CSE 421

Bellman Ford – Linear Programming

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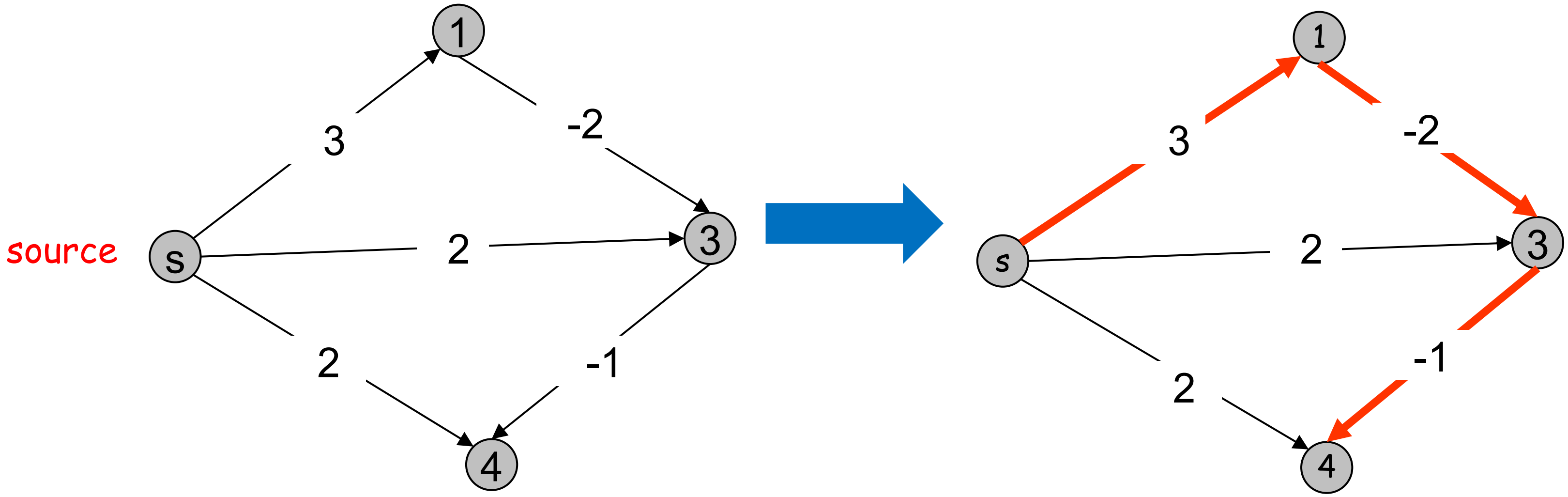
Shortest Paths with Negative Edge Weights

Shortest Paths with Neg Edge Weights

Given a weighted directed graph $G = (V, E)$ and a source vertex s, where the weight of edge (u,v) is $c_{u,v}$

Goal: Find the shortest path from s to all vertices of G.

Recall that Dikjstra's Algorithm fails when weights are negative

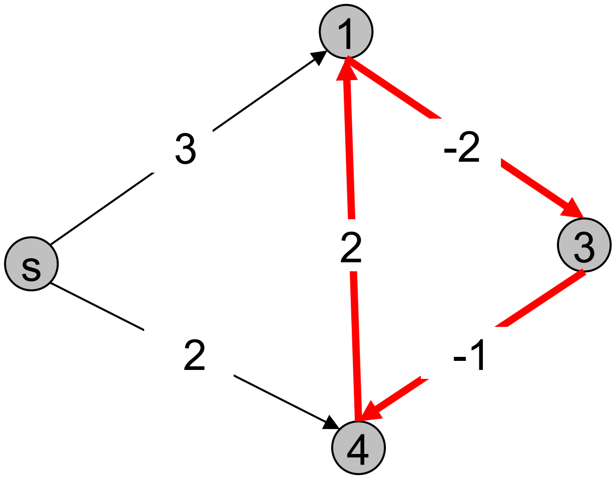


Impossibility on Graphs with Neg Cycles

Observation: No solution exists if G has a negative cycle.

This is because we can minimize the length by going over the cycle again and again.

So, suppose G does not have a negative cycle.



DP for Shortest Path

Def: Let $OPT(v, i)$ be the length of the shortest s - v path with at most i edges.

Let us characterize $OPT(v, i)$.

Case 1: $OPT(v, i)$ path has less than i edges.

- Then, $OPT(v, i) = OPT(v, i - 1)$.

Case 2: $OPT(v, i)$ path has exactly i edges.

- Let $s, v_1, v_2, \dots, v_{i-1}, v$ be the $OPT(v, i)$ path with i edges.
- Then, $s, v_1, \dots, v_{i-1}$ must be the shortest s - v_{i-1} path with at most $i - 1$ edges. So,

$$OPT(v, i) = OPT(v_{i-1}, i - 1) + c_{v_{i-1}, v}$$

DP for Shortest Path

Def: Let $OPT(v, i)$ be the length of the shortest s - v path with at most i edges.

$$OPT(v,i) = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{if } v \neq s, i = 0 \\ \min\left(OPT(v, i-1), \min_{u:(u,v)\text{ an edge}} OPT(u, i-1) + c_{u,v}\right) & \end{cases}$$

So, for every v, $OPT(v, ?)$ is the shortest path from s to v.

But how long do we have to run?

Since G has no negative cycle, it has at most $n-1$ edges. So, $OPT(v, n-1)$ is the answer.

Bellman Ford Algorithm

```
for v=1 to n
   if v ≠ s then
      M[v,0]=∞
M[s,0]=0.

for i=1 to n-1
   for v=1 to n
      M[v,i]=M[v,i-1]
      for every edge (u,v)
         M[v,i]=min(M[v,i], M[u,i-1]+c_{u,v})
```

Running Time: $O(nm)$

Can we test if G has negative cycles?

Bellman Ford Algorithm

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         M[v,i]=min(M[v,i], M[u,i-1]+c_{u,v})
```

Running Time: $O(nm)$

Can we test if G has negative cycles?

Yes, run for i=1…2n and see if the M[v,n-1] is different from M[v,2n]

System of Linear Equations

Find a solution to

$$x_3 - x_1 = 4$$
$$x_3 - 2x_2 = 3$$
$$x + 2x_2 + x_3 = 7$$

Can be solved by Gaussian elimination method

Linear Programming

Optimize a linear function subject to linear inequalities

$$
\begin{aligned}
\max \quad & 3x_1 + 4x_3 \\
s.t., \quad & x_1 + x_2 \le 5 \\
& x_3 - x_1 = 4 \\
& x_3 - x_2 \ge -5 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

- We can have inequalities,
- We can have a linear objective functions

Applications of Linear Programming

Generalizes: Ax=b, 2-person zero-sum games, shortest path, max-flow, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- We can solve linear programming in polynomial time.
- Useful for approximation algorithms
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: IBM CPLEX, Gorubi in Python, CVX in Matlab, ….
- CPLEX can solve LPs with millions of variables/constraints in minutes

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alory and (h)appiness per pound:

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat, 0.2lb of fruits we will be $0.5\ h_m + 0.2\ h_f$ happy

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

- You should eat 1500 calaroies to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned} \max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\ s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \leq 20 \\ & x_v c_v + x_m c_m + x_f c_f + x_d c_d \leq 1500 \\ & x_v, x_m, x_f, x_d \geq 0 \end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

How to Design an LP?

- Define the set of variables
- Put constraints on your variables,
  - should they be nonnegative?
- Write down the constraints
  - If a constraint is not linear try to approximate it with a linear constraint
- Write down the objective function
  - If it is not linear approximation with a linear function
- Decide if it is a minimize/maximization problem

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$
\begin{aligned}
\max \quad & \sum_{e \text{ out of } s} x_e \\
s.t. \quad & \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e && \forall v \neq s, t \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}
$$

Q: Do we get exactly the same properties as Ford Fulkerson?

A: Not necessarily, the max-flow may not be integral

Example 3: Min Cost Max Flow

Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$ for each gallon of water that we send through e.

Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathrm{E}} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e && \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}
$$

Summary (Linear Programming)

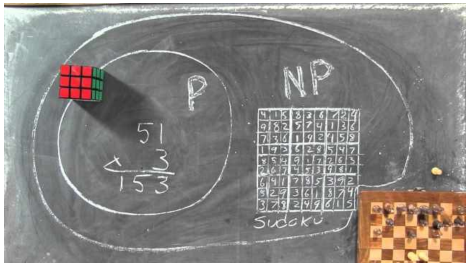
- Linear programming is one of the biggest advances in 20th century

- It is being used in many areas of science: Mechanics, Physics, Operations Research, and in CS: AI, Machine Learning, Theory, …

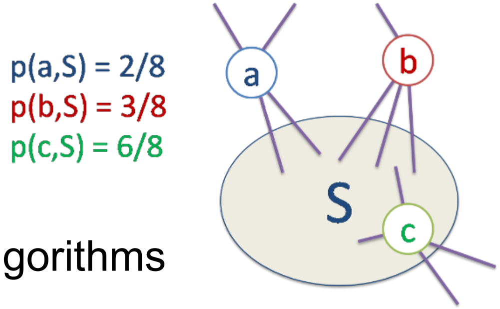
- Almost all problems that we talked can be solved with LPs, Why not use LPs?
  - Combinatorial algorithms are typically faster
  - They exhibit a better understanding of worst case instances of a problem
  - They give certain structural properties, e.g., Integrality of Max-flow when capacities are integral

- There is rich theory of LP-duality which generalizes max-flow min-cut theorem

What is next?



- CSE 431 (Complexity Course)
  - How to prove lower bounds on algorithms?



- CSE 521 (Graduate Algorithms Course)
  - How to design streaming algorithms?
  - How to design algorithms for high dimensional data?
  - How to use matrices/eigenvalues/eigenvectors to design algorithms
  - How to use LPs to design algorithms?

- CSE 525 (Graduate Randomized Algorithms Course)
  - How to use randomization to design algorithms?
  - How to use Markov Chains to design algorithms?