Stable Marriage

KLEINBERG/TARDOS, CHAPTER 1.

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Algorithms

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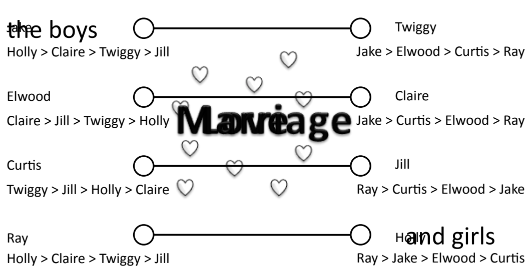
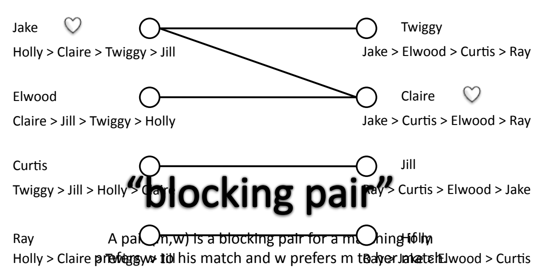
Romance

Love, marriage, & bipartite graphs



When is marriage stable?



When is marriage stable?

“blocking pair”

A pair (m,w) is a blocking pair for a matching if m prefers w to his match and w prefers m to her match.

“stability”

A matching is stable if there is no blocking pair.

Questions…

Question 1. Do stable matchings always exist?
Question 2. Are they easy to find?
Question 3. Does the courtship ritual work?
Question 4. Are stable matchings unique?
Question 5. If not, who benefits?

Courtship Ritual

- Algorithm proposed by Gale-Shapley in 1962

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman)
{
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

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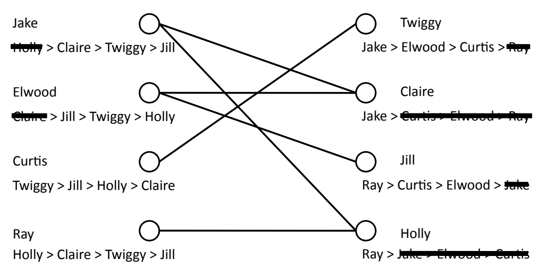
The courtship ritual



Cupid's Theorems

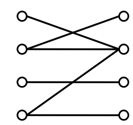
Theorem. Courtship algorithm terminates.

Theorem. Resulting marriage is stable.

Corollary. Stable marriages always exist and are easy to find.

We will prove these facts next.

Step 1: Consider "state" of algorithm



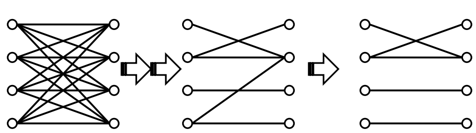
An edge from a boy to every girl who hasn't yet rejected him.

* Completely defines state of boys' preference lists

Step 1: Define states algorithm can be in

"Transitions"

A rejected proposal, i.e., a deleted edge or crossed-off name.



Start state:
complete bipartite graph

Step 2: proof of termination

Theorem.

The courtship ritual terminates.

Proof.

1. Define a "measure of progress": the number of edges in the bipartite graph (i.e., the number of names on boys' lists)
2. Note this is strictly decreasing and always non-negative.

Step 3: defining useful invariants

The girls' options only ever improve!

Invariant

For every girl G and boy B, if G is crossed off B's list it is because she has a suitor she prefers.

Proof Sketch.

B only crosses off G when she rejects him in favor of someone else.

Step 4: everyone marries

Theorem.

Everyone gets married.

Proof.

By contradiction. Suppose boy B isn't married.

1. Then he has crossed everyone off his list.
2. So every girl has a suitor she prefers to B.
3. Also, all girls are married to a unique boy.
4. But number of girls equals number of boys.

Step 5: proof of stability

Theorem.

The resulting matching is stable.

Proof.

Consider boy B and girl G that are not married to each other.

1. Suppose G was crossed off B's list. Then G prefers husband to B, so won't elope with B.
2. Suppose G is on B's list. Then B didn't propose to G yet, so B prefers wife to G, so won't elope with G.

Questions and answers…

Question 1. Do stable matchings always exist?

Question 2. Are they easy to find?

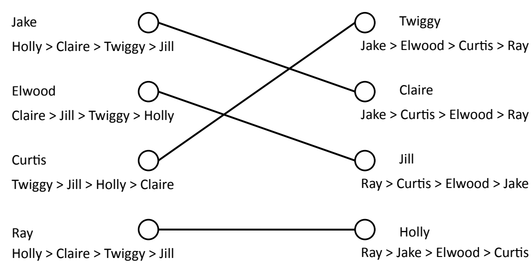
Question 3. Does the courtship ritual work?

Yes, yes, yes.

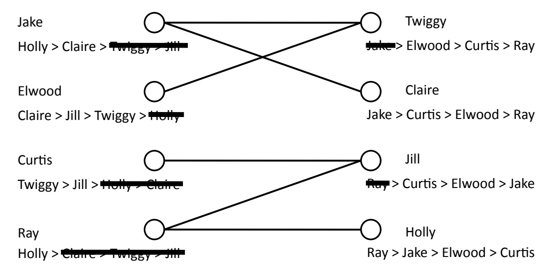
Question 4. Are stable matchings unique????

Question 5. If not, who benefits????

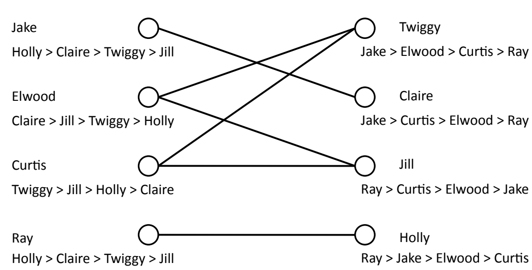
Recall …



In an alternate universe

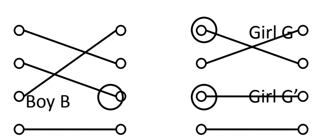


Conclusion: not unique



Stable spouses

In general, there are many stable marriages.



"stable spouse"

A person P is a stable spouse of a person P' if P is married to P' in some stable matching.

Boys are happier than girls

way

Theorem.

Boys marry their most preferred stable wife….
… and girls get their least preferred stable husband!

Proof.

Proof.

Of 1st claim, by contradiction. Suppose some boy isn't married to favorite stable spouse.

1. He must have proposed to her and been refused.
2. Let B be 1st boy to lose his favorite stable wife G.
3. Then G must have had a proposal from a boy B' she preferred to B.
4. Since B' has not yet crossed off his favorite stable wife, B' must love G more than any stable wife.
5. But then B' and G will elope in marriage which matches B to G, contradicting stability of wife G.

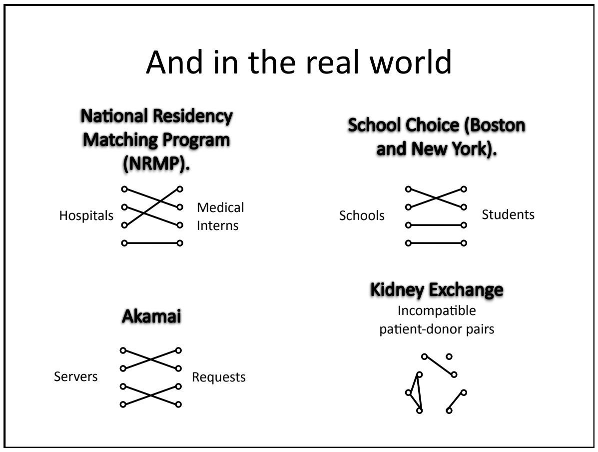
Proof.

Of 2nd claim, by contradiction. Suppose there is a stable matching in which girl G gets worse husband.

1. Let B be her husband when boys propose and B' be her husband in worse matching.
2. Then G prefers B to B' by assumption.
3. Furthermore, by 1st claim, G is favorite stable wife of B, so B prefers G to wife in worse matching.
4. But then B and G will elope in worse matching.

But of course … symmetry

If girls propose, then they will get
their favorite stable husbands.


And in the real world
National Residency Matching Program (NRMP).
Hospitals
Medical Interns
School Choice (Boston and New York).
Schools
Students
Akamai
Servers
Requests
Kidney Exchange
Incompatible patient-donor pairs