CSE 421: Introduction to Algorithms

DFS - DAGs

Shayan Oveis Gharan

HW1 Grade

Q: I received low grade in HW1 what should I do?

- Understand what was your mistake. Did you understand the problem statement correctly?
- Show up to office hours and ask for hints or to explain your solution
- Review materials of 311 on proofs/induction
- Do exercises from the book

Q: My HW1 grade is low, will I be able to receive 4.0?

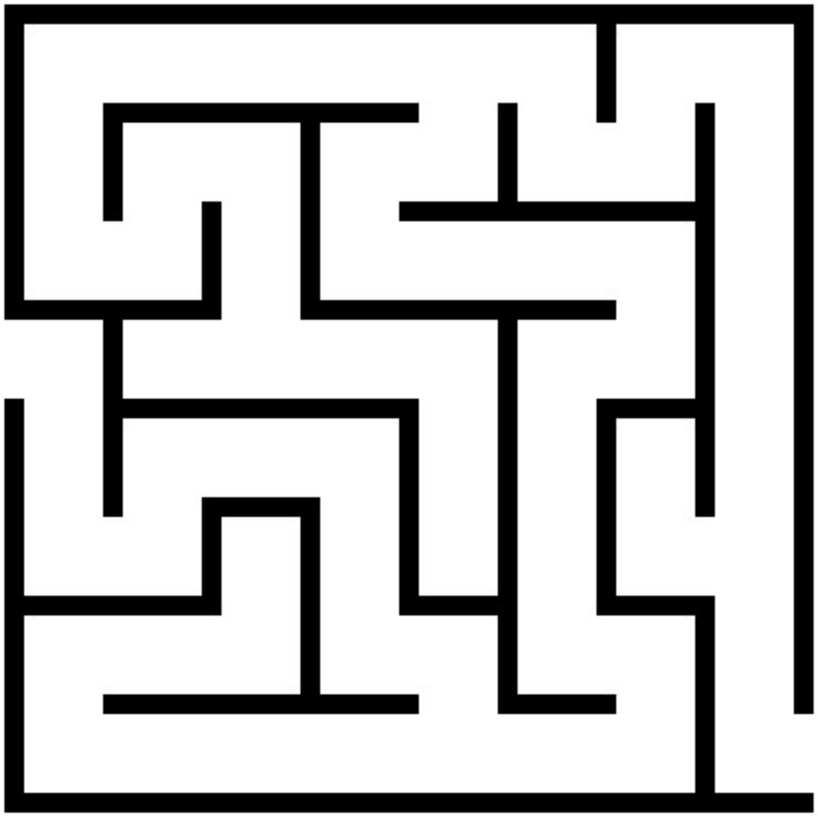
- Yes, I usually look at your progress. Many students are behind at beginning but by practice they catch up and receive 4.0

Q: I have filled out a regrade request, but was not convinced, what should I do?

- Show up to my office hour and discuss your solution

Depth First Search

Follow the first path you find as far as you can go; back up to last unexplored edge when you reach a dead end, then go as far you can



Naturally implemented using recursive calls or a stack

DFS(s) – Recursive version

Global Initialization: mark all vertices undiscovered

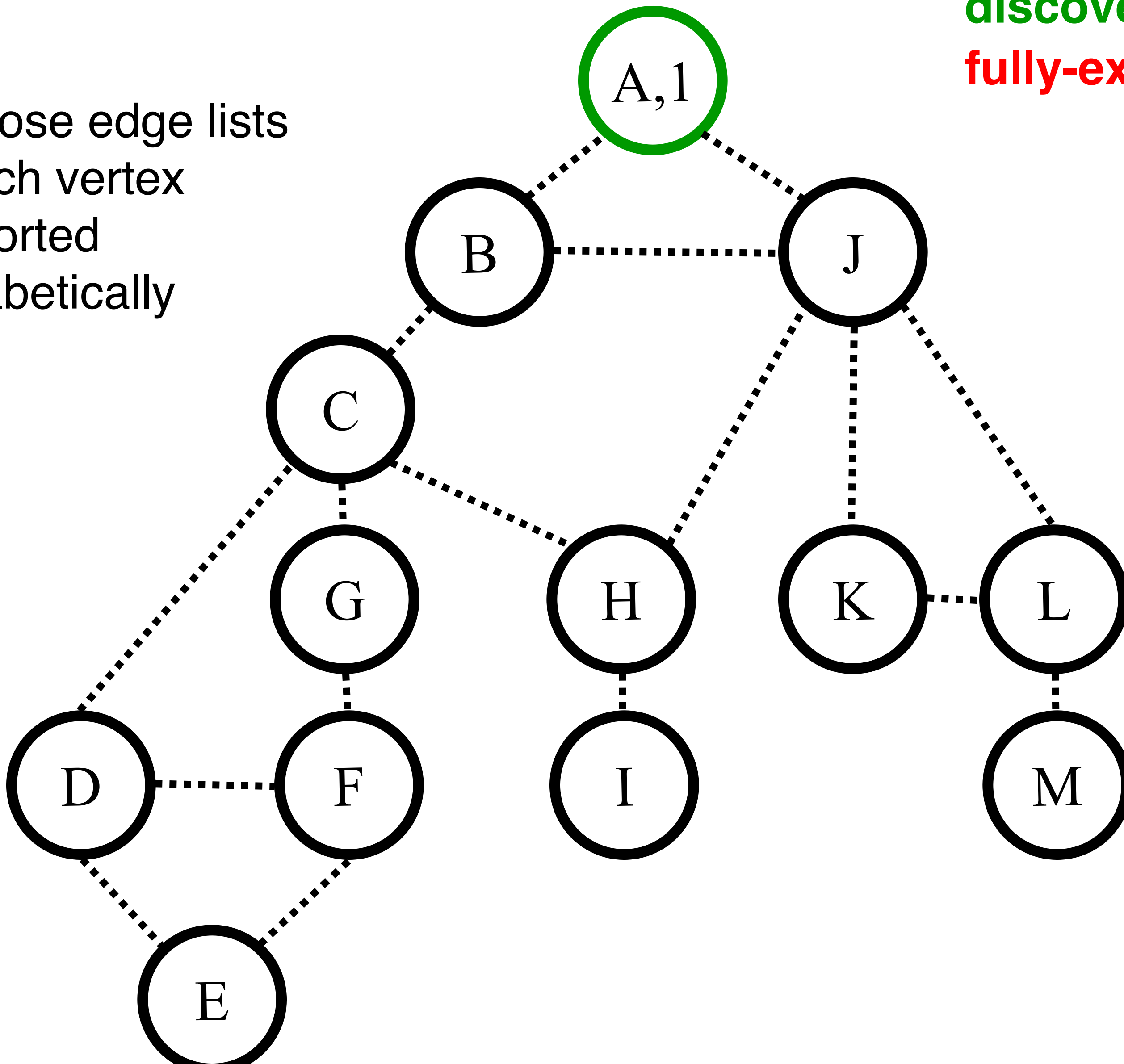
```
DFS(v)
   Mark v discovered

   for each edge {v,x}
       if (x is undiscovered)
           Mark x discovered
           DFS(x)

   Mark v full-discovered
```

DFS(A)

Color code:

undiscovered
discovered
fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack (Edge list):

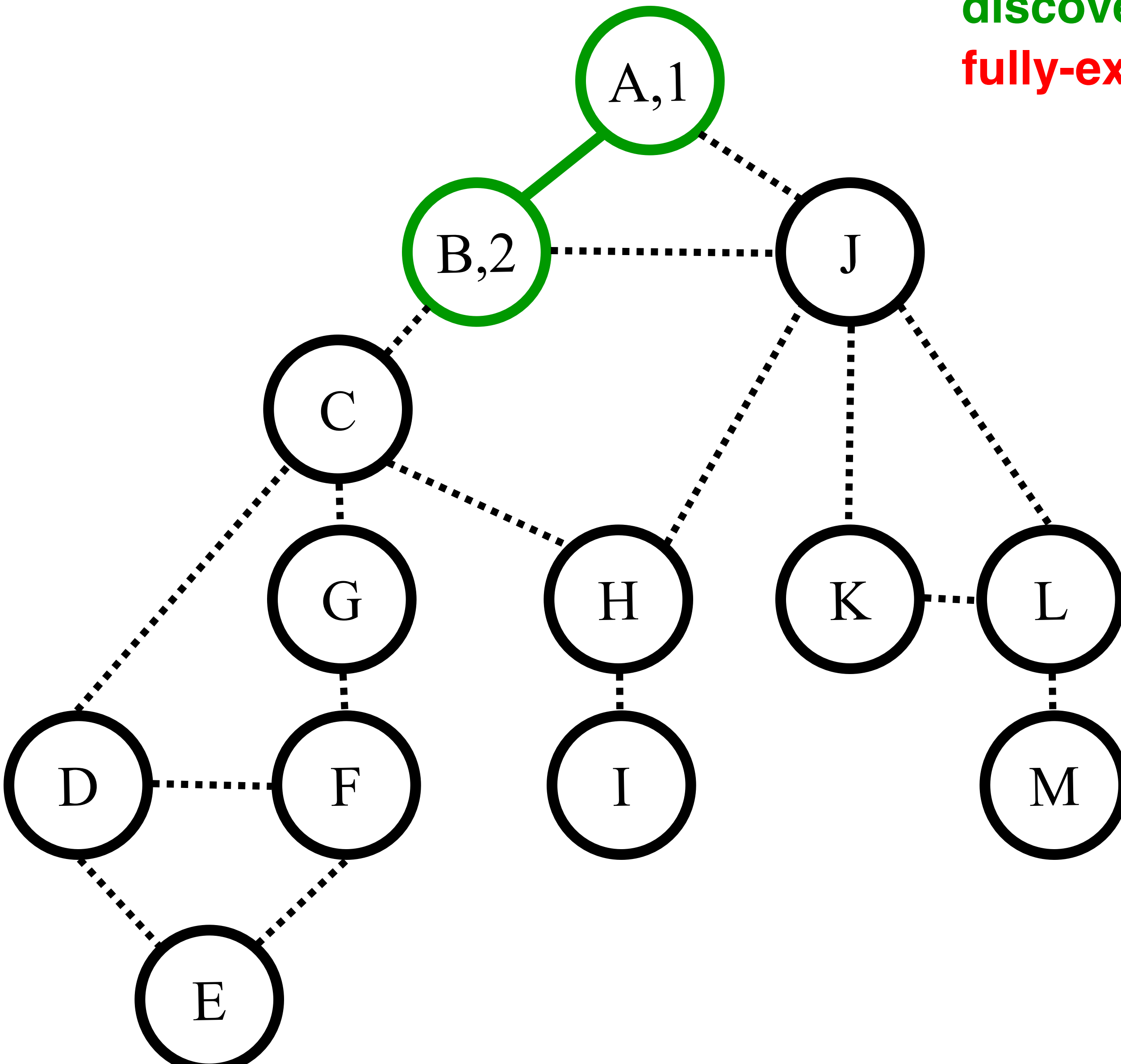
A (B,J)

st[] = {1}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

st[] =
{1,2}

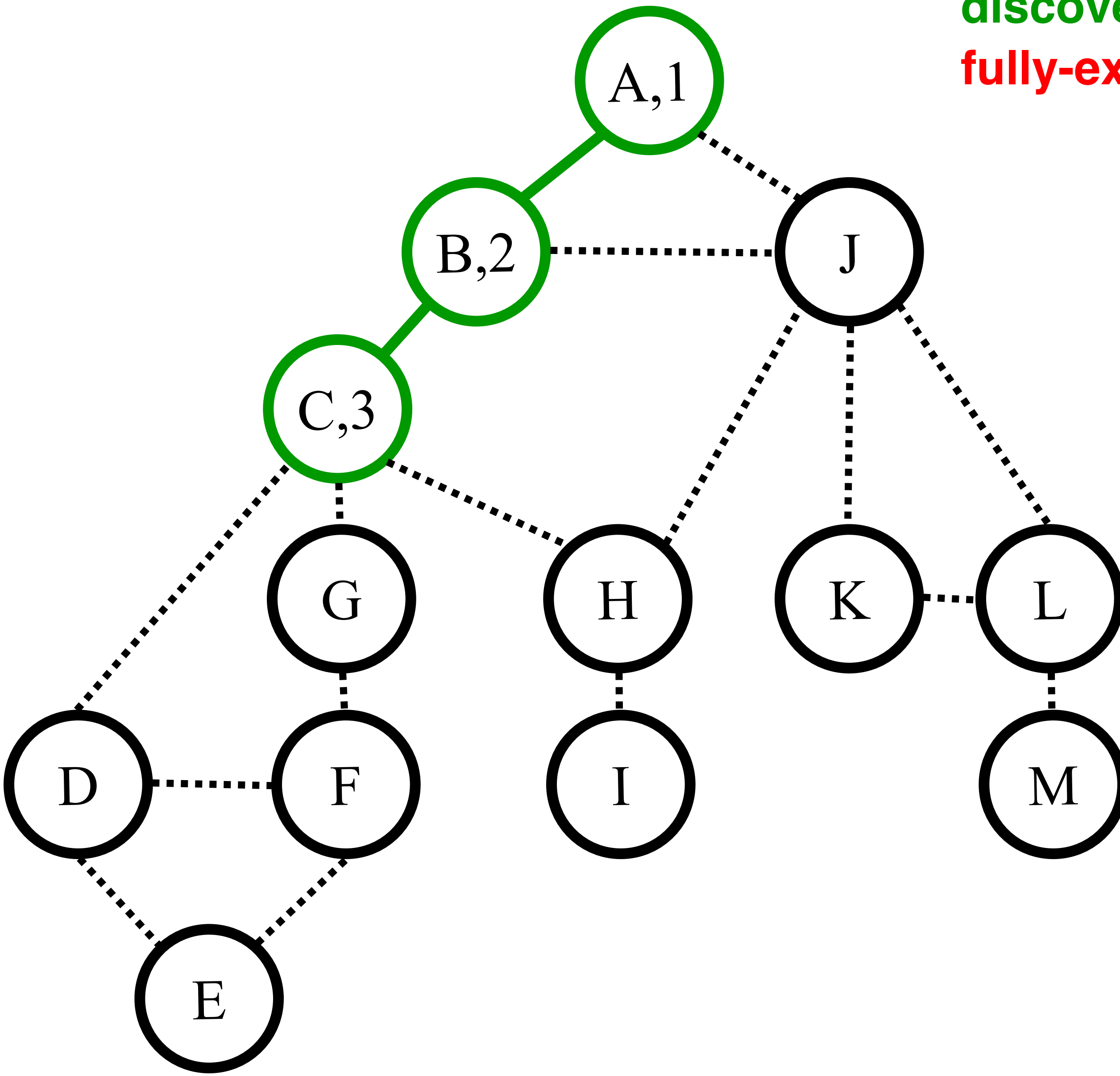
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

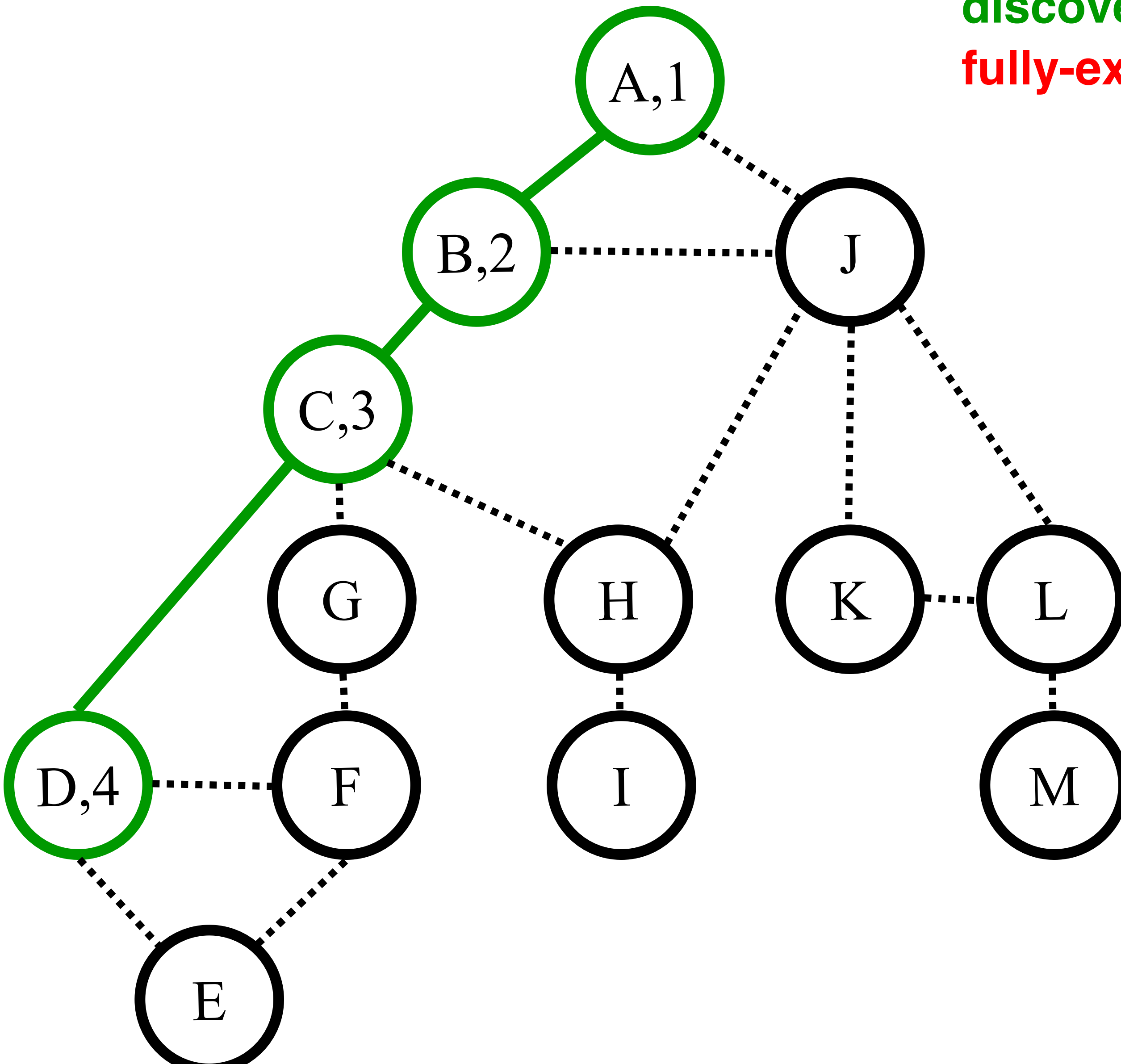
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)

st[] =
{1,2,3}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)

st[] =
{1,2,3,4}

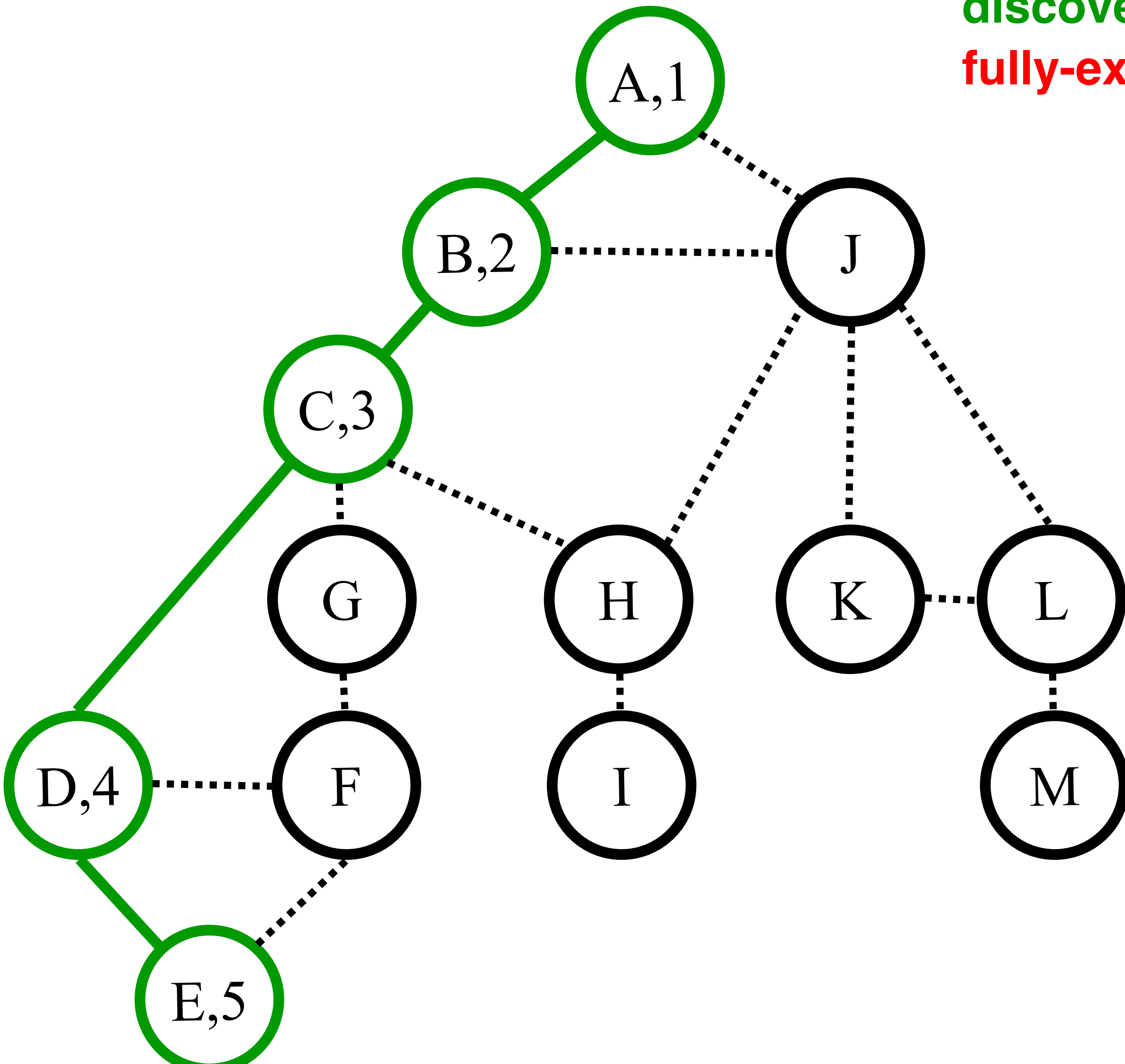
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)
E (D,F)

st[] =
{1,2,3,4,5}

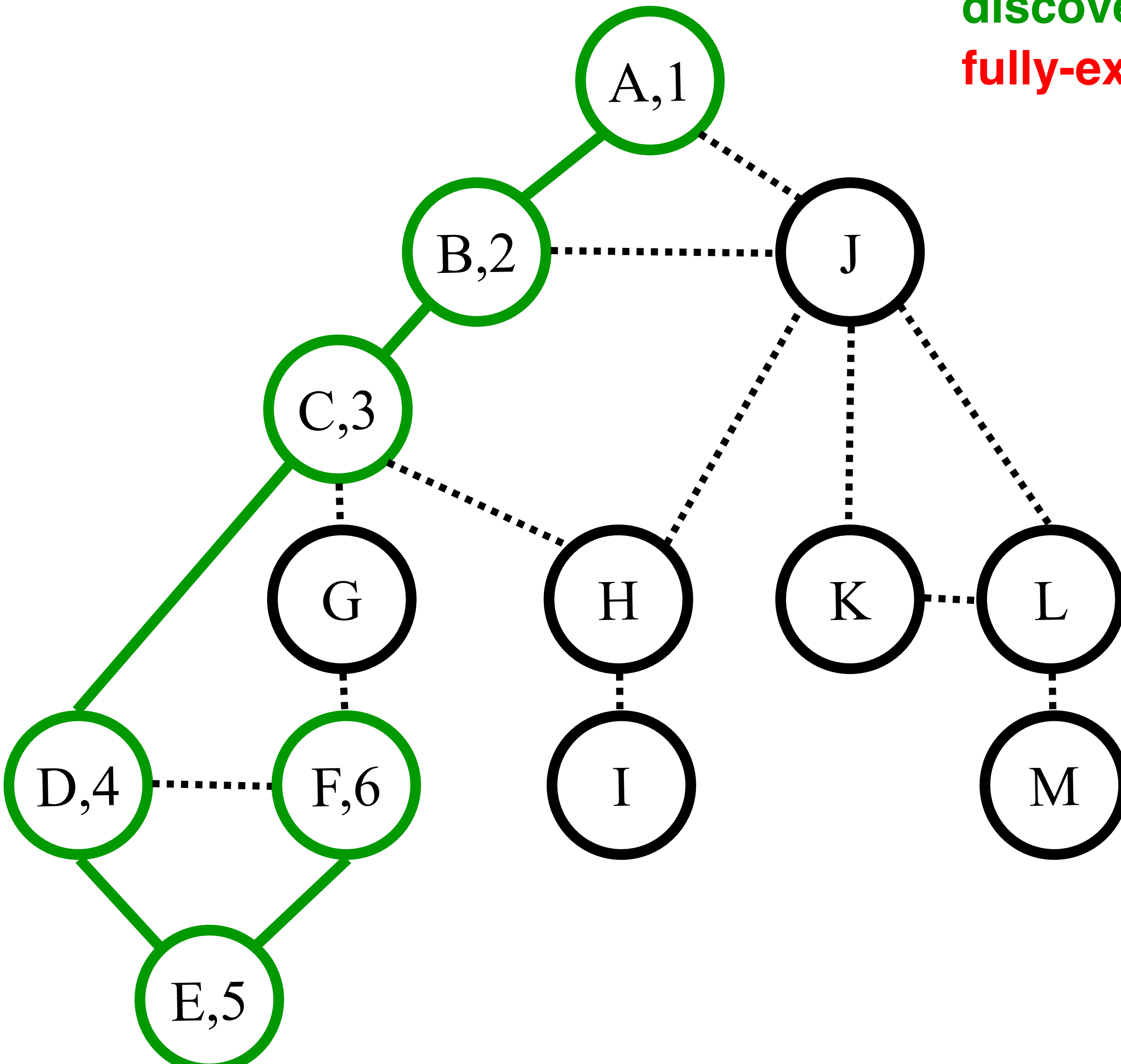
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

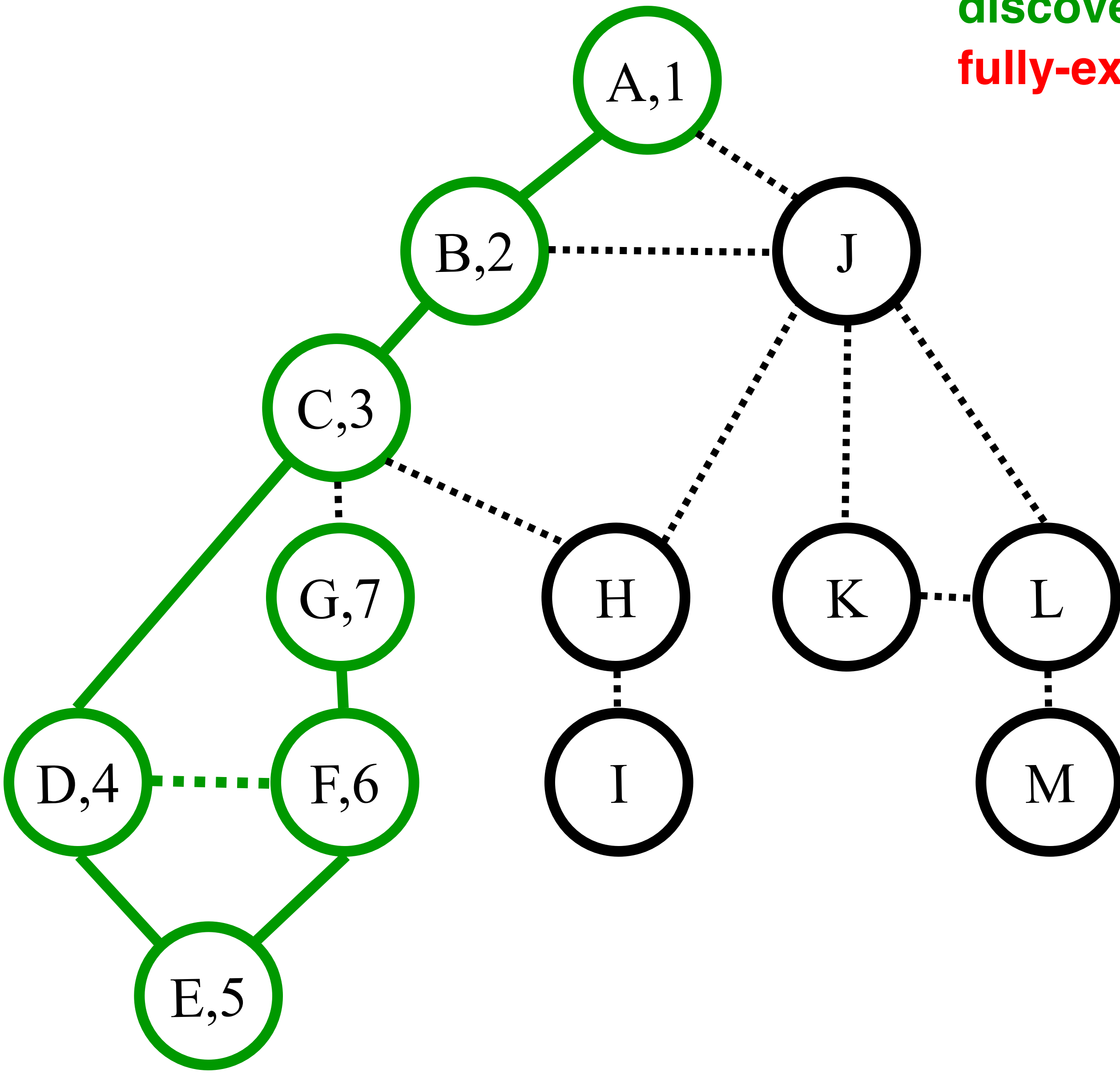
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

st[] =
{1,2,3,4,5,
6}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

st[] =
{1,2,3,4,5,
6,7}

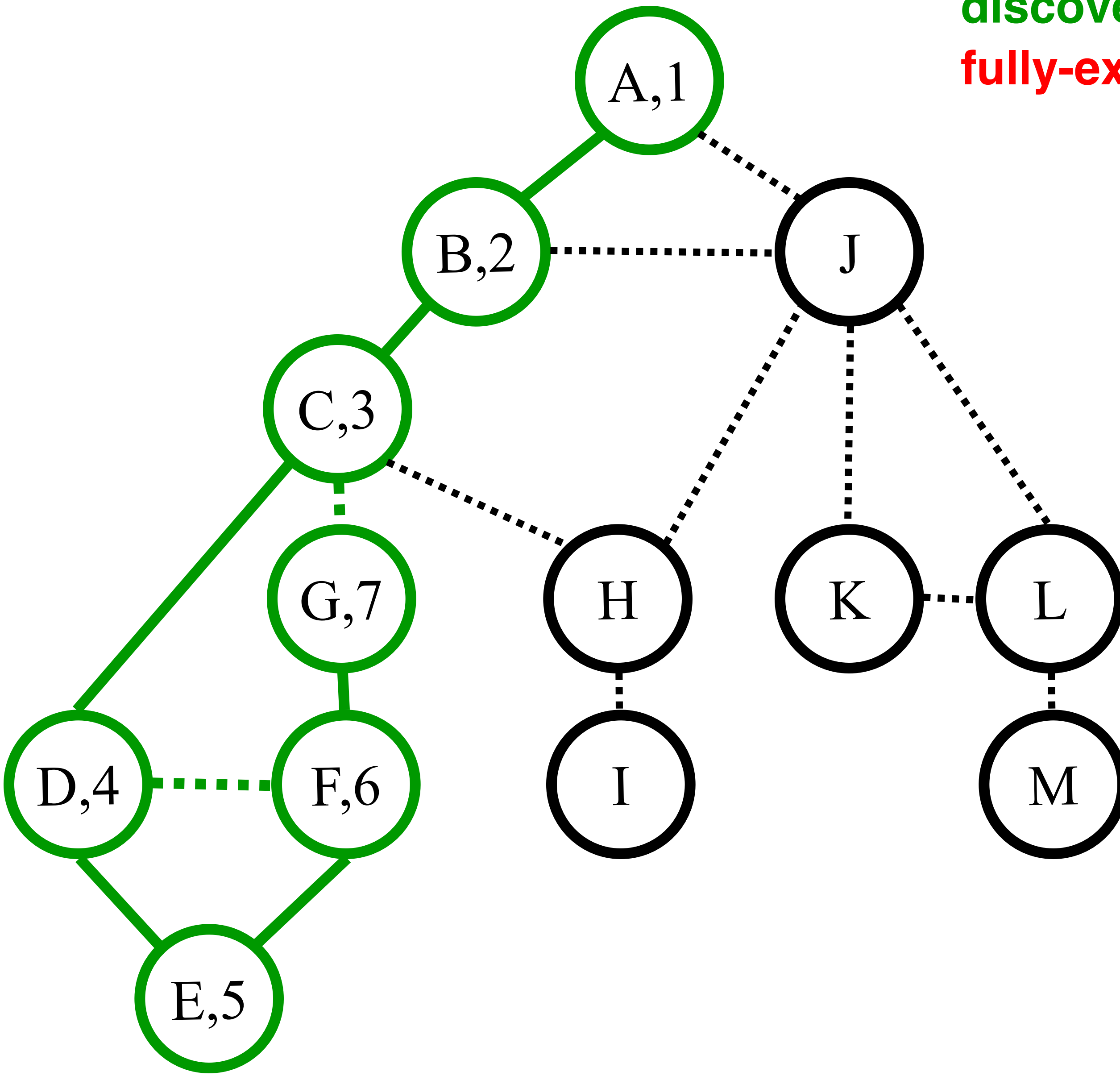
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

st[] =
{1,2,3,4,5,
6,7}

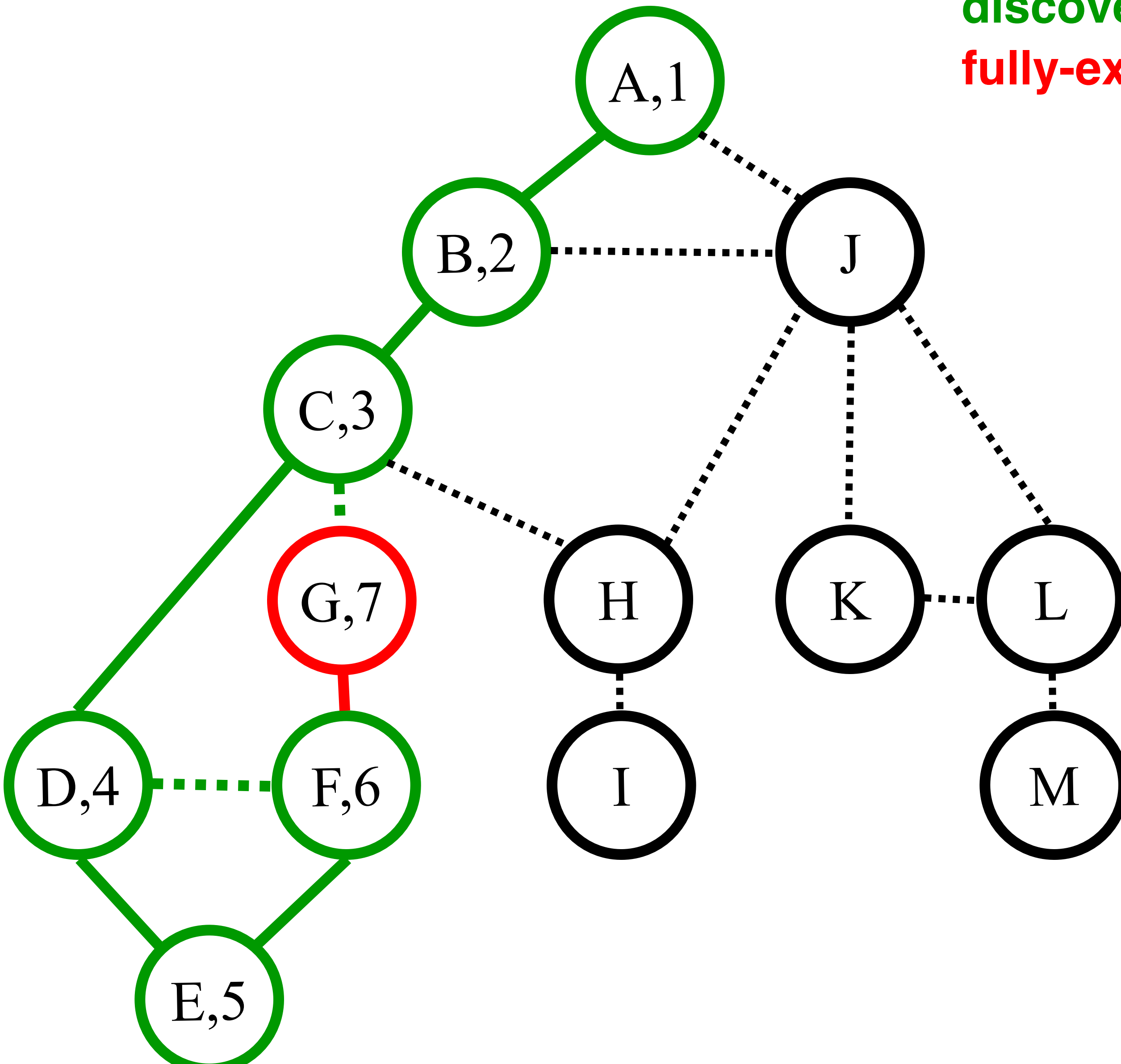
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

st[] =
{1,2,3,4,5,
6}

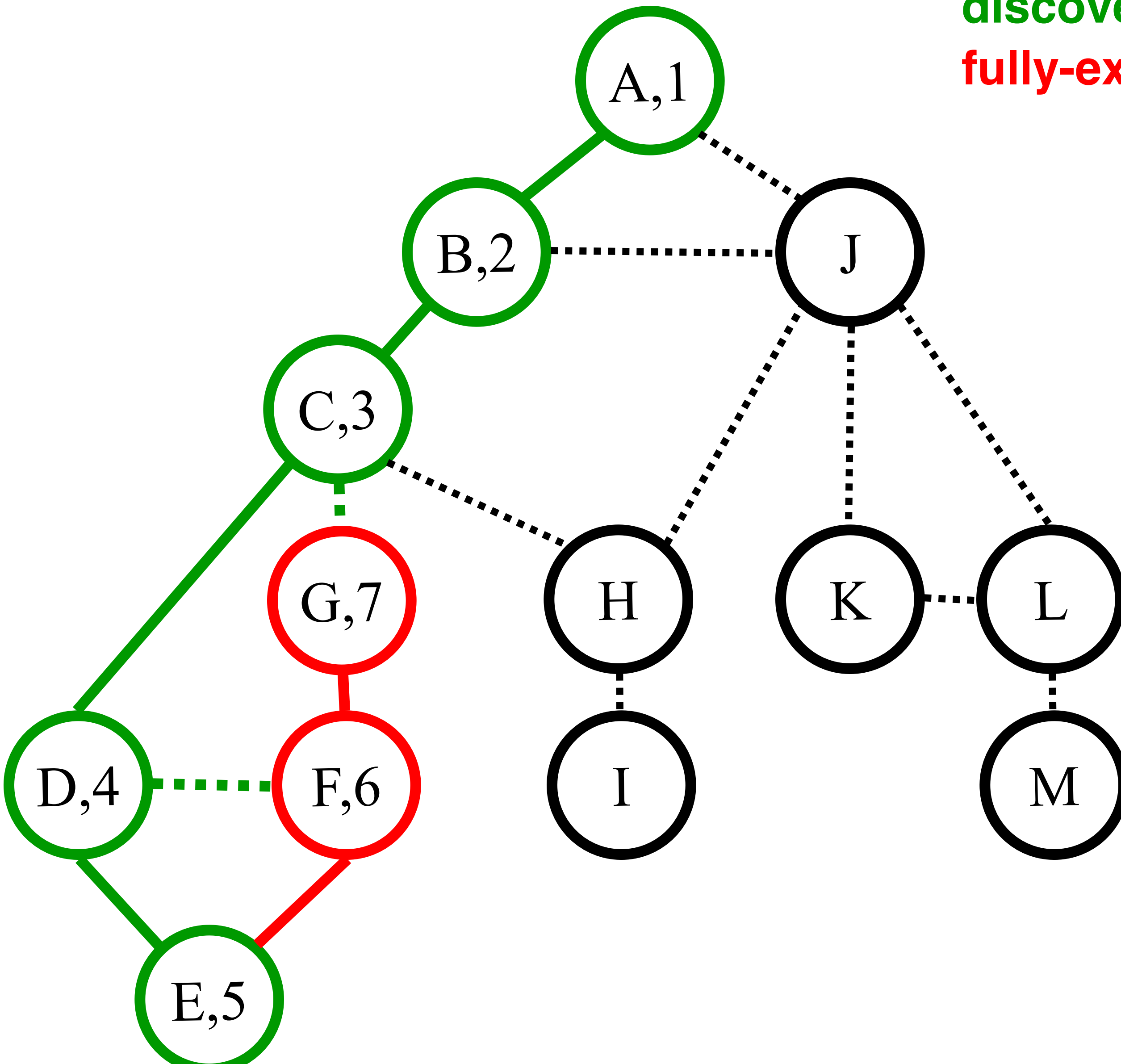
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

st[] =
{1,2,3,4,5}

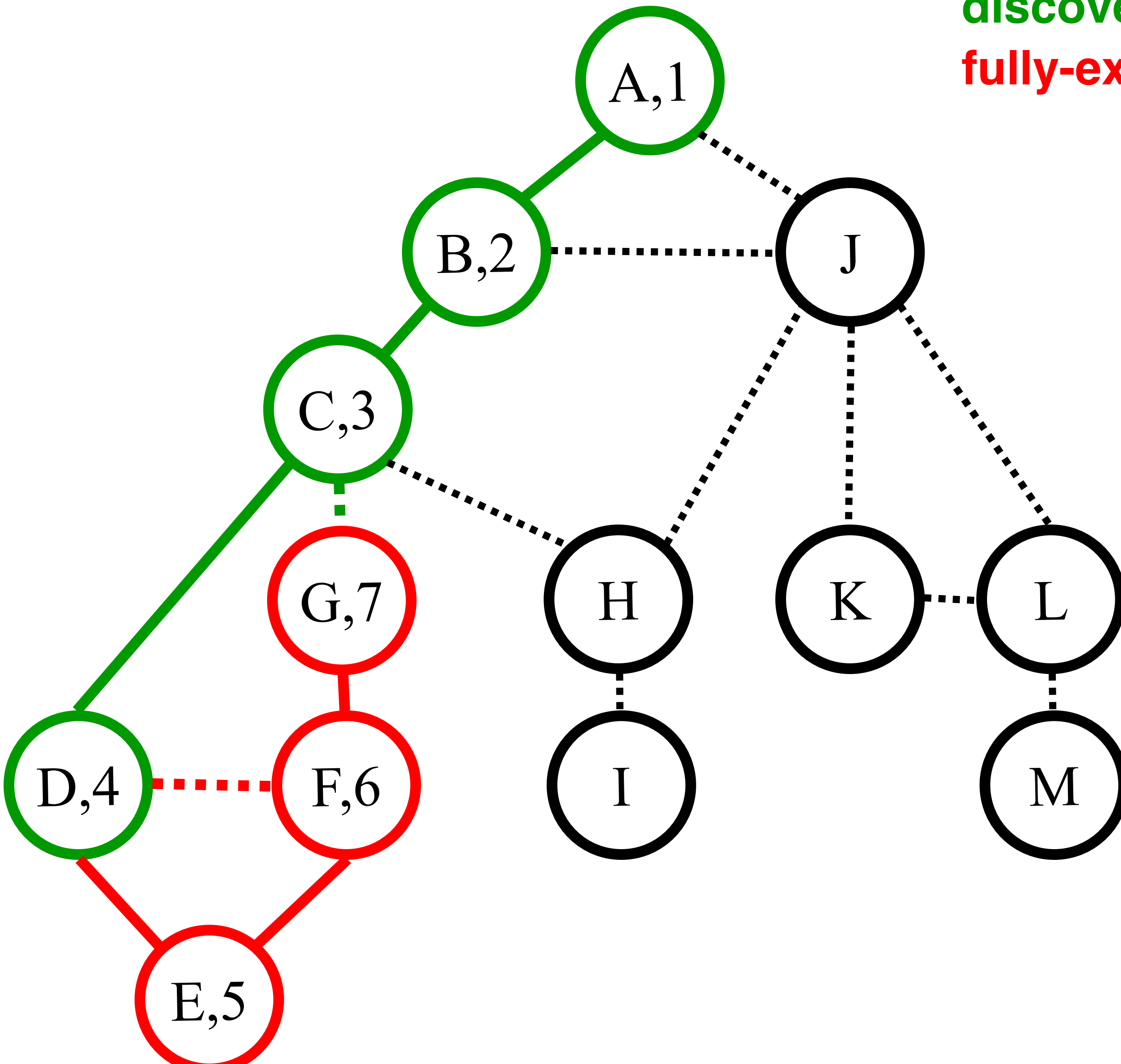
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

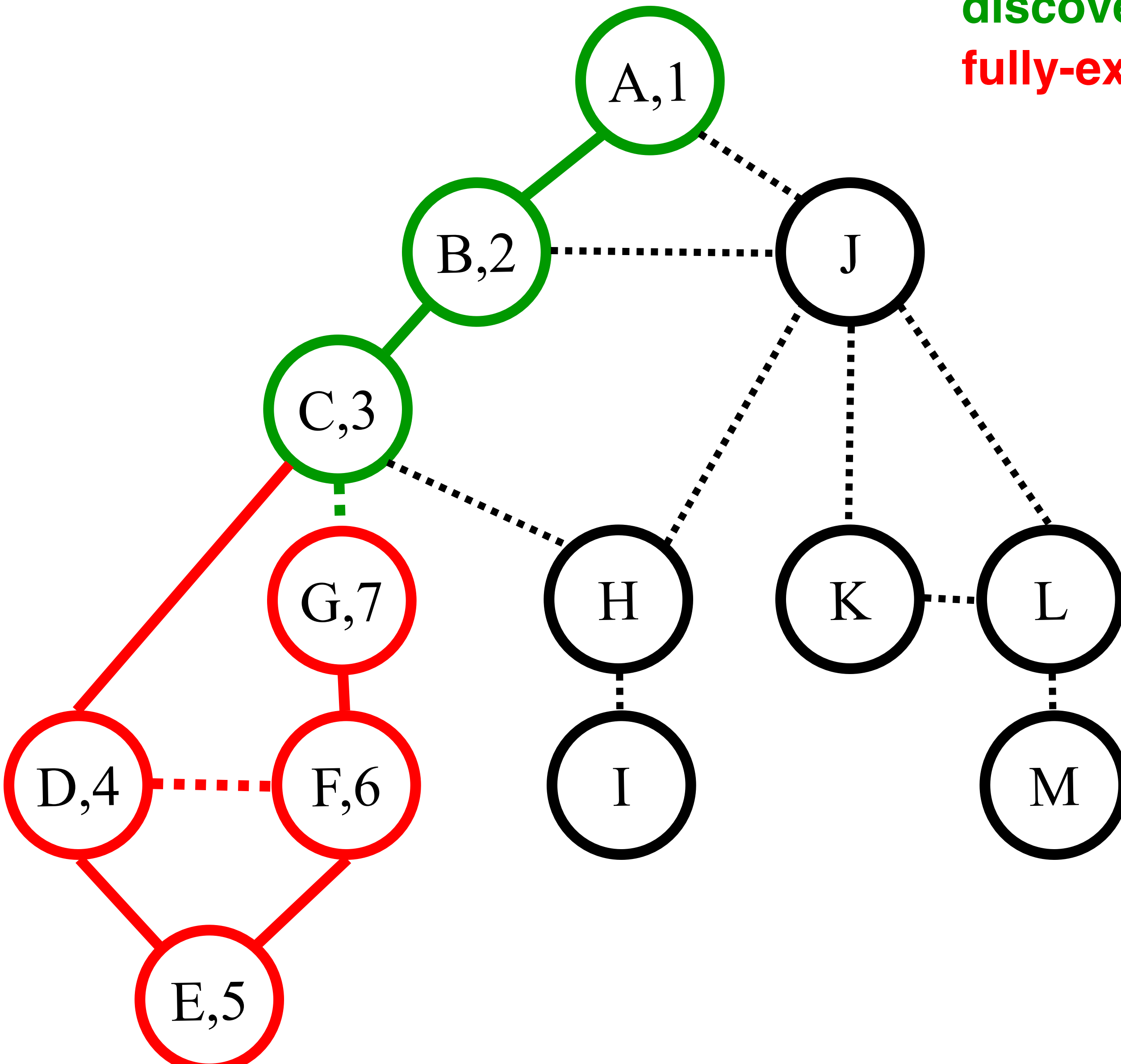
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

st[] =
{1,2,3,4}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

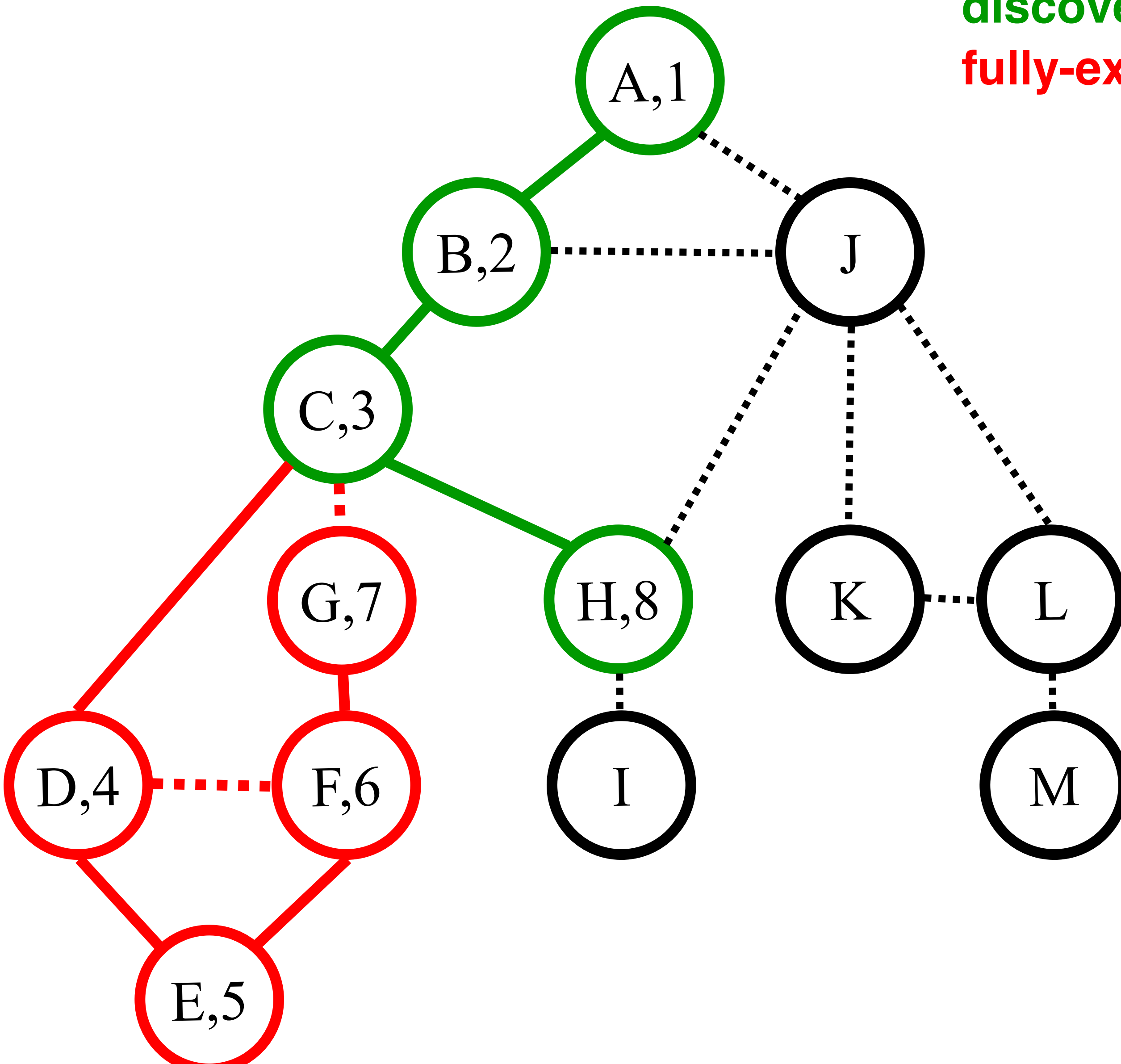
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)

st[] =
{1,2,3}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)

st[] =
{1,2,3,8}

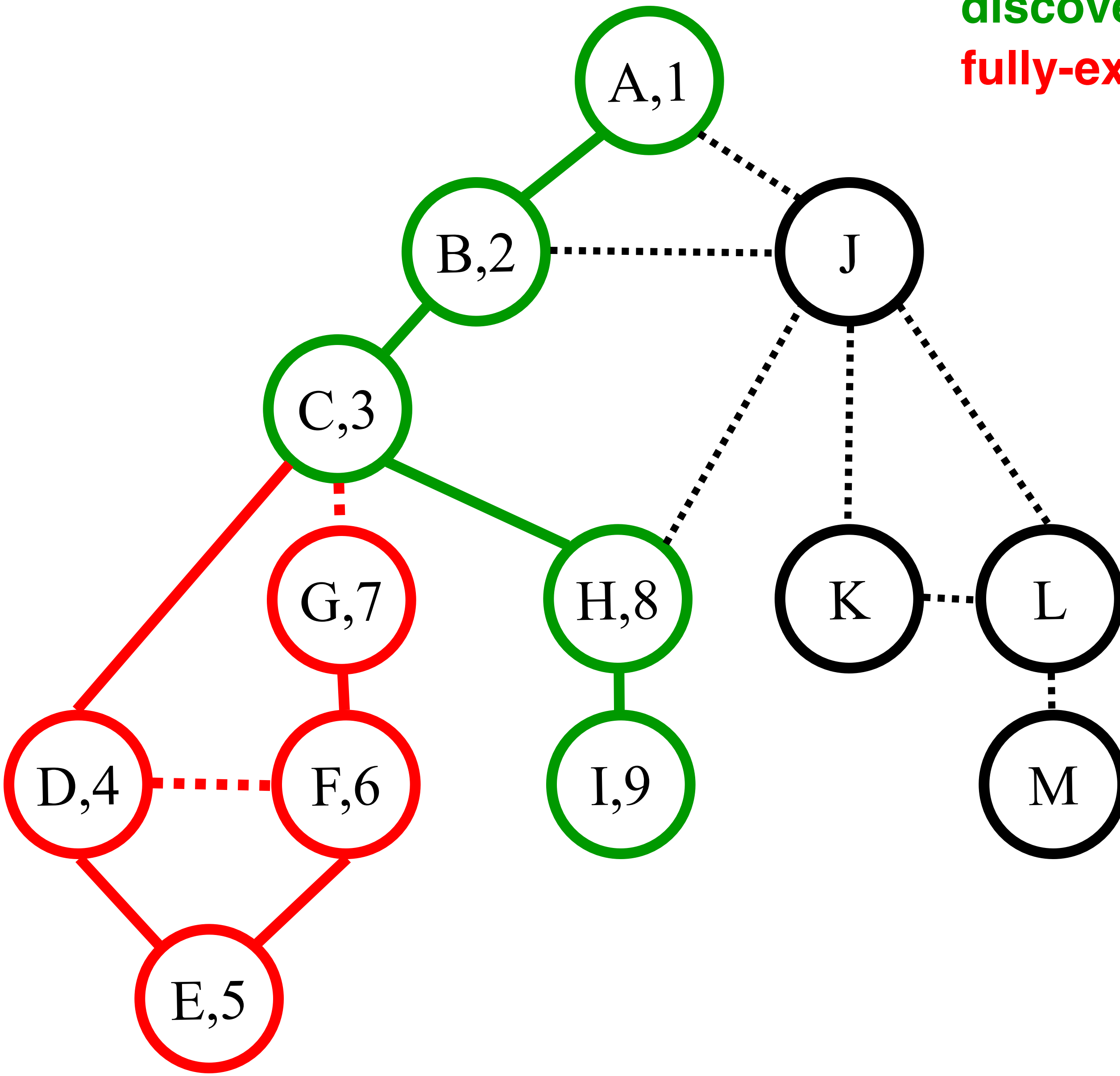
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

st[] =
{1,2,3,8,9}

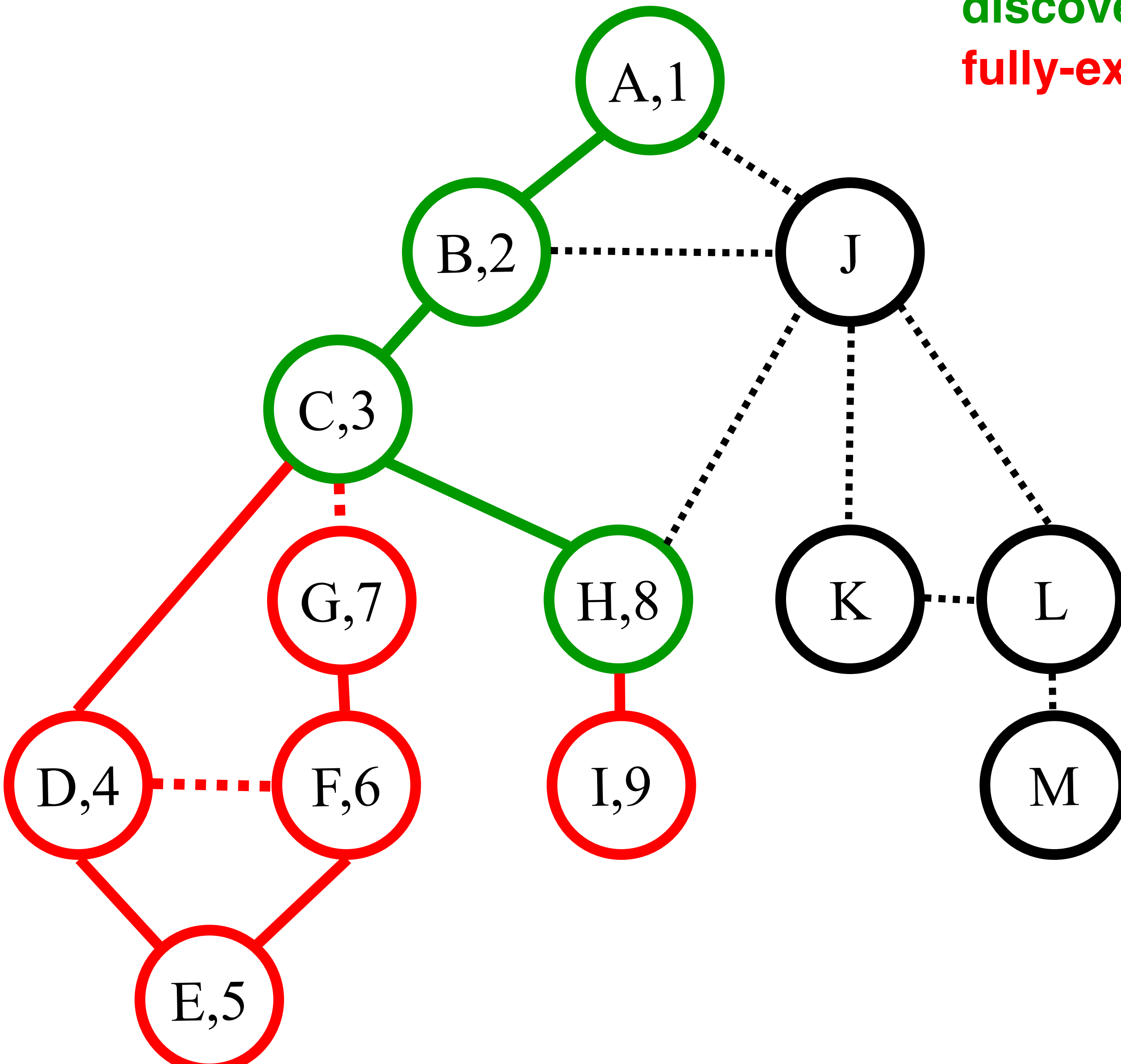
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

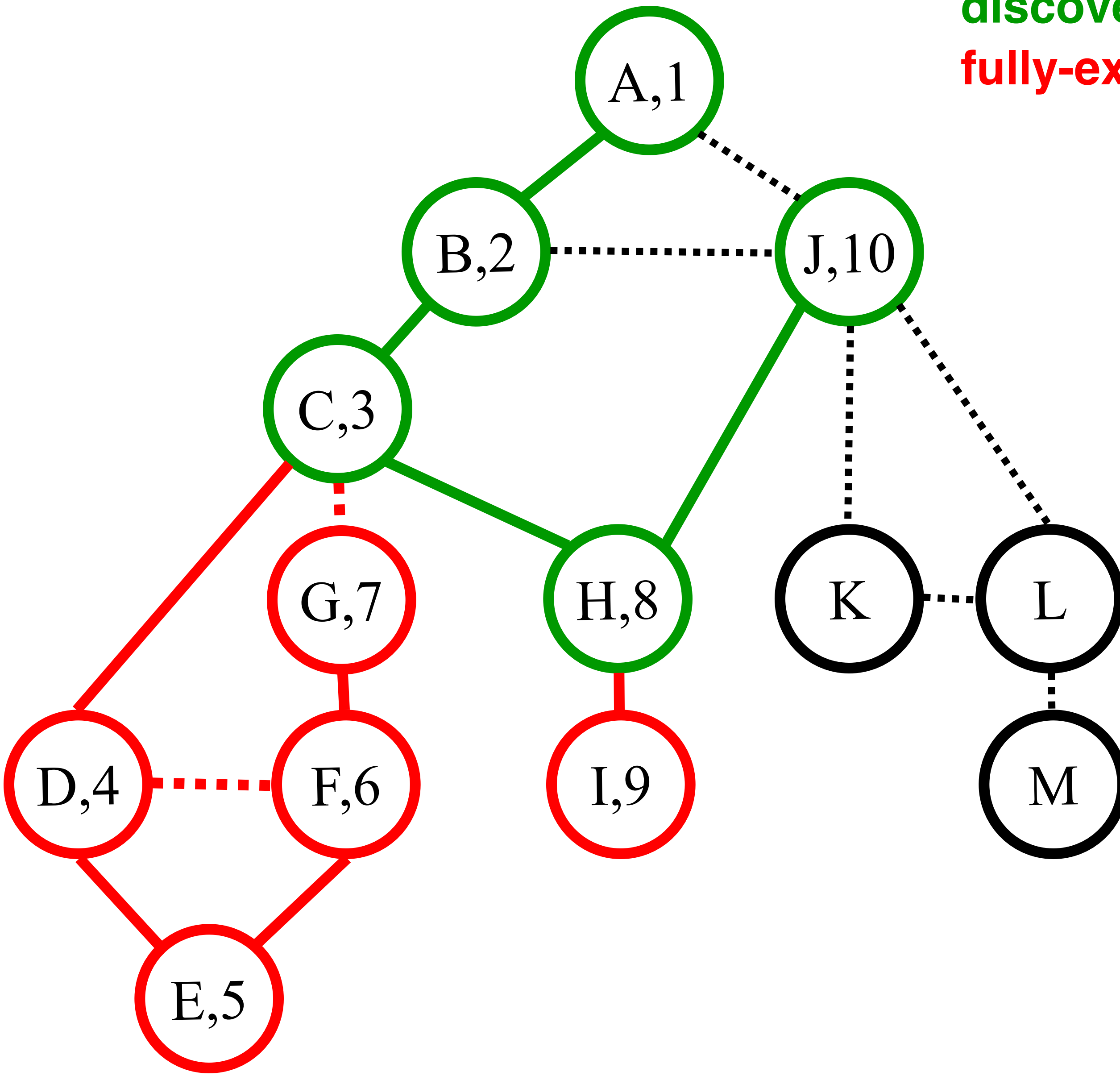
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

st[] = {1,2,3,8}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

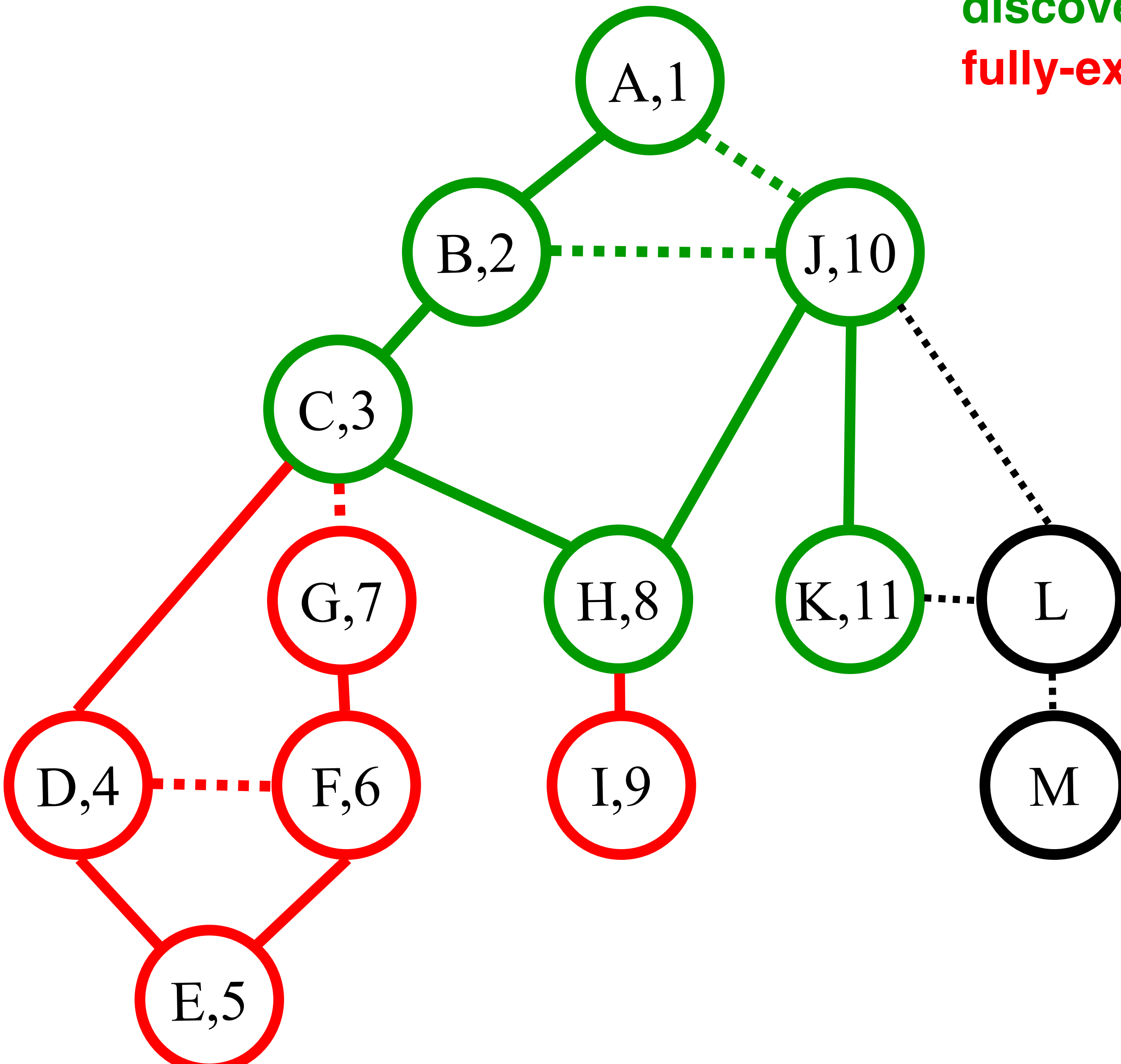
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

st[] = {1,2,3,8,10,11}

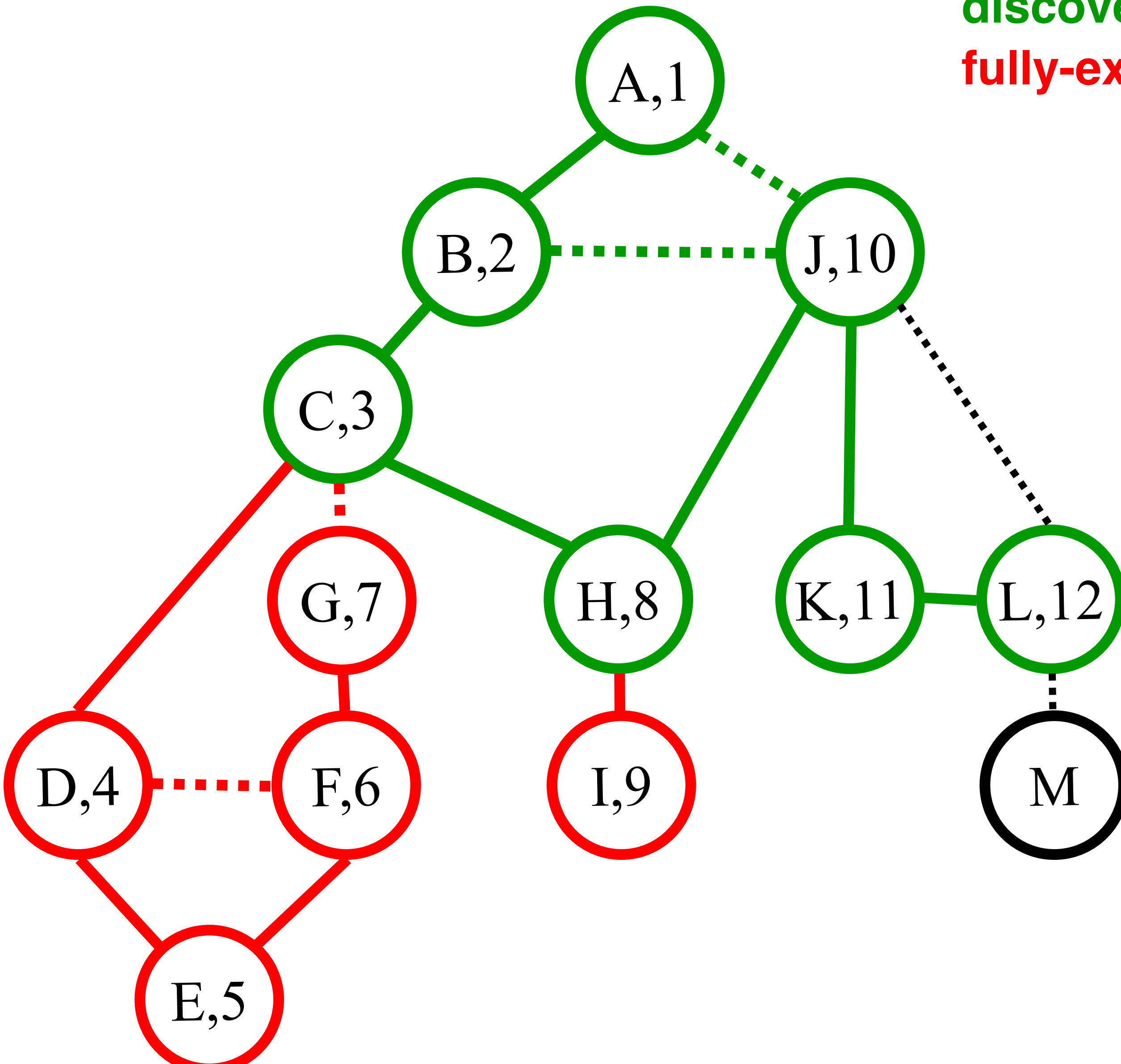
DFS(A)

Color code:

undiscovered

discovered

fully-explored

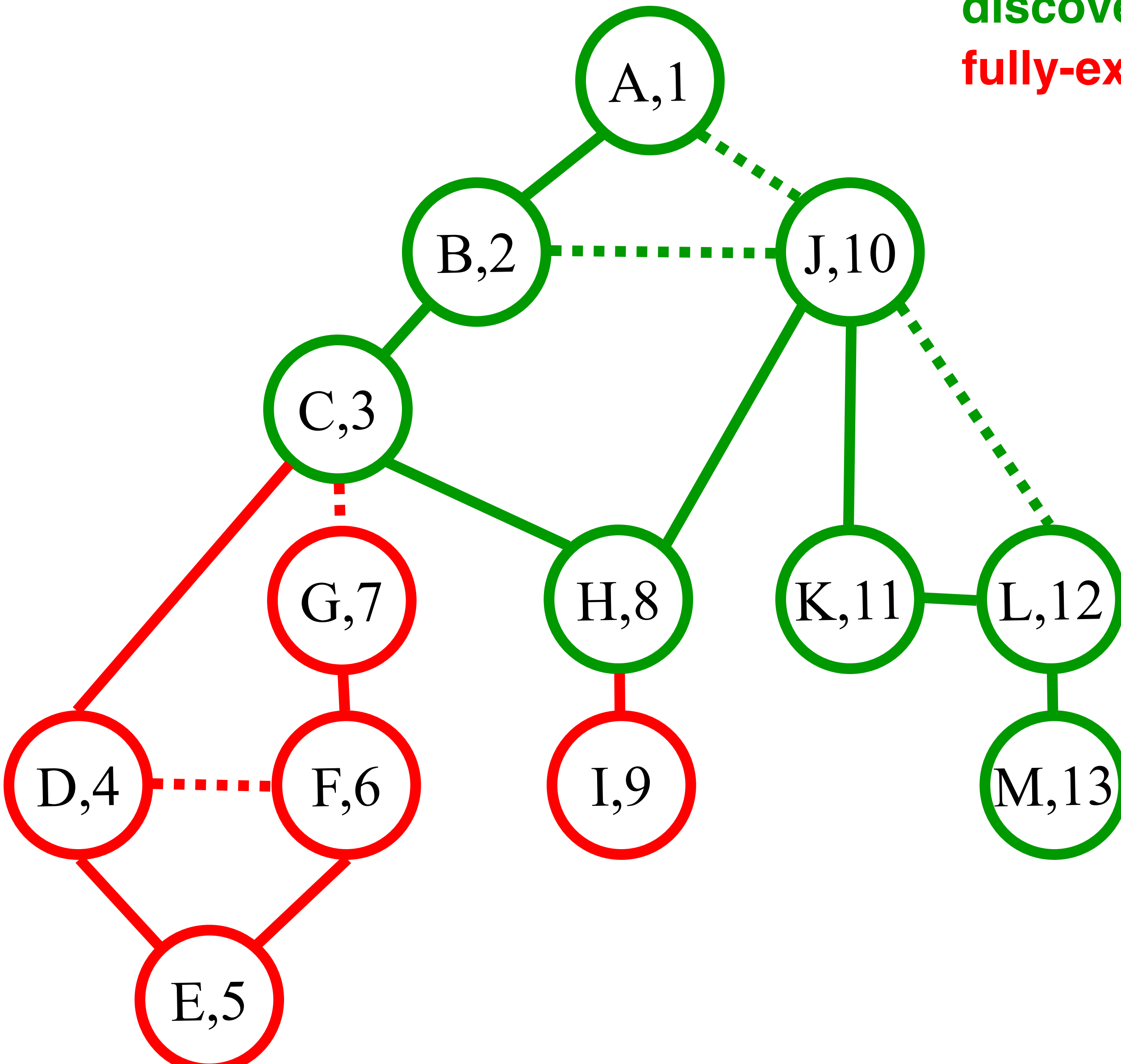


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (J,K,M)

st[] = {1,2,3,8,10,11,12}

DFS(A)



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

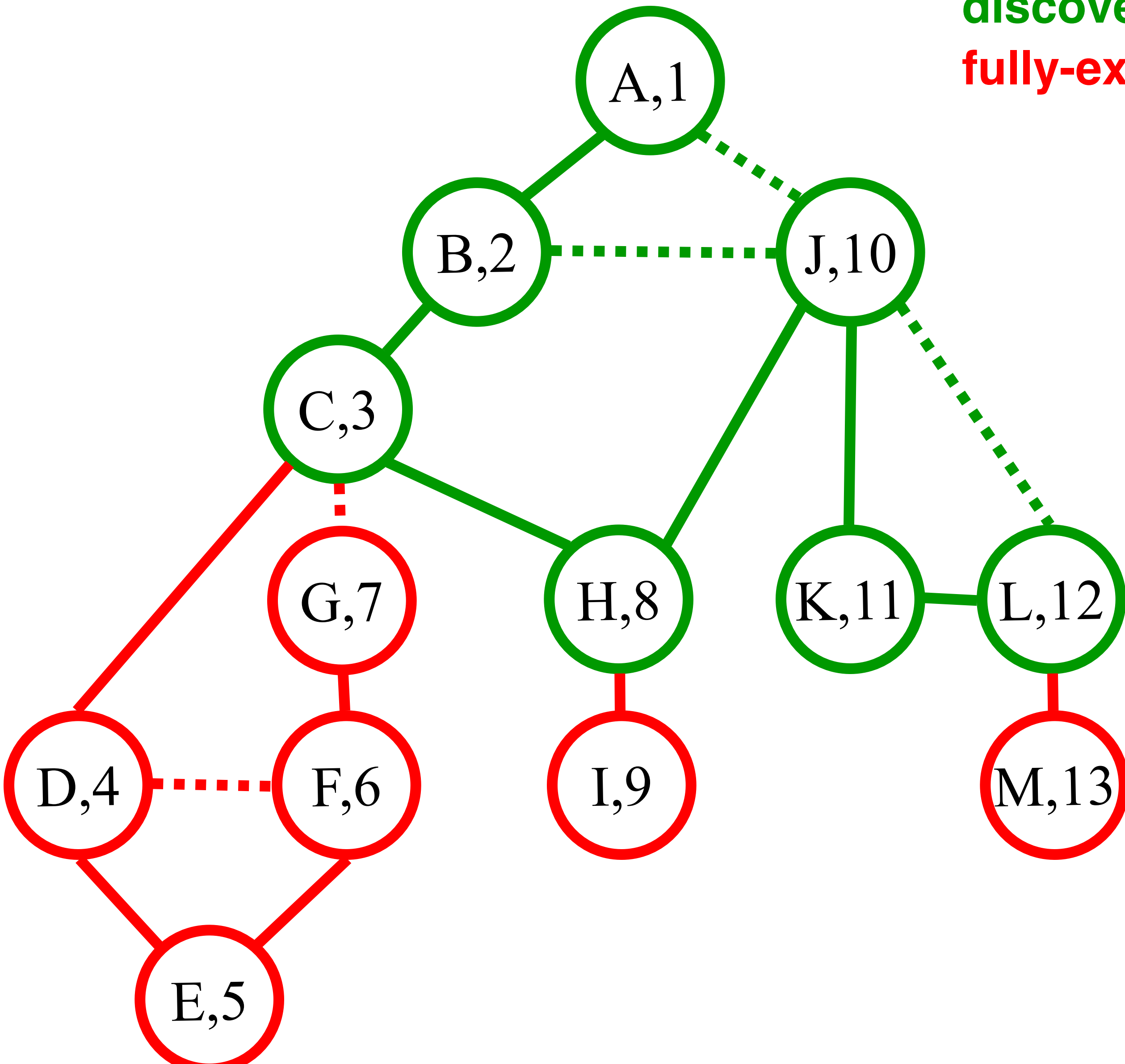
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
M (L)

st[] = {1,2,3,8,10,11,12,13}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

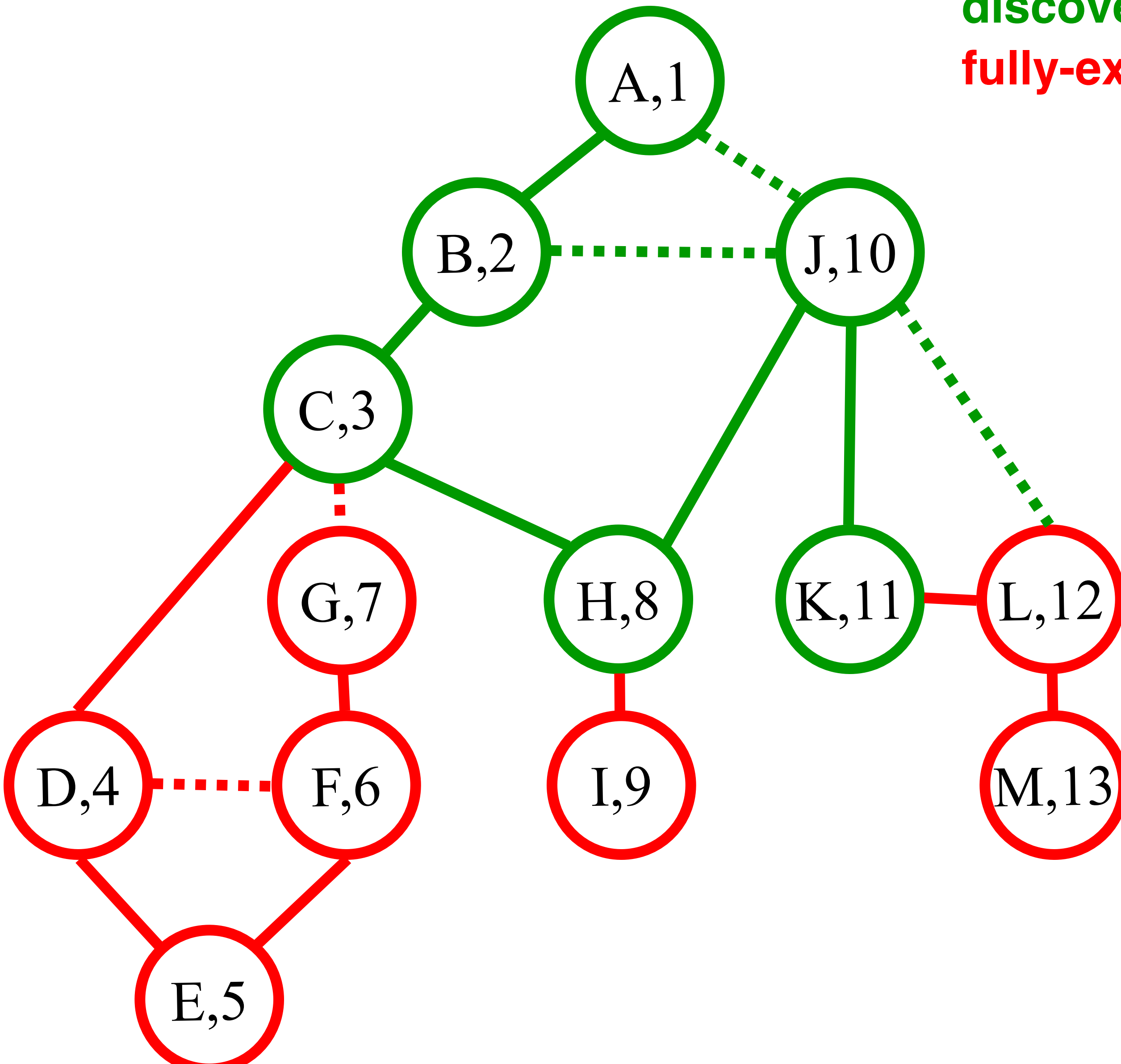
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)

st[] = {1,2,3,8,10,11,12}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

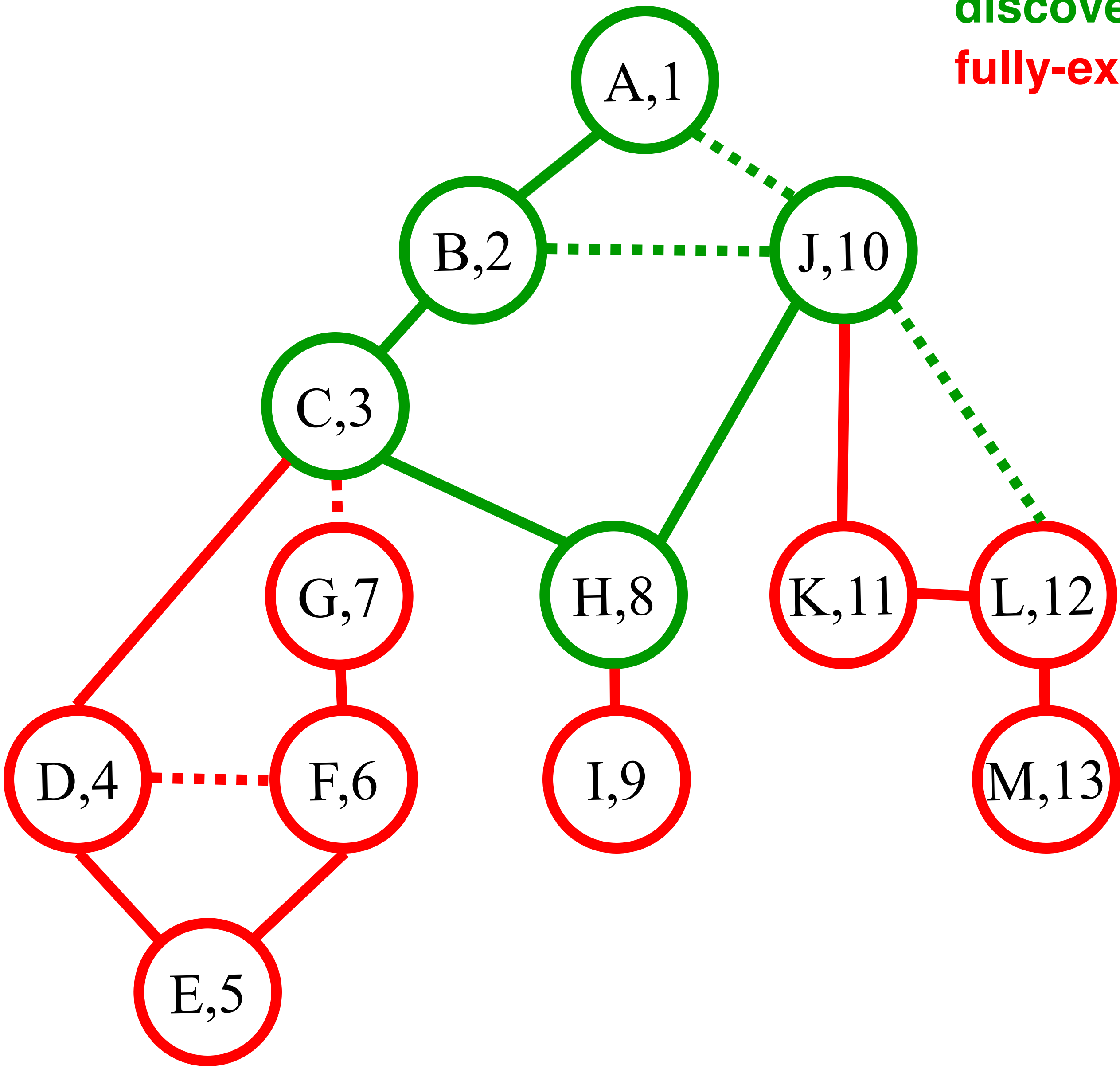
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)

st[] = {1,2,3,8,10,11}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

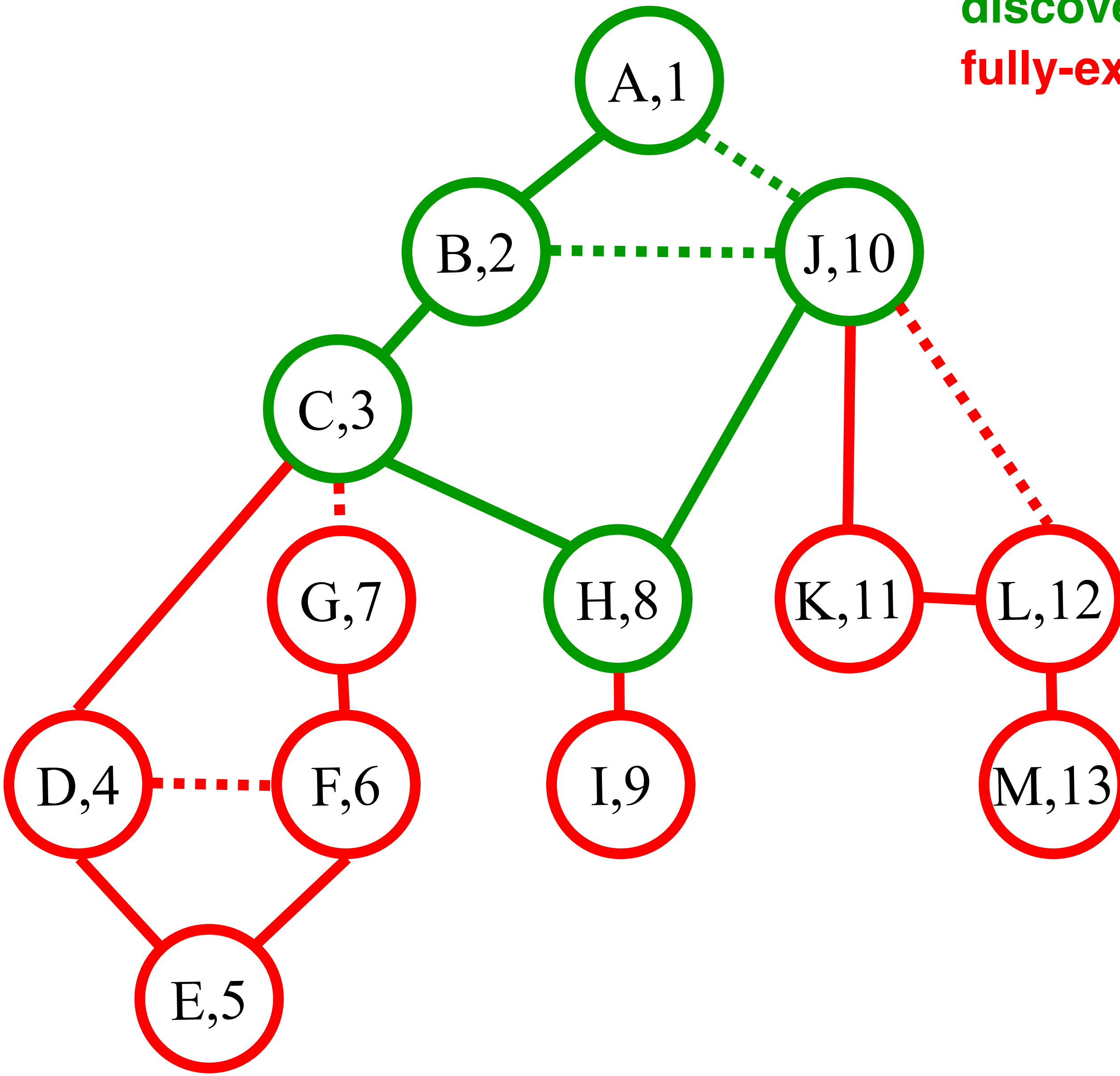
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

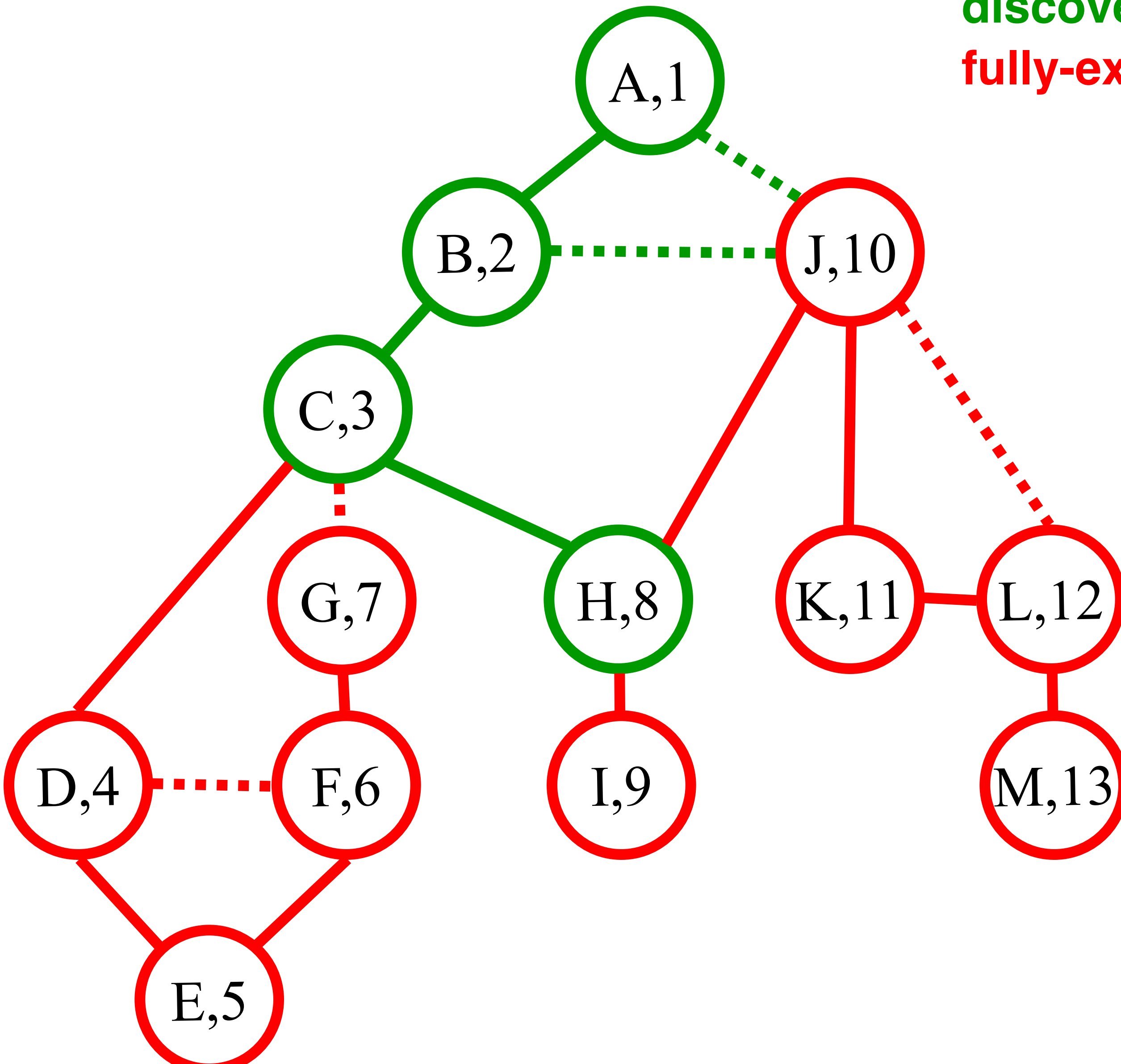
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

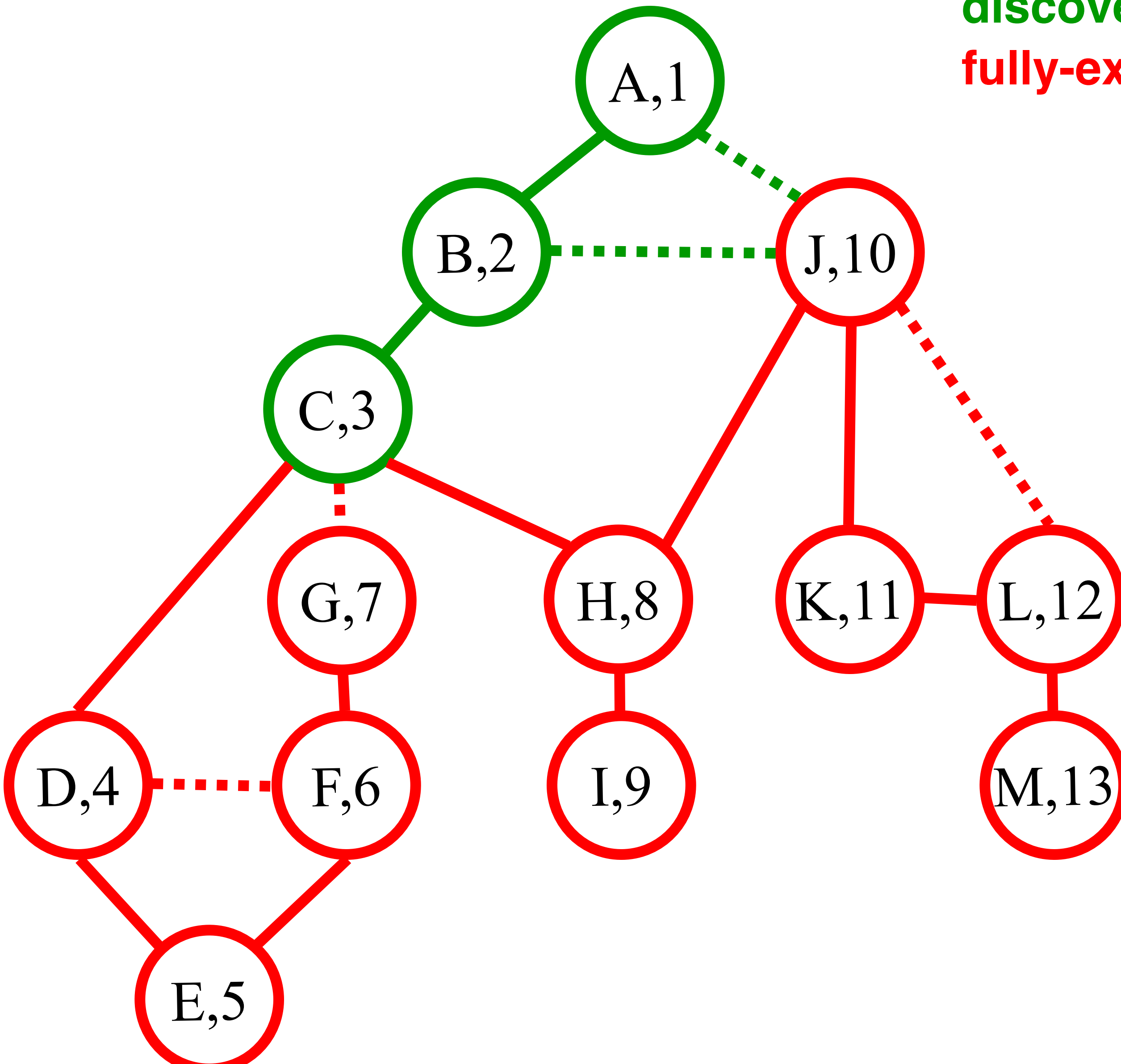
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

st[] =
{1,2,3,8}

DFS(A)

Color code:

undiscovered
discovered
fully-explored

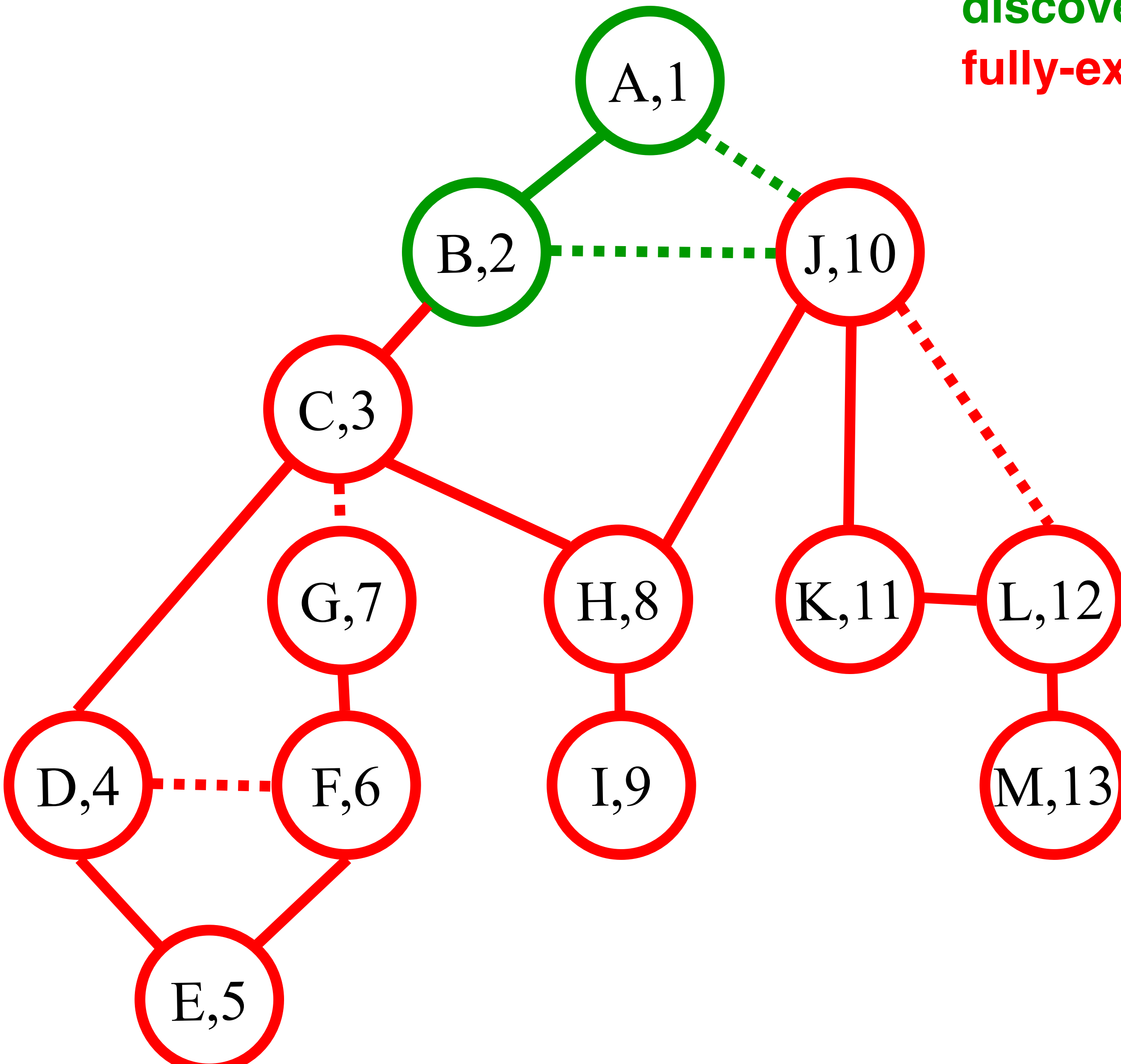


Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)

st[] =
{1,2,3}

DFS(A)



Color code:

undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

st[] =
{1,2}

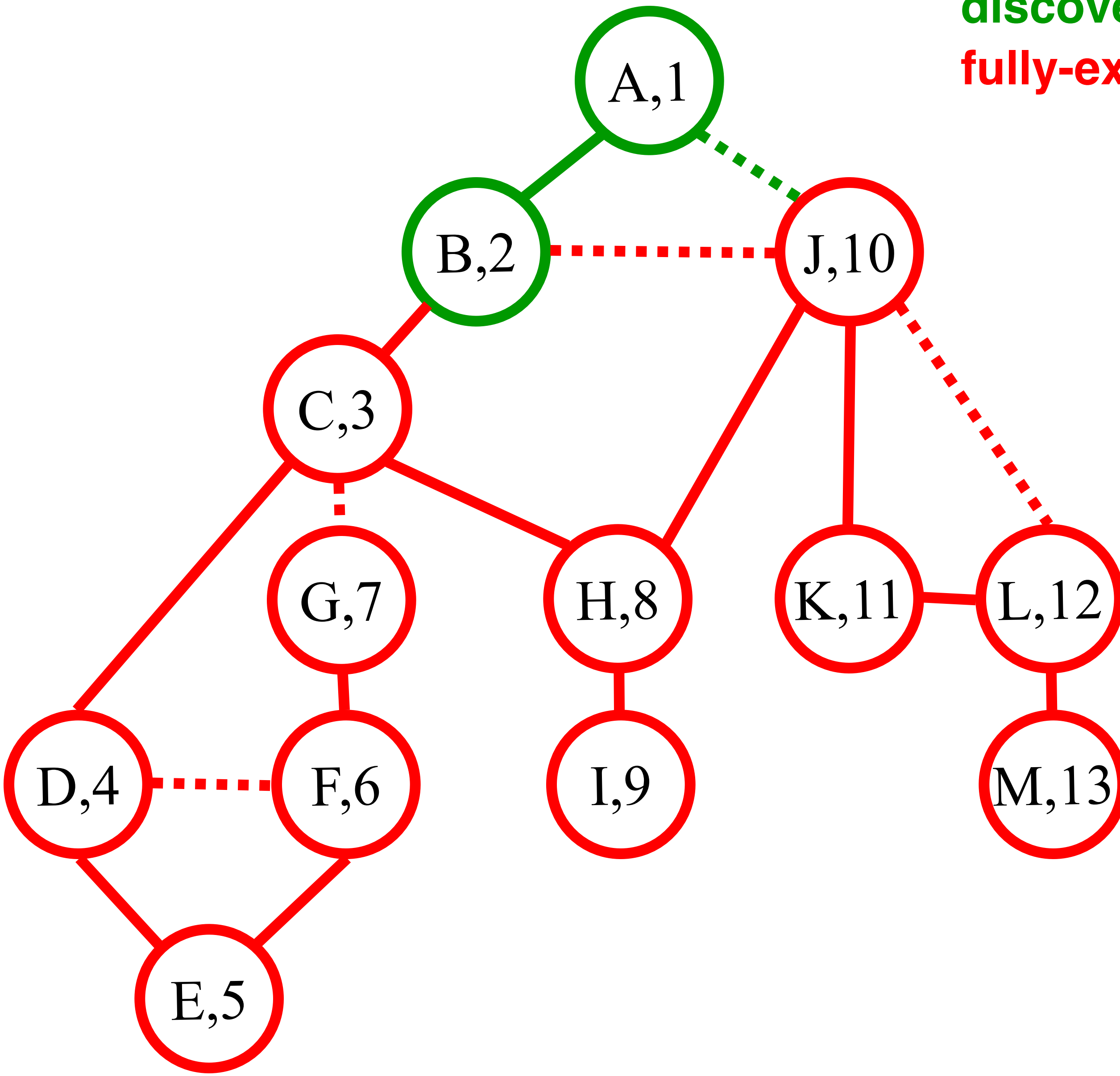
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

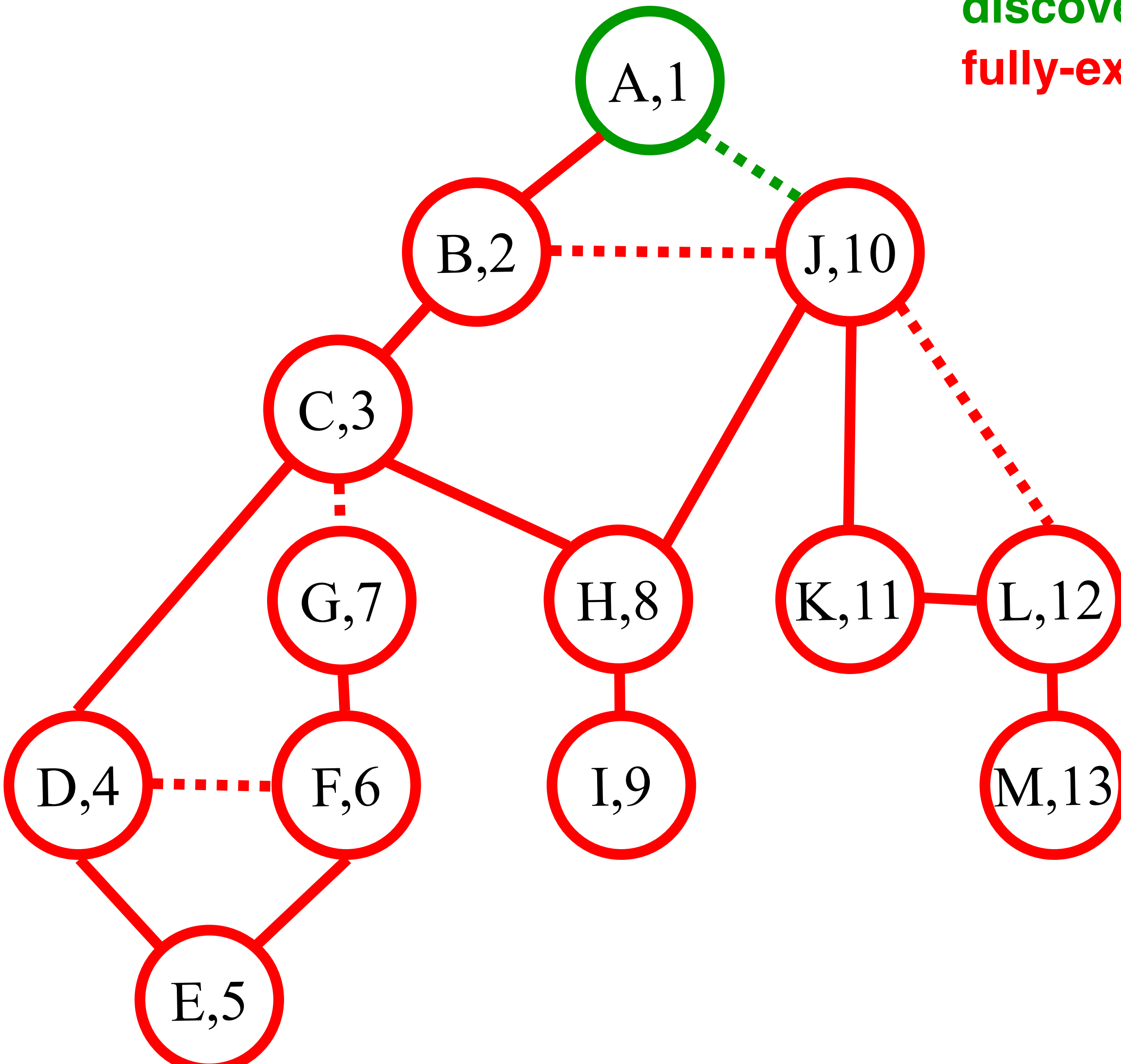
A (B,J)
B (A,C,J)

st[] =
{1,2}

DFS(A)

Color code:

undiscovered
discovered
fully-explored

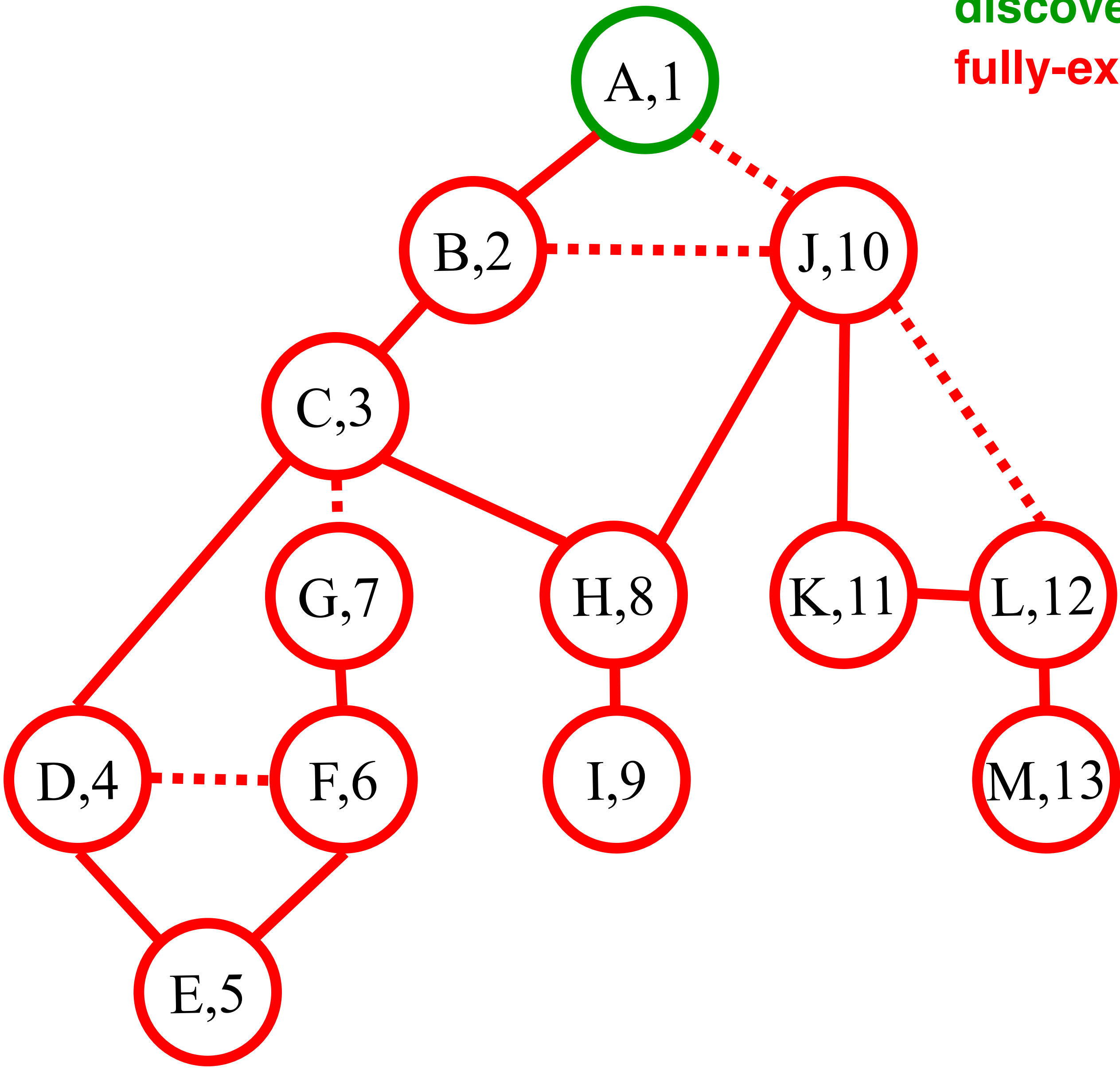


Call Stack:
(Edge list)

A (~~B~~,J)

st[] =
{1}

DFS(A)



Color code:

undiscovered
discovered
fully-explored

Call Stack:
(Edge list)

A (B,J)

st[] =
{1}

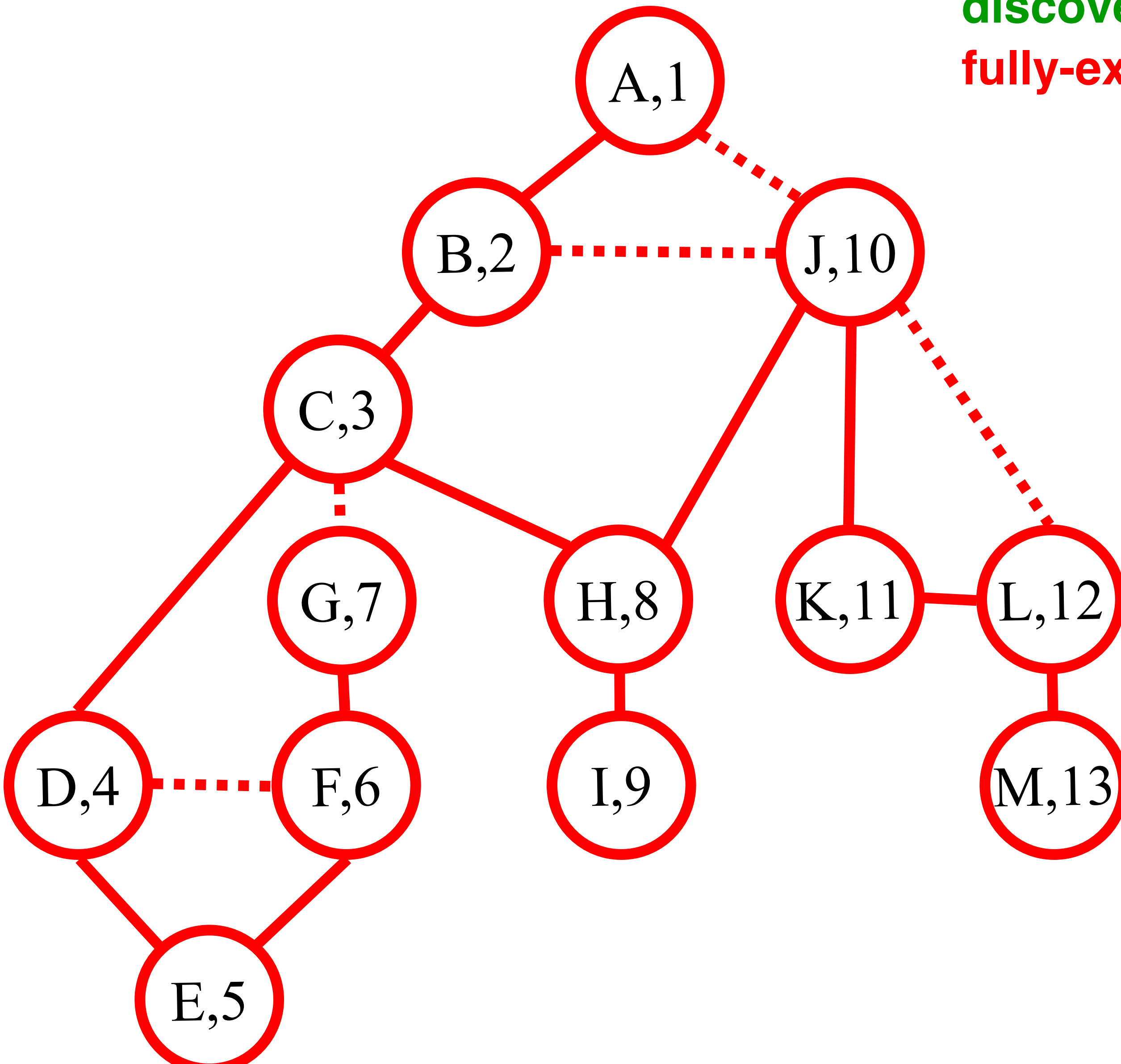
DFS(A)

Color code:

undiscovered

discovered

fully-explored

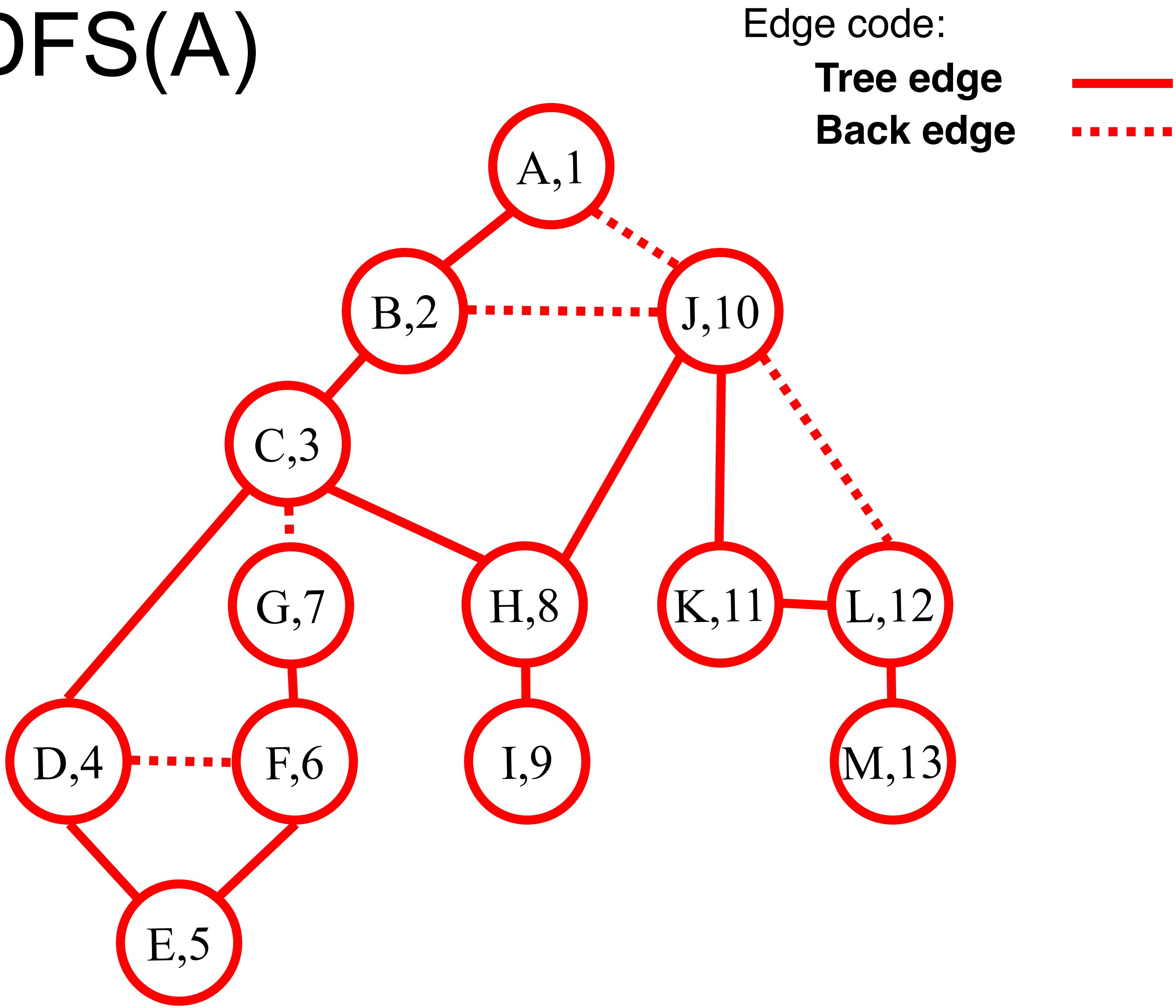


Call Stack:
(Edge list)

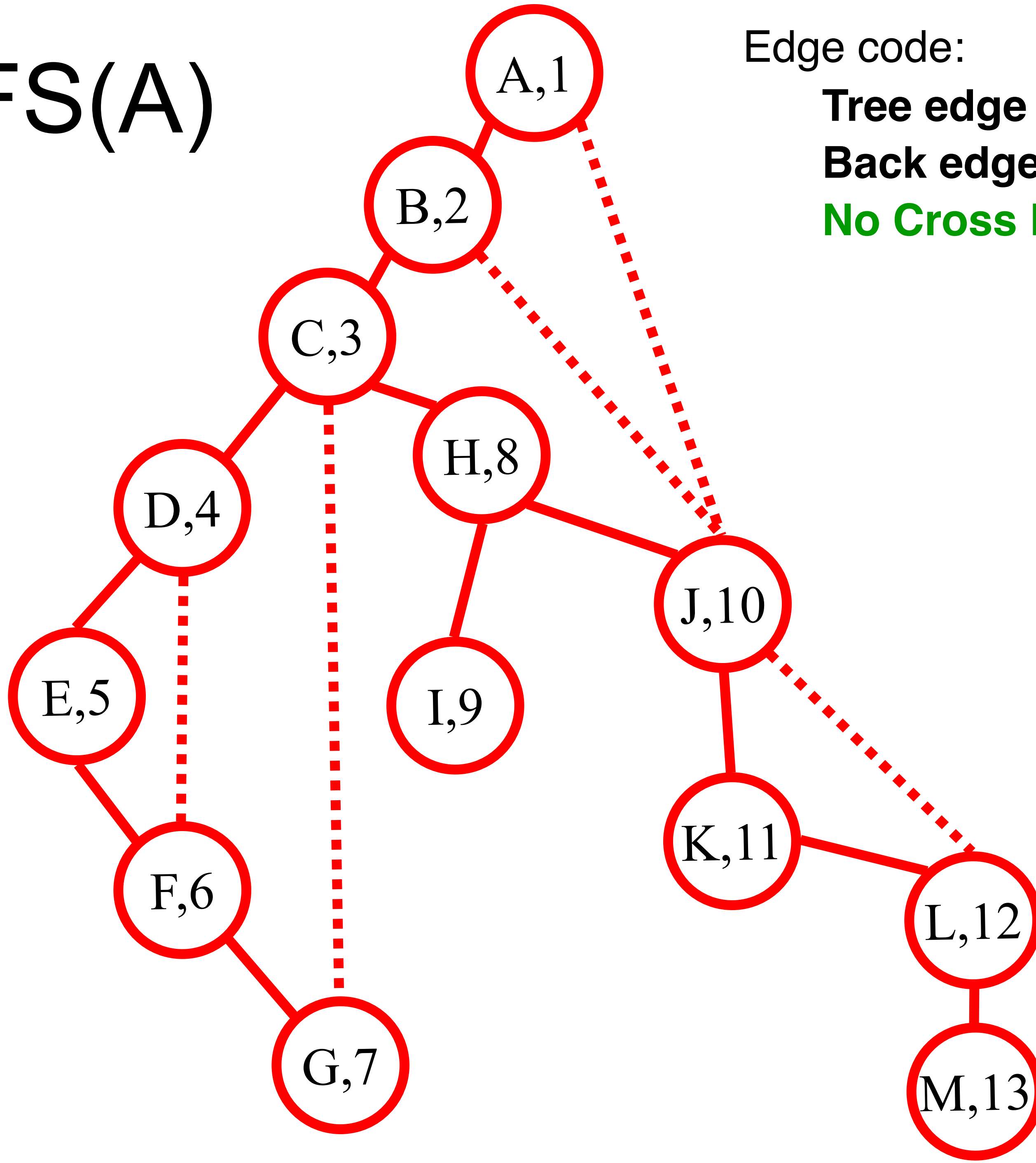
TA-DA!!

st[] = {}

DFS(A)


Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5

DFS(A)



Edge code:

Tree edge

Back edge

No Cross Edges!

Properties of (undirected) DFS

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x

  So, we can use DFS to find connected components

- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

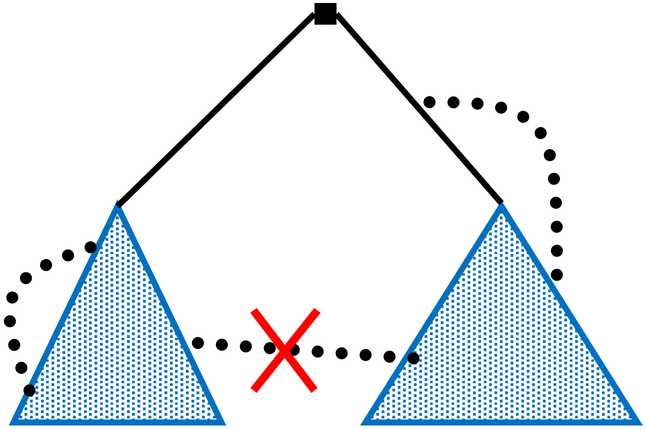
Unlike the BFS tree:

- The DFS spanning tree isn't minimum depth
- Its levels don't reflect min distance from the root
- Non-tree edges never join vertices on the same or adjacent levels

Non-Tree Edges in DFS

All non-tree edges join a vertex and one of its descendants/ancestors in the DFS tree

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor



Non-Tree Edges in DFS

Obs: During DFS(x) every vertex marked visited is a descendant of x in the DFS tree

Lemma: For every edge $\{x, y\}$, if $\{x, y\}$ is not in DFS tree, then one of x or y is an ancestor of the other in the tree.

Proof:

One of x or y is visited first, suppose WLOG that x is visited first and therefore DFS(x) was called before DFS(y)

Since $\{x, y\}$ is not in DFS tree, y was fully-explored when the edge {x,y} was examined during DFS(x)

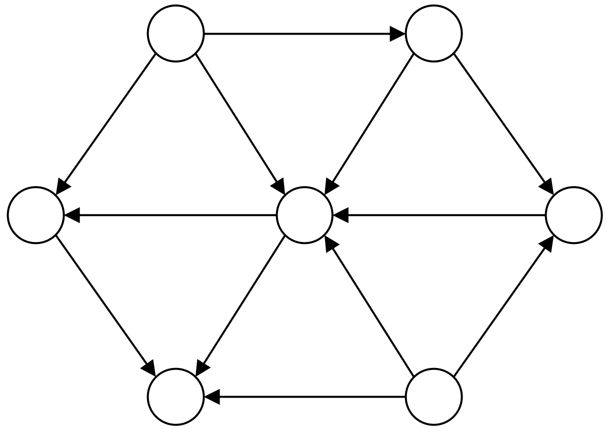
- Therefore y was visited during the call to DFS(x) so y is a descendant of x.

DAGs and Topological Ordering

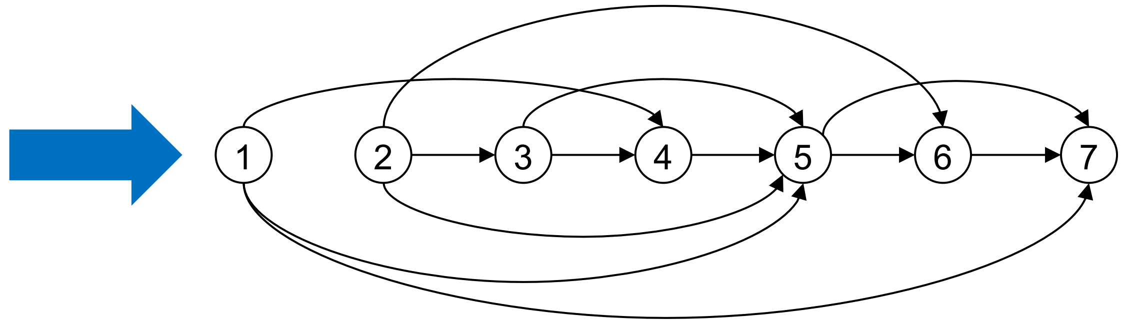
Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Def: A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering of that DAG– all edges left-to-right

DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

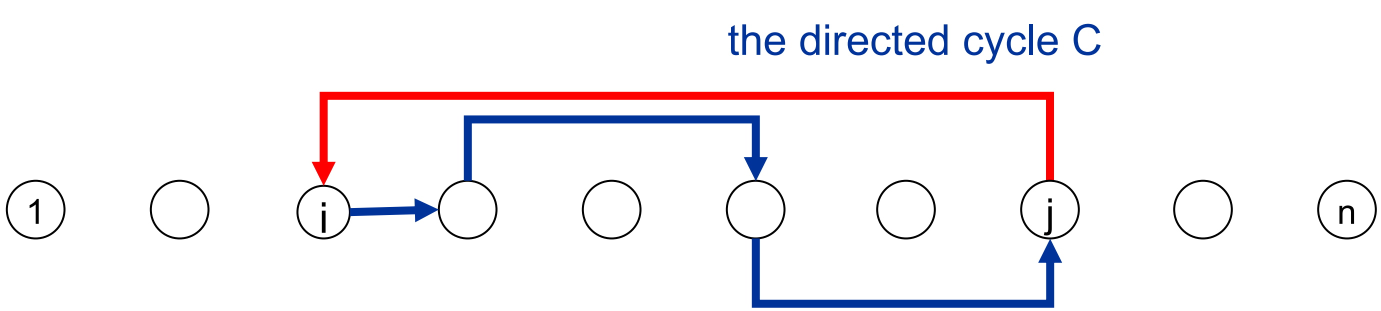
Pf. (by contradiction)

Suppose that G has a topological order $1, 2, \ldots, n$ and that G also has a directed cycle C.

Let i be the lowest-indexed node in C, and let j be the node just before i; thus (j, i) is an (directed) edge.

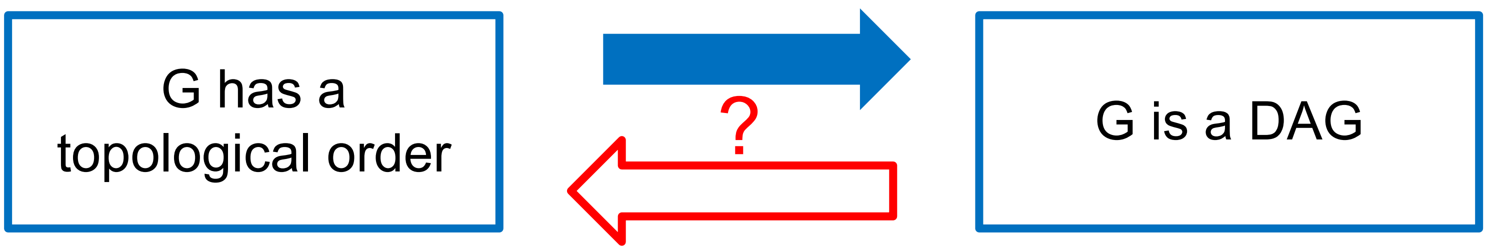
By our choice of i, we have $i < j$.

On the other hand, since (j, i) is an edge and $1, \ldots, n$ is a topological order, we must have $j < i$, a contradiction



the supposed topological order: 1,2,…,n

DAGs: A Sufficient Condition



Every DAG has a source node

Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

Pf. (by contradiction)

Suppose that G is a DAG and and it has no source

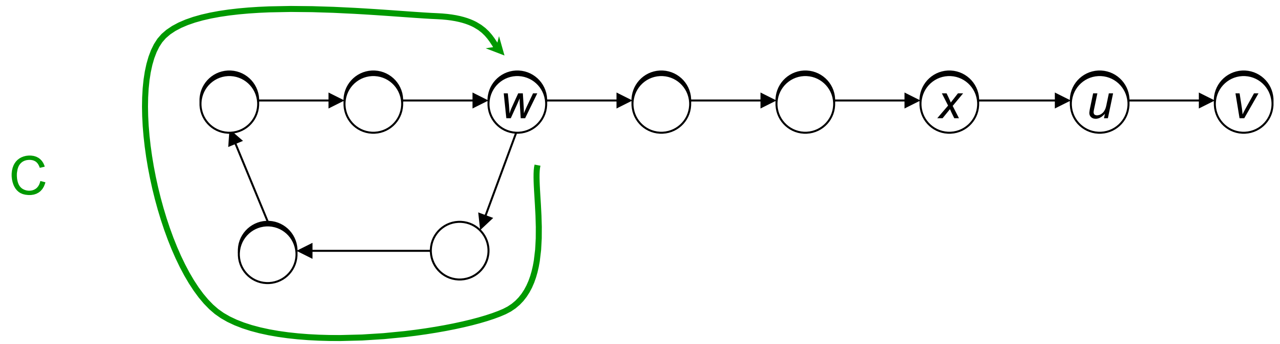
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Is this similar to a previous proof?

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

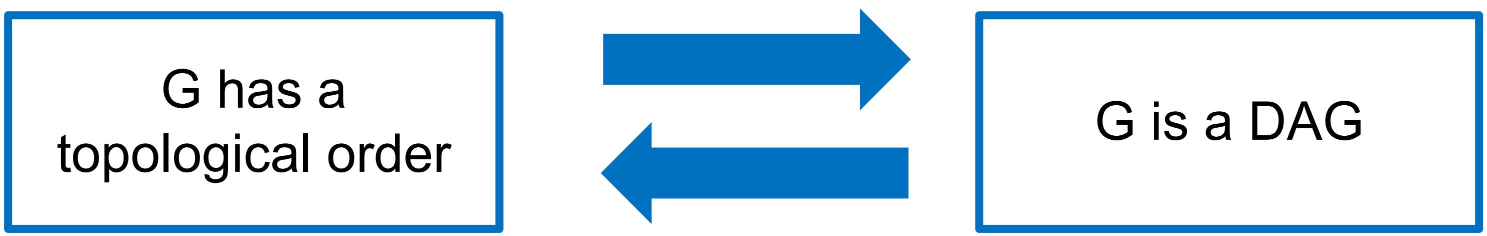
IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with $n > 1$ nodes, find a source node v.
$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

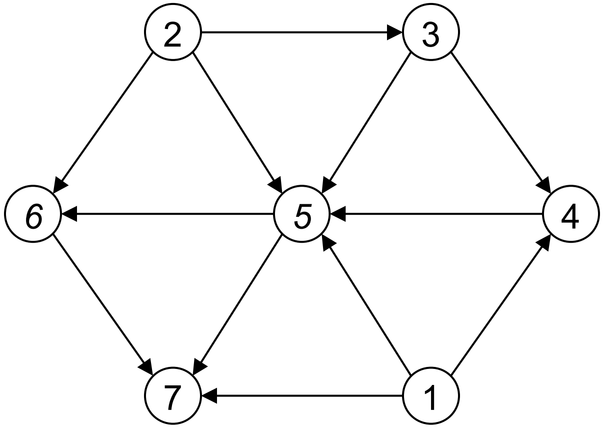
Reminder: Always remove vertices/edges to use IH

By IH, $G - \{ v \}$ has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges.

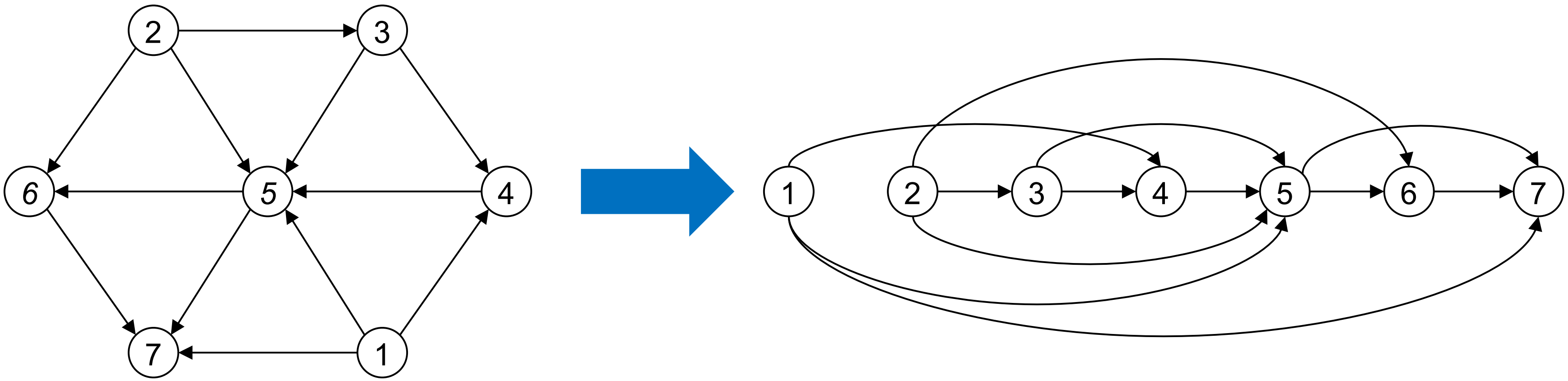
A Characterization of DAGs



Topological Order Algorithm: Example



Topological Order Algorithm: Example



Topological order: 1, 2, 3, 4, 5, 6, 7

Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w) — $O(m + n)$

S = S ∪ {w} for all w with count[w]=0

Main loop:

while S not empty

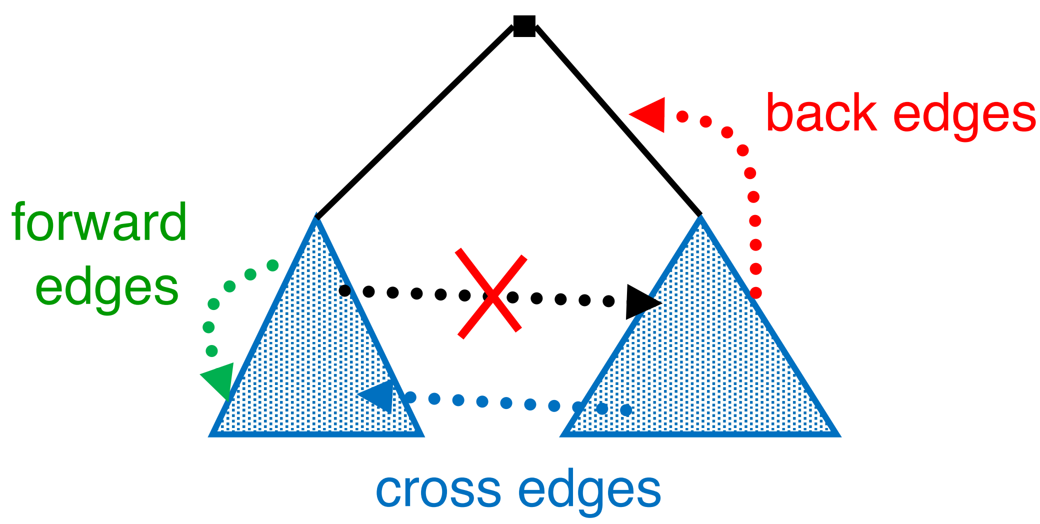
- remove some v from S
- make v next in topo order — $O(1)$ per node
- for all edges from v to some w — $O(1)$ per edge
  - decrement count[w]
  - add w to S if count[w] hits 0

Correctness: clear, I hope

Time: $O(m + n)$ (assuming edge-list representation of graph)

DFS on Directed Graphs

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree



Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: $m = O(n^2)$, often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort

Greedy Algorithms

Greedy Strategy

Goal: Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Cashier's algorithm: At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: $2.89.

Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.

Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.

Greedy Algorithms Outline

Pros
- Intuitive
- Often simple to design (and to implement)
- Often fast

Cons
- Often incorrect!

Proof techniques:
- Stay ahead
- Structural
- Exchange arguments