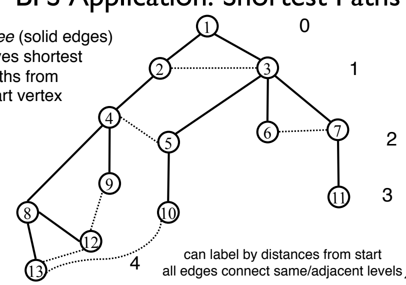

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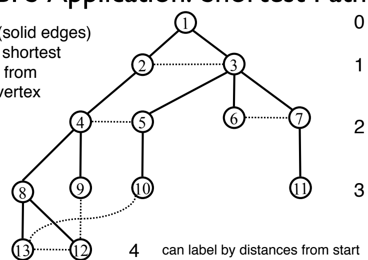


can label by distances from start
all edges connect same/adjacent levels

Why fuss about trees?

Trees are simpler than graphs

Ditto for algorithms on trees vs algs on graphs

So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure

E.g., BFS finds a tree s.t. level-jumps are minimized

DFS (below) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

Want to answer questions of the form:

given vertices u and v, is there a path from u to v?

Set up one-time data structure to answer such questions efficiently.

Graph Search Application: Connected Components

Want to answer questions of the form:

given vertices u and v, is there a path from u to v?

Idea: create array A such that

A[u] = smallest numbered vertex that is connected to u. Question reduces to whether A[u]=A[v]?

Q: Why not create 2-d array Path [u,v]?

Graph Search Application: Connected Components

```
initial state: all v undiscovered
for v = 1 to n do
    if state(v) != fully-explored then
        BFS(v): setting A[u] ←v for each u found
        (and marking u discovered/fully-explored)
    endif
endfor
```

Total cost: O(n+m)

each edge is touched a constant number of times (twice)

works also with DFS

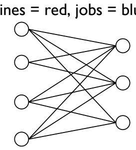
3.4 Testing Bipartiteness

Bipartite Graphs

Def. An undirected graph G = (V, E) is *bipartite (2-colorable)* if the nodes can be colored red or blue such that no edge has both ends the same color.

"bi-partite" means "two parts." An equivalent definition: G is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

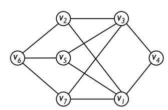
Applications.
- Stable marriage: men = red, women = blue
- Scheduling: machines = red, jobs = blue



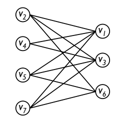
a bipartite graph

Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?
- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

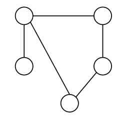


another drawing of G

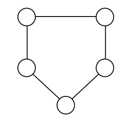
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

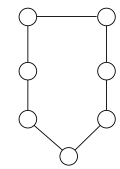
Pf. Impossible to 2-color the odd cycle, let alone G.



bipartite (2-colorable)



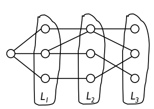
not bipartite (not 2-colorable)



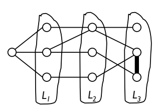
not bipartite (not 2-colorable)

Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.
- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)

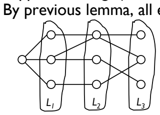


Case (ii)

Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.
- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)
- Suppose no edge joins two nodes in the same layer.
- By previous lemma, all edges join nodes on adjacent levels.



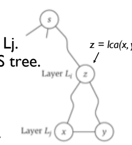
Case (i)

Bipartition:
- red = nodes on odd levels,
- blue = nodes on even levels.

Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.
- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

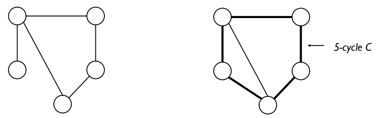
Pf. (ii)
- Suppose (x, y) is an edge & x, y in same level Lj.
- Let z = their lowest common ancestor in BFS tree.
- Let Li be level containing z.
- Consider cycle that takes edge from x to y, then tree from y to z, then tree from z to x.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd.



Obstruction to Bipartiteness

Cor: A graph G is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic–it *finds* a coloring or odd cycle.



bipartite (2-colorable)

not bipartite (not 2-colorable)

3.6 DAGs and Topological Ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications

- Course prerequisites: course v_i must be taken before v_j
- Compilation: must compile module v_i before v_j
- Computing workflow: output of job v_i is input to job v_j
- Manufacturing or assembly: sand it before you paint it…
- Spreadsheet evaluation order: if A7 is "=A6+A5+A4", evaluate them first

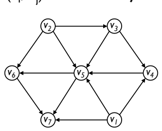
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

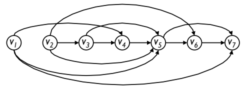
Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A <u>topological order</u> of a directed graph G = (V, E) is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

E.g., $\forall$edge (v_i, v_j), finish v_i before starting v_j



a DAG



a topological ordering of that DAG– all edges left-to-right

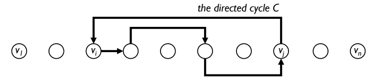
Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

if all edges go L→R, you can't loop back to close a cycle

Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C.
Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.
By our choice of i, we have $i < j$.
On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.



the supposed topological order: $v_1, \ldots, v_n$

Directed Acyclic Graphs

Lemma.
If G has a topological order, then G is a DAG.

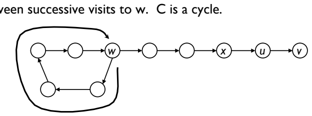
Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)
- Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
- Then, since u has at least one incoming edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice. ← Why must this happen?
- Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



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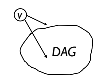
Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)
- Base case: true if n = 1.
- Given DAG on n > 1 nodes, find a node v with no incoming edges.
- G - { v } is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, G - { v } has a topological ordering.
- Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges. ▪

```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```



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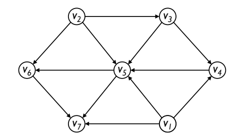
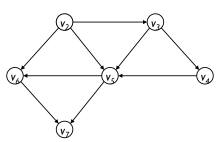
Topological Ordering Algorithm: Example



Topological order:

63

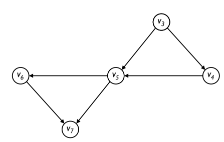
Topological Ordering Algorithm: Example



Topological order: v_1

64

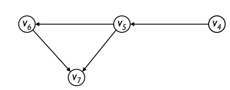
Topological Ordering Algorithm: Example



Topological order: v_1, v_2

65

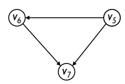
Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3

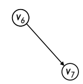
66

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

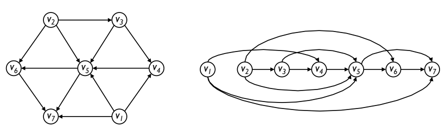


Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological Ordering Algorithm: Example



Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Topological Sorting Algorithm

Maintain the following:
  count[w] = (remaining) number of incoming edges to node w
  S = set of (remaining) nodes with no incoming edges
Initialization:
  count[w] = 0 for all w
  count[w]++ for all edges (v,w)
  S = S ∪ {w} for all w with count[w]==0
} O(m + n)
Main loop:
  while S not empty
    remove some v from S
    make v next in topo order
    for all edges from v to some w
    decrement count[w]
    add w to S if count[w] hits 0
} O(1) per node
O(1) per edge
Correctness: clear, I hope
Time: O(m + n) (assuming edge-list representation of graph)

Depth-First Search

Follow the first path you find as far as you can go

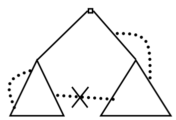
Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!



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DFS(v) – Recursive version

Global Initialization:
    for all nodes v, v.dfs# = -1   // mark v "undiscovered"
    dfscounter = 0

DFS(v)
    v.dfs# = dfscounter++   // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)   // tree edge (x previously undiscovered)
            DFS(x)
        else …   // code for back-, fwd-, parent,
                                   // edges, if needed
                                   // mark v "completed," if needed

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Why fuss about trees (again)?

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ ancestor

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Exercise: modify to compute vertex numbering

DFS(v) - explicit stack

Global Initialization: mark all vertices "undiscovered"

DFS(v)
    mark v "discovered"
    push (v,1) onto empty stack
    while stack not empty
        (u,i) = pop(stack)
        for (; i ≤ # of neighbors of u; i++)
            x = i^{th} edge on u's edge list
            if (x is undiscovered)
            mark x "discovered"
            push (u,i+1)   // save info to resume with u's next edge,
            u = x   // after exploring from x,
            i = 1   // (starting with its first edge)
        mark u completed

Idea: stack of unfinished vertices, plus pointers into their edge lists to say what work remains to finish.

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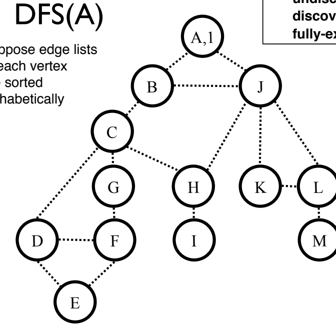
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:
undiscovered
discovered
fully-explored



Call Stack: (Edge list):

A (B,J)

77

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:
undiscovered
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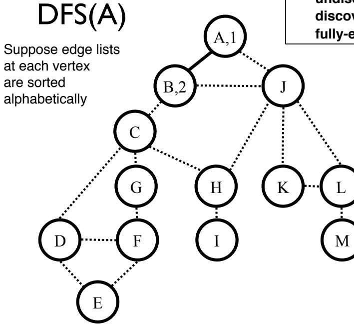
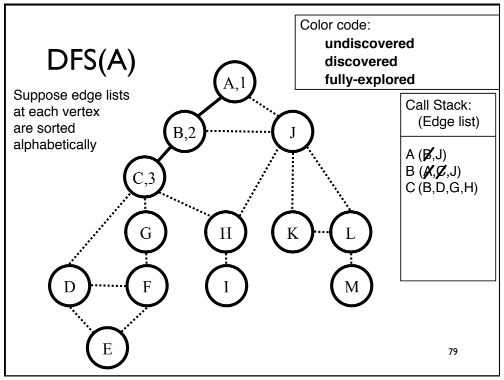
fully-explored



Call Stack: (Edge list)

A (~~B~~,J)
B (A,C,J)

78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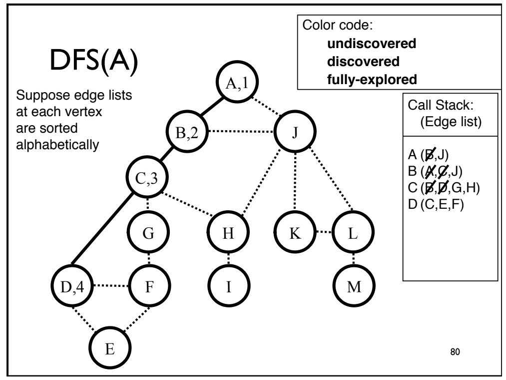
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)
A,1
B,2
J
C,3
G
H
K
L
D
F
I
M
E
79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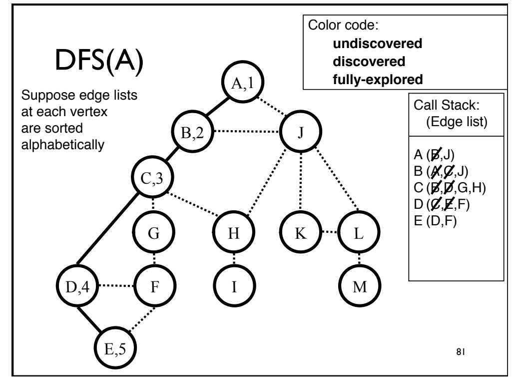
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)
A,1
B,2
J
C,3
G
H
K
L
D,4
F
I
M
E
80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Call Stack: (Edge list)
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B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)
A,1
B,2
J
C,3
G
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K
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D,4
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I
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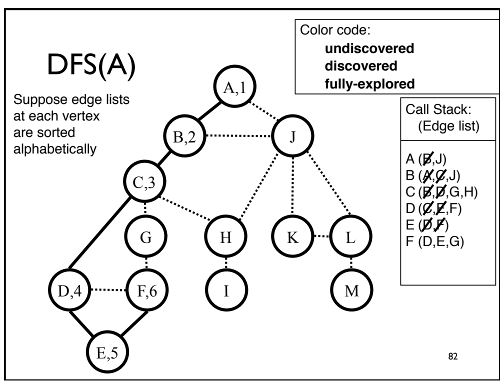
E,5
81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B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)
A,1
B,2
J
C,3
G
H
K
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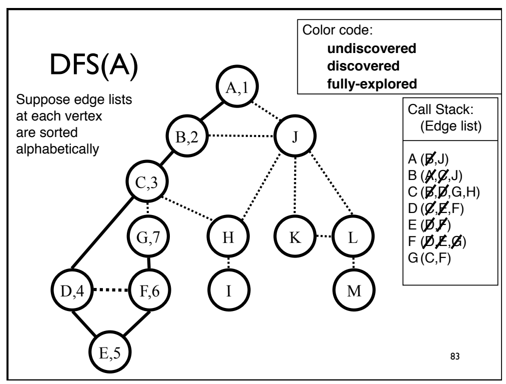
D,4
F,6
I
M
E,5
82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C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)
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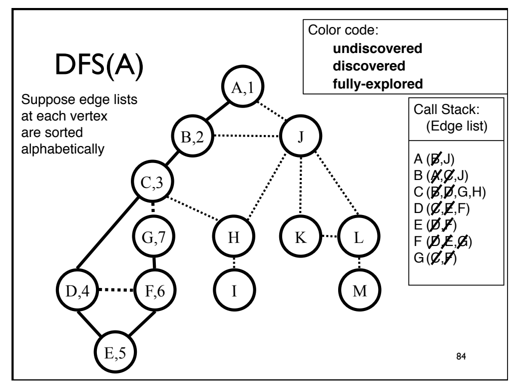
B,2
J
C,3
G,7
H
K
L
D,4
F,6
I
M
E,5
83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B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)
A,1
B,2
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C,3
G,7
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K
L
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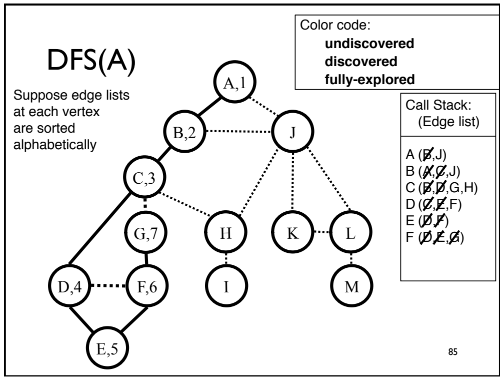
F,6
I
M
E,5
84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D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
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C,3
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E,5
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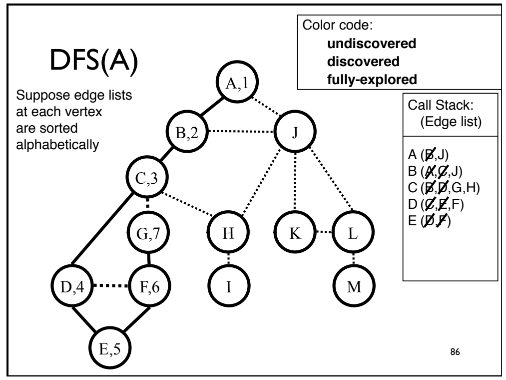
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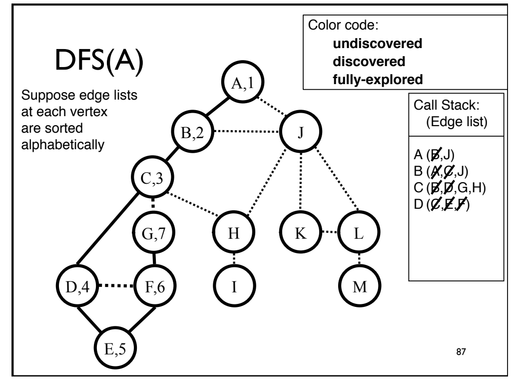
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
A,1
B,2
J
C,3
G,7
H
K
L
D,4
F,6
I
M
E,5
86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discovered
fully-explored
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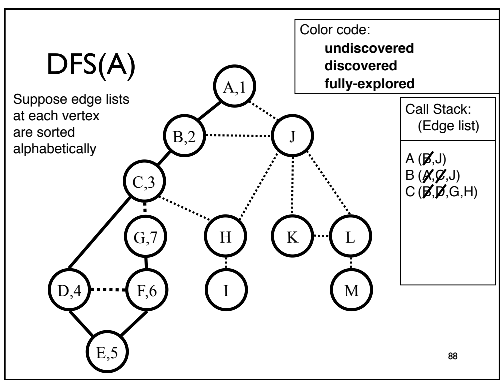
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)
A,1
B,2
J
C,3
G,7
H
K
L
D,4
F,6
I
M
E,5
87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discovered
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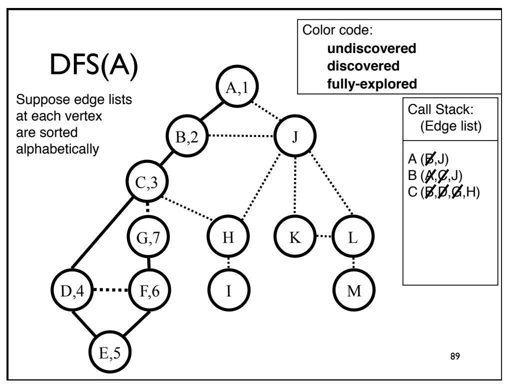
I
M
E,5
88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fully-explored
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C (~~B~~,~~D~~,~~G~~,H)
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J
C,3
G,7
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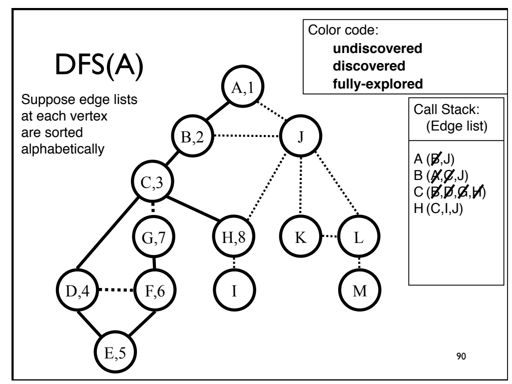
D,4
F,6
I
M
E,5
89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discovered
fully-explored
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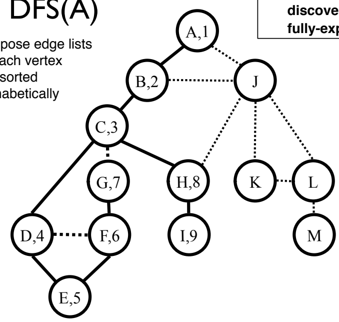
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)
A,1
B,2
J
C,3
G,7
H,8
K
L
D,4
F,6
I
M
E,5
90

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

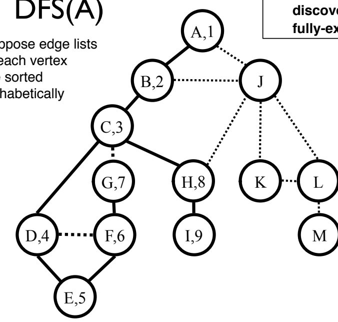
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

91

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

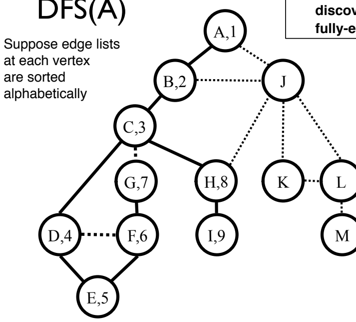
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (~~H~~)

92

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
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fully-explored

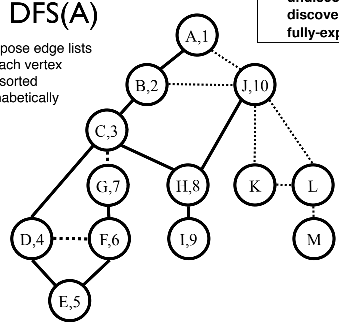
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

93

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

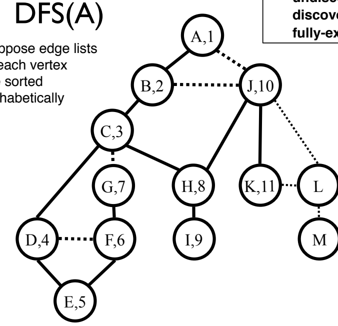
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

94

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

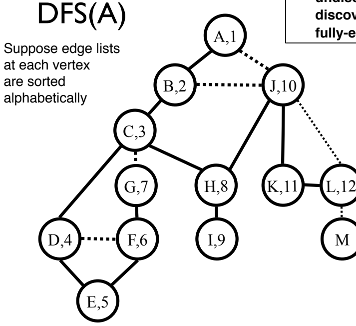
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

95

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

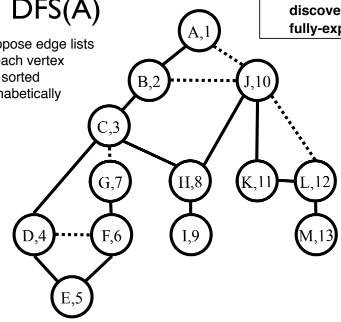
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (J,K,M)

96

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



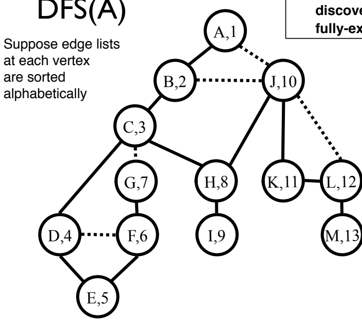
Color code:
undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)
M (L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



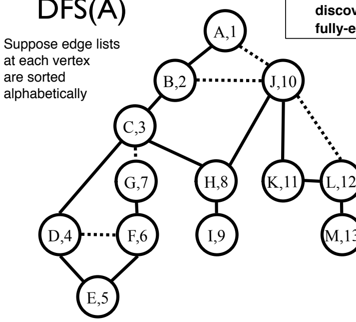
Color code:
undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



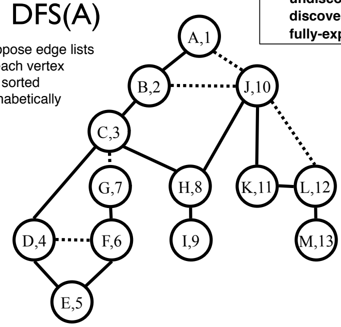
Color code:
undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



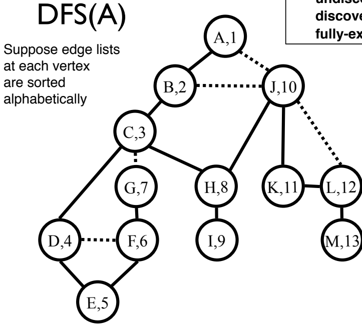
Color code:
undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



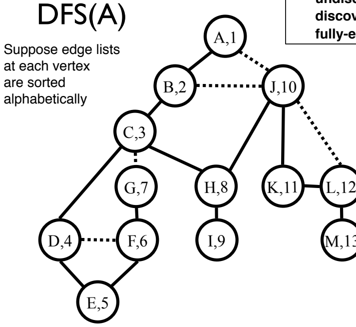
Color code:
undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,~~L~~)

DFS(A)

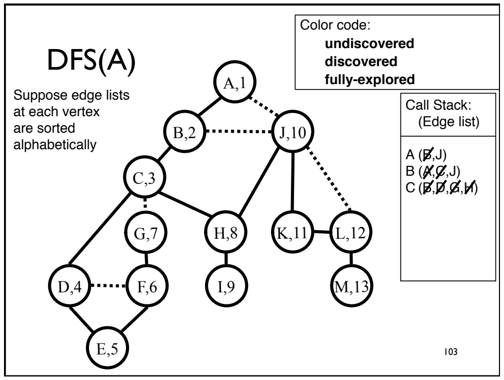
Suppose edge lists at each vertex are sorted alphabetically

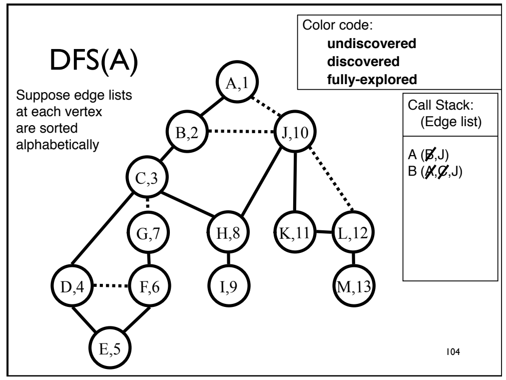


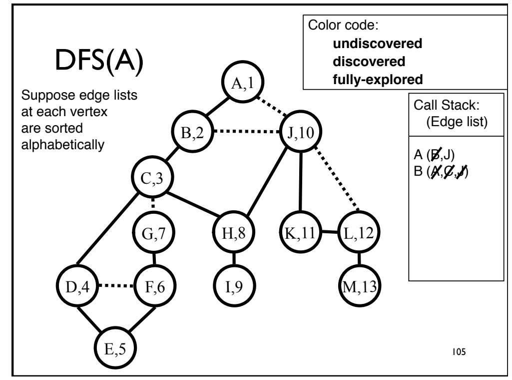
Color code:
undiscovered
discovered
fully-explored

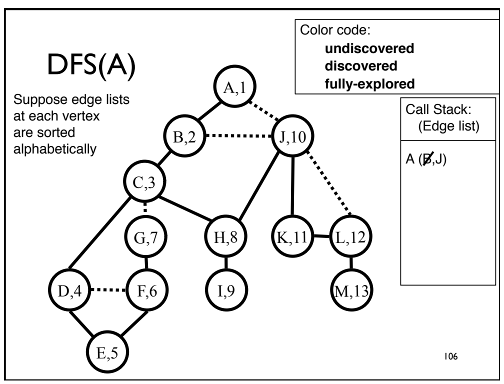
Call Stack:
(Edge list)

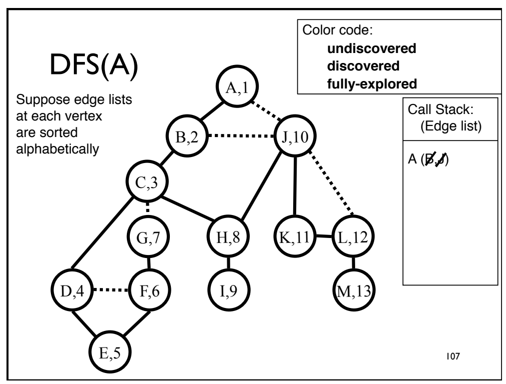
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

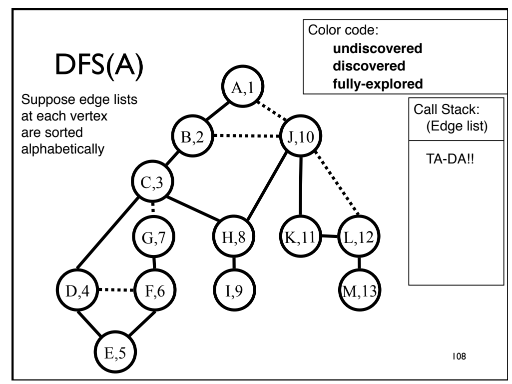

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
103

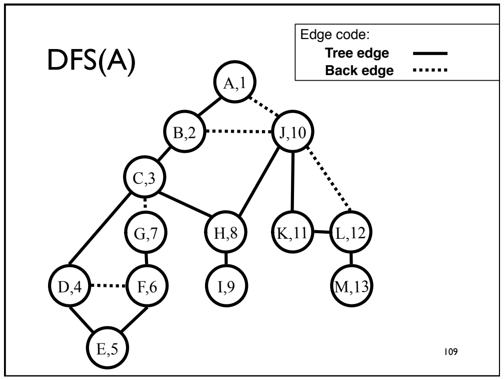

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,J)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
104

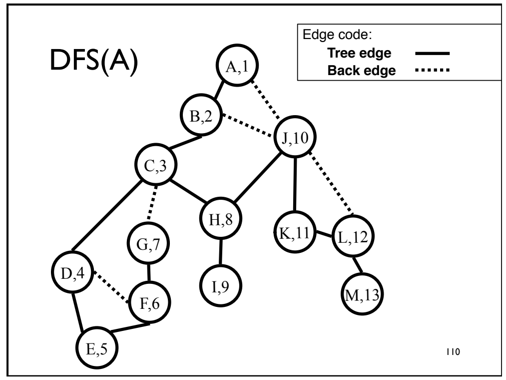

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,J)
B (~~A~~,~~C~~,~~J~~)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
105

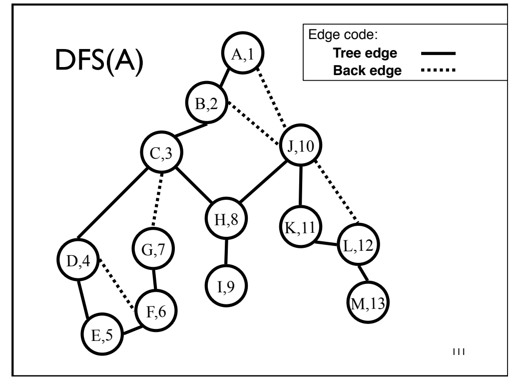

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,J)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
106

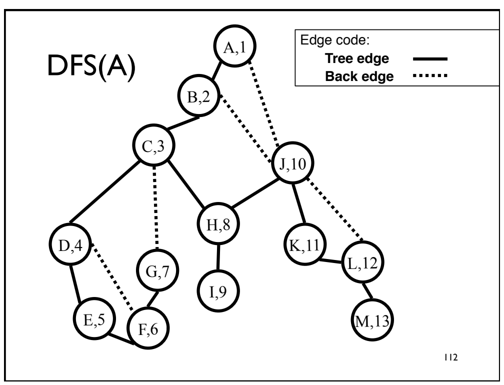

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (~~B~~,~~J~~)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
107

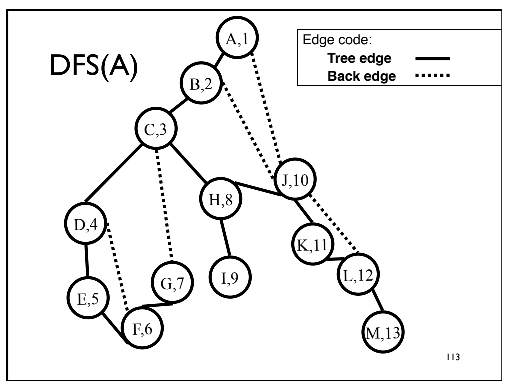

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
TA-DA!!
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
108

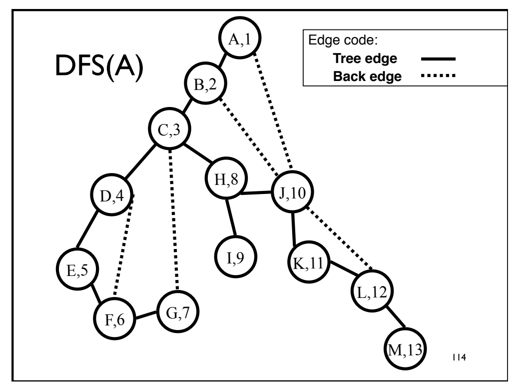

DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
109


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
H,8
K,11
L,12
G,7
D,4
F,6
I,9
M,13
E,5
110

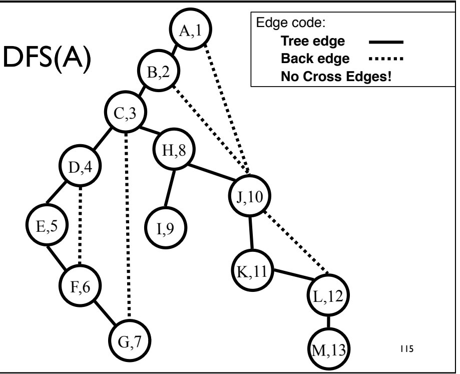

DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
H,8
K,11
L,12
D,4
G,7
F,6
I,9
M,13
E,5
111


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
J,10
H,8
D,4
K,11
L,12
G,7
I,9
E,5
F,6
M,13
112


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
J,10
H,8
D,4
K,11
L,12
G,7
I,9
E,5
F,6
M,13
113


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
H,8
J,10
D,4
I,9
K,11
E,5
L,12
F,6
G,7
M,13
114

DFS(A)



115

Properties of (Undirected) DFS(v)

Like BFS(v):
- DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

Unlike the BFS tree:
- the DF spanning tree isn't minimum depth
- its levels don't reflect min distance from the root
- non-tree edges never join vertices on the same or adjacent levels

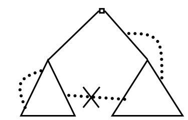
BUT…

116

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!



117

DFS(v) – Recursive version

```
Global Initialization:
   for all nodes v, v.dfs# = -1   // mark v "undiscovered"
   dfscounter = 0

DFS(v)
   v.dfs# = dfscounter++          // v "discovered", number it
   for each edge (v,x)
      if (x.dfs# = -1)            // tree edge (x previously undiscovered)
         DFS(x)
      else …                      // code for back-, fwd-, parent,
                                  // edges, if needed
                                  // mark v "completed," if needed
```

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Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple"--only descendant/ancestor

119

A simple problem on trees

Given: tree T, a value L(v) defined for every vertex v in T
Goal: find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).
How?

120

A simple problem on trees

Given: tree T, a value L(v) defined for every vertex v in T
Goal: find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).
How? Depth first search, using:

$$M(v) = \begin{cases} L(v) & \text{if } v \text{ is a leaf} \\ \min(L(v),\ \min_{w \text{ a child of v}} M(w)) & \text{otherwise} \end{cases}$$

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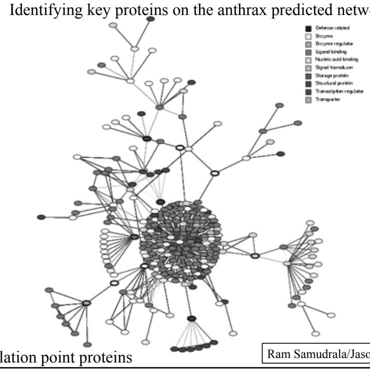
Application: Articulation Points

A node in an undirected graph is an **articulation point** iff removing it disconnects the graph

articulation points represent vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

122

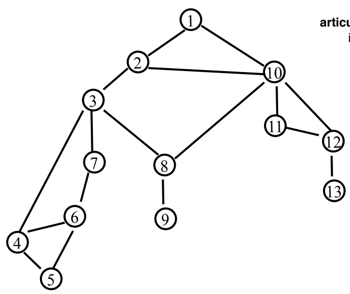
Identifying key proteins on the anthrax predicted network



Articulation point proteins

Ram Samudrala/Jason McDermott

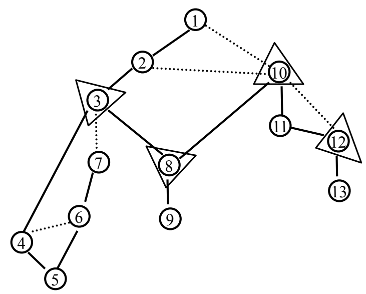
Articulation Points



articulation point
iff its removal disconnects the graph

124

Articulation Points



125

Brainstorming

draw a graph, ~ 10 nodes, A-J
redraw as via DFS, starting at "E"
add dsf#s & tree/back edges (solid/dashed)
find cycles
give alg to find cycles via dfs; does G have any?

find articulation points
what do cycles have to do with articulation points?
alg to find articulation points via DFS???

126

Simple Case: Artic. Pts in a tree

Leaves – never articulation points

Internal nodes – always articulation points

Root – articulation point if and only if two or more children

Non-tree: extra edges remove some articulation points (which ones?)

127

Articulation Points from DFS

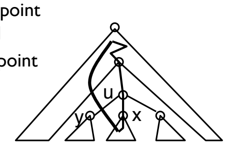
Root node is an articulation point iff it has more than one child

Leaf is never an articulation point

non-leaf, non-root node u is an articulation point

⇕

∃ some child y of u s.t. no non-tree edge goes above u from y or below



If removal of u does NOT separate x, there must be an exit from x's subtree. How? Via back edge.

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Articulation Points: the "LOW" function

trivial

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.

critical

LOW(v) =
min ({dfs#(v)} ∪ {LOW(w) | w a child of v } ∪ { dfs#(x) | {v,x} is a back edge from v })

v articulation point iff…

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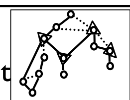
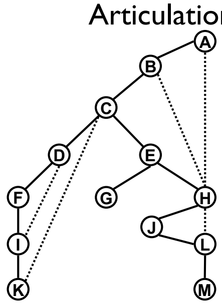
Articulation Points

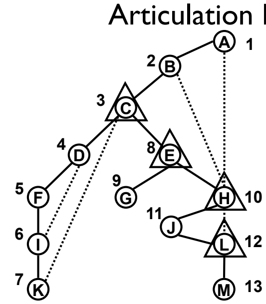




Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

130

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
I	6	3
J	11	10
K	7	3
L	12	10
M	13	13

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DFS(v) for Finding Articulation Points

```
Global initialization: v.dfs# = -1 for all v.
DFS(v)
 v.dfs# = dfscounter++
 v.low = v.dfs#                    // initialization
 for each edge {v,x}
     if (x.dfs# == -1)  // x is undiscovered
         DFS(x)
         v.low = min(v.low, x.low)
         if (x.low >= v.dfs#)
             print "v is art. pt., separating x"
     else if (x is not v's parent)
         v.low = min(v.low, x.dfs#)
```

Except for root. Why?

Equiv: "if({v,x} is a back edge)" Why?

Summary

Graphs –abstract relationships among pairs of objects

Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected

Representation – edge list, adjacency matrix

Nodes vs Edges – m = O(n^2), often less

BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer

DFS – recursion/stack; all edges ancestor/descendant

Algorithms – connected components, bipartiteness, topological sort, articulation points

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3' UTR

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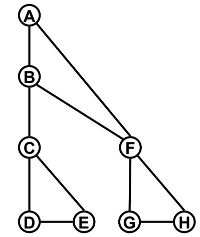
Articulation Points

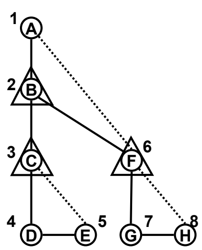


Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		

AP's:
BCC's:
1)
2)
3)
4)

135

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	3
D	4	3
E	5	3
F	6	1
G	7	6
H	8	6

AP's: C, B, F
BCC's:
1) C--D, D--E, E--C
2) B--C
3) A--B, B--F, F--A
4) F--G, G--H, H--F

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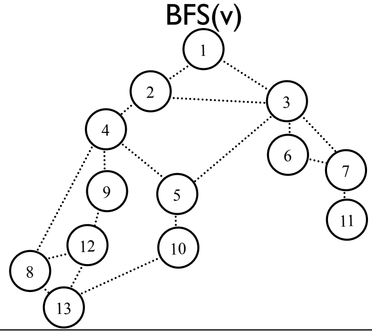
BFS(v)



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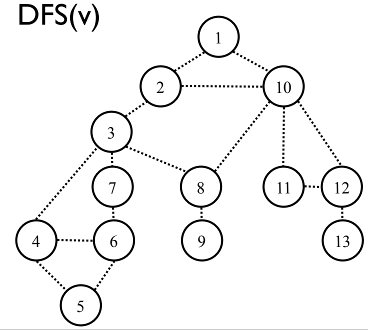
DFS(v)



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Articulation Points from DFS

- Every interior vertex of a tree is an articulation point
  - Non-tree edges eliminate articulation points
- Leaves are never articulation points
- Root node is an articulation point iff it has more than one child

non-leaf, non-root node u is an articulation point	⇔	no non-tree edge goes above u from a sub-tree below some child of u

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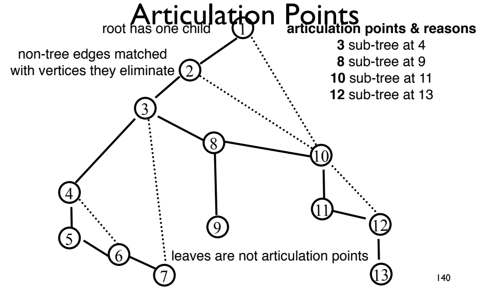
DFS Application: Articulation Points



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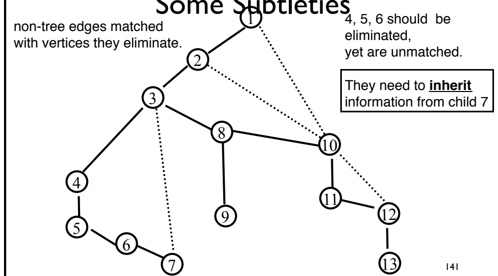
Articulation Points: Some Subtleties



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DFS Vertex Numbering

If u is an ancestor of v in the DFS tree, then dfs#(u) < dfs#(v).

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