CSE 421: Introduction to Algorithms

Stable Matching

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Summary

Stable matching problem: Given n companies and n applicants, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm: Guarantees to find a stable matching for any problem instance.
- Q: How to implement GS algorithm efficiently?
- Q: If there are multiple stable matchings, which one does GS find?
- Q: How many stable matchings are there?

Company Optimal Assignments

Definition: Company c is a valid partner of applicant a if there exists some stable matching in which they are matched.

Company-optimal matching: Each company receives the best valid partner (according to its preferences).

- Not that each company receives its most favorite applicant.

Claim: All executions of GS yield a company-optimal matching, which is a stable matching!

- So, output of GS is unique!!
- No reason a priori to believe that company-optimal matching is perfect, let alone stable.

Company Optimality

S
(c, a)
(c', a')
$\ldots$

Claim: GS matching **S*** is company-optimal.

Proof: (by contradiction)

Suppose some company is paired with someone other than its best valid partner. Companies propose in decreasing order of preference $\Rightarrow$ some company is rejected by a valid partner.

Let c be the first such rejection, and let a be its best valid partner.

Let **S** be a stable matching where c and a are matched.

In building **S***, when c is rejected, a is assigned to a company, say c' whom she prefers to c.

Let c' be a' partner in **S**.

In building **S***, c' is not rejected by any valid partner at the point when c is rejected by a. Thus, c' prefers a to a'.

But a prefers c' to c.

Thus (c', a) is unstable in **S**. ■

since this is the first rejection by a valid partner

Company Optimality Summary

Company-optimality: In version of GS where companies propose, each comapny receives the best valid partner.

> a is a valid partner of c if there exist some stable matching where c and a are paired

Q: Does company-optimality come at the expense of the applicants?

Applicant Pessimality

Applicant-pessimal assignment: Each applicant receives the worst valid partner.

Claim. GS finds applicant-pessimal stable matching **S***.

Proof.

Suppose (c, a) matched in **S***, but c is not the worst valid partner for a.

There exists stable matching **S** in which a is paired with a company, say c', whom she likes less than c.

Let a' be c partner in **S**.

c prefers a to a'. ← company-optimality of **S***

Thus, (c, a) is an unstable in **S**.

■

Summary

- Stable matching problem: Given **n** men and **n** women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

- GS algorithm finds a stable matching in $O(n^2)$ time. ✓
- GS algorithm finds man-optimal woman pessimal matching ✓
- Q: How many stable matching are there?

Lessons Learned

- Powerful ideas learned in course.
  - Isolate underlying structure of problem.
  - Create useful and efficient algorithms.

- Potentially deep social ramifications. [legal disclaimer]
  - Always try to propose first!

How many stable Matchings?

We already show every instance has at least 1 stable matchings.

There are instances with about 2.24^n stable matchings for

[Karlin-O-Weber'17]: Every instance has at most 131072^n stable matchings
[Palmer-Palvolgyi'20]: Every instance has at most 4.47^n stable matchings

[Research-Question]:
Is there an "efficient" algorithm that chooses a uniformly random stable matching of a given instance.

Extensions: Matching Residents to Hospitals

Comapnies ≈ hospitals, Applicants ≈ med school residents.

- Variant 1: Some participants declare others as unacceptable.
- Variant 2: Unequal number of companies and applicants.
- Variant 3: A hospital wants to hire multiple residents

e.g. A resident not interested in Cleveland

An analogous version of GS algorithm works!

Induction: Intro 1

Prove that for all $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Def $P(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Base Case: $P(1)$ holds: $1 = 1(1+1)/2$

IH: $P(n-1)$ holds.

IS: Goal to prove $P(n)$.

$$\begin{aligned} 1 + \cdots + n &= (1 + \cdots + n - 1) + n \\ &= \left(\frac{(n-1)n}{2}\right) + n \qquad \text{By IH} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

Induction: Intro

Prove that every instance of stable matchings with n companies and n applicants where some participant declare others as unacceptable has at most $n! = n(n-1)\dots 21$ many perfect matchings.