CSE 421: Introduction to Algorithms

NP-completeness

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Computational Complexity

- Classify problems according to the amount of computational resources used by the best algorithms that solve them

- Recall:
  - worst-case running time of an algorithm
    - **max** # steps algorithm takes on any input of size **n**

Relative Complexity of Problems

- Want to ***compare*** the complexity of problems
  - Want to be able to say

    "Problem **B** is solvable in polynomial time $\Rightarrow$ problem **A** is solvable in polynomial time"

    "Problem **B** is at least as hard as problem **A**"

Polynomial Time Reduction

- **Definition:** **A** $\leq_P$ **B** iff there is an algorithm for **A** using a 'black box' (subroutine/method) that solves **B** that
  - Uses only a polynomial number of steps
  - Makes only a polynomial number of calls to a method for **B**

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  poly time algorithm for **B** $\Rightarrow$ poly time algorithm for **A**
  - Not only is the number of calls polynomial but the size of the inputs on which the calls are made is polynomial!

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  if you can prove there is no fast algorithm for **A**, then that proves there is no fast algorithm for **B**

Why the name reduction?

- Weird: it maps an easier problem into a harder one

- Maxwell reduced the problem of analyzing electricity & magnetism to solving partial differential equations
  - solving partial differential equations in general is a much harder problem than solving E&M problems
  - but we know that we won’t need anything else

A nerd joke

- An engineer
  - is placed in a kitchen with an empty kettle on the table and told to boil water; she fills the kettle with water, puts it on the stove, turns on the gas and boils water.

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  - she is next confronted with a kettle full of water sitting on the counter and told to boil water: she empties the kettle in the sink, places the empty kettle on the table and says, “I’ve reduced this to an already solved problem”.

A Special kind of Polynomial-Time Reduction

We will always use a restricted form of $\mathbf{A} \leq_{\mathbf{P}} \mathbf{B}$ often called a Karp or many-one reduction

Definition: $\mathbf{A} \leq_{\mathbf{P}}^{1} \mathbf{B}$ iff there is an algorithm for **A** given a black box solving **B** that on input **x**

- Runs for polynomial time computing $\mathbf{y}=\mathbf{f}(\mathbf{x})$
- Makes **1** call to the black box for **B** on input **y**
- Returns the answer that the black box gave

We say that the function **f** is the reduction

Reductions by Simple Equivalence

- Independent-Set:
  - Given a graph $G=(V,E)$ and an integer k,
    - is there a subset U of V with $|U| \geq k$ such that no two vertices in U are joined by an edge?
- Clique:
  - Given a graph $G=(V,E)$ and an integer k
    - is there a subset U of V with $|U| \geq k$ such that every pair of vertices in U is joined by an edge?
- Show: Independent-Set $\leq_P$ Clique

Independent-Set $\leq_P$ Clique

- Given:
  - (G,k) as input to Independent-Set where G=(V,E)
- Transform (G,k) to (G',k) where
  - G'=(V,E') has the same vertices as G but E' consists of **precisely** those edges on V that are not edges of G
- U is an independent set in G

  $\Leftrightarrow$ U is a clique in G'

Clique $\leq_P$ Independent-Set

- Given:
  - **(G,k)** as input to Clique where **G=(V,E)**
- Transform **(G,k)** to **(G',k)** where
  - **G'=(V,E')** has the same vertices as **G** but **E'** consists of **precisely** those edges on **V** that are not edges of **G**
- **U** is an clique in **G**

  ⇔ **U** is a independent set in **G'**

More Reductions

- Show: Independent Set $\leq_P$ Vertex-Cover
- Vertex-Cover:
  - Given a graph **G=(V,E)** and an integer **k**,
    - is there a subset **W** of **V** with **|W| ≤ k** such that every edge of **G** has at least one endpoint in **W**? (i.e. **W** covers all edges of **G**)?
- Independent-Set:
  - Given a graph **G=(V,E)** and an integer **k**,
    - is there a subset **U** of **V** with **|U| ≥ k** such that no two vertices in **U** are joined by an edge?

Reduction Idea

- **Claim:** In a graph $G=(V,E)$, for $S \subseteq V$
  S is an independent set $\Leftrightarrow$ $V-S$ is a vertex cover
- Proof:
  - $\Rightarrow$ Let S be an independent set in G
    - Then for every edge $e \in E$,
      S contains at most one endpoint of e
    - At least one endpoint of e must be in $V-S$
    - $V-S$ is a vertex cover

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    - $V-S$ is a vertex cover
  - $\Leftarrow$ Let $W=V-S$ be a vertex cover of G
    - Then S does not contain both endpoints of any edge (else W would miss that edge)
    - S is an independent set

Reduction

- Map **(G,k)** to **(G,n-k)**
  - Previous lemma proves correctness

- Clearly polynomial time

- Just as for Clique, we also can show
  - Vertex-Cover $\leq_P$ Independent Set
    - Map **(G,k)** to **(G,n-k)**

Reductions from a Special Case to a General Case

- Show: Vertex-Cover $\leq_P$ Set-Cover
- Vertex-Cover:
  - Given a graph $G=(V,E)$ and an integer k,
    - is there a subset W of V with $|W| \leq k$ such that every edge of G has at least one endpoint in W? (i.e. W covers all edges of G)?
- Set-Cover:
  - Given a set U of n elements, a collection $S_1,\ldots,S_m$ of subsets of U, and an integer k
    - does there exist a collection of at most k sets whose union is equal to U?

The Simple Reduction

- Transformation **f** maps $(G=(V,E),k)$ to $(U,S_1,\ldots,S_m,k')$
  - $U \leftarrow E$
  - For each vertex $v \in V$ create a set S_v containing all edges that touch v
  - $k' \leftarrow k$
- Reduction **f** is clearly polynomial-time to compute
- We need to prove that the resulting algorithm gives the right answer!

Proof of Correctness

- Two directions:
  - If the answer to Vertex-Cover on (**G**,**k**) is YES then the answer for Set-Cover on **f**(**G**,**k**) is YES
    - If a set **W** of **k** vertices covers all edges then the collection $\{S_v \mid v \in W\}$ of **k** sets covers all of **U**

Proof of Correctness

- Two directions:
  - If the answer to Vertex-Cover on (**G**,**k**) is YES then the answer for Set-Cover on **f**(**G**,**k**) is YES
    - If a set **W** of **k** vertices covers all edges then the collection $\{S_v \mid v \in W\}$ of **k** sets covers all of **U**
  - If the answer to Set-Cover on **f**(**G**,**k**) is YES then the answer for Vertex-Cover on (**G**,**k**) is YES
    - If a subcollection $S_{v_1},\ldots,S_{v_k}$ covers all of **U** then the set $\{v_1,\ldots,v_k\}$ is a vertex cover in **G**.

Decision problems

- Computational complexity usually analyzed using decision problems
  - answer is just 1 or 0 (yes or no).
- Why?
  - much simpler to deal with
  - *deciding* whether **G** has a path from s to t, is certainly no harder than *finding* a path from s to t in G, so a *lower* bound on deciding is also a lower bound on finding
  - Less important, but if you have a good decider, you can often use it to get a good finder.

Polynomial time

- Define **P** (polynomial-time) to be
  - the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.

Beyond P?

- There are many other natural, practical problems for which we don't know any polynomial-time algorithms; e.g.,
  - Independent-Set, Clique, Vertex-Cover, Set-Cover
  - decisionTSP:
    - Given a weighted graph **G** and an integer **k**,
      - does there exist a tour that visits all vertices in **G** having total weight at most **k**?

Satisfiability

- Boolean variables $x_1,\ldots,x_n$
  - taking values in $\{0,1\}$. 0=false, 1=true
- Literals
  - x_i or $\neg x_i$ for $i=1,\ldots,n$
- Clause
  - a logical OR of one or more literals
  - e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$
- CNF formula
  - a logical AND of a bunch of clauses
- **k**-CNF formula
  - All clauses have exactly **k** variables

Satisfiability

- CNF formula example

  $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
- If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - the one above is, the following isn't
  - $x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$
- 3-SAT: Given a CNF formula F with 3 variables per clause, is it satisfiable?

Common property of these problems

- There is a special piece of information, a **short certificate** or proof, that allows you to **efficiently verify** (in polynomial-time) that the YES answer is correct. This certificate might be very hard to find

- e.g.
  - DecisionTSP: the tour itself,
  - Independent-Set, Clique: the set **U**
  - 3-SAT: an assignment that makes **F** true.

The complexity class NP

NP consists of all decision problems where

- You can **verify** the **YES** answers efficiently (in polynomial time) given a short (polynomial-size) ***certificate***

and

- **No fake certificate** can fool your polynomial time verifier into saying **YES** for a **NO** instance

More Precise Definition of NP

A decision problem A is in **NP** iff there is

- a polynomial time procedure **VerifyA**(.,.) and
- a polynomial **p**

s.t.

- for every input **x** that is a **YES** for A there is a string **t** with $|\mathbf{t}| \leq \mathbf{p}(|\mathbf{x}|)$ with **VerifyA**(**x**,**t**) = **YES**

and

- for every input **x** that is a **NO** for A there does **not** exist a string **t** with with $|\mathbf{t}| \leq \mathbf{p}(|\mathbf{x}|)$ with **VerifyA**(**x**,**t**) = **YES**

- A string **t** on which **VerifyA**(**x**,**t**)=YES is called a ***certificate*** for **x** or a ***proof*** that **x** is a YES input

Example: CLIQUE is in NP

procedure **Verify**(**x**,**t**)

if

x is a well-formed representation of a graph **G** = (**V**, **E**) and an integer **k**,

and

t is a well-formed representation of a vertex subset **U** of **V** of size **k**,

and

U is a clique in **G**,

then output "**YES**"

else output "**I'm not convinced**"

Is it correct?

For every **x** = (**G**,**k**) such that **G** contains a **k**-clique, there is a certificate **t** that will cause **Verify**(**x**,**t**) to say **YES**,

- **t** = a list of the vertices in such a **k**-clique

And no fake certificate **t** can fool **Verify**(**x**,**t**) into saying **YES** if either

- **x** isn't well-formed (the uninteresting case)
- **x** = (**G**,**k**) but **G** does not have any cliques of size **k** (the interesting case)

NP problems can be amusing

- Sudoku
  - Is there a solution where this square has value 4?
  - Certificate = full filled in table
    - Easy to check

9			5					
6	2		7			5		
		5				6		7
		6			4			
2				3			9	
	8						1	
4								8
			1	8		4		
7							2	

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- All **NP** problems could be solved by solving general $\mathbf{n^2} \times \mathbf{n^2}$ version of Sudoku!

Keys to showing that a problem is in NP

1. What's the output? (must be **YES**/**NO**)
2. What must the input look like?
3. Which inputs need a **YES** answer?
   - Call such inputs **YES** inputs/**YES** instances
4. For every given **YES** input, is there a certificate (i.e., a hint) that would help?
   - OK if some inputs don't need a certificate
5. For any given **NO** input, is there a fake certificate that would trick you?

Solving NP problems without hints

- The only **obvious algorithm** for most of these problems is **brute force**:
  - try all possible certificates and check each one to see if it works.
  - *Exponential* time:
    - 2^n truth assignments for n variables
    - $n!$ possible TSP tours of n vertices
    - $\binom{n}{k}$ possible k element subsets of n vertices
    - etc.

What We Know

- Nobody knows if all problems in **NP** can be done in polynomial time; i.e., does **P=NP**?
  - one of the most important open questions in all of science.
  - huge practical implications
- Every problem in **P** is in **NP**
  - one doesn't even need a certificate for problems in **P** so just ignore any hint you are given
- Every problem in **NP** is in exponential time

NP-hardness & NP-completeness

- Some problems in **NP** seem hard
  - people have looked for efficient algorithms for them for hundreds of years without success
- However
  - nobody knows how to **prove** that they are really hard to solve, i.e. **P$\neq$ NP**

Problems in NP that seem hard

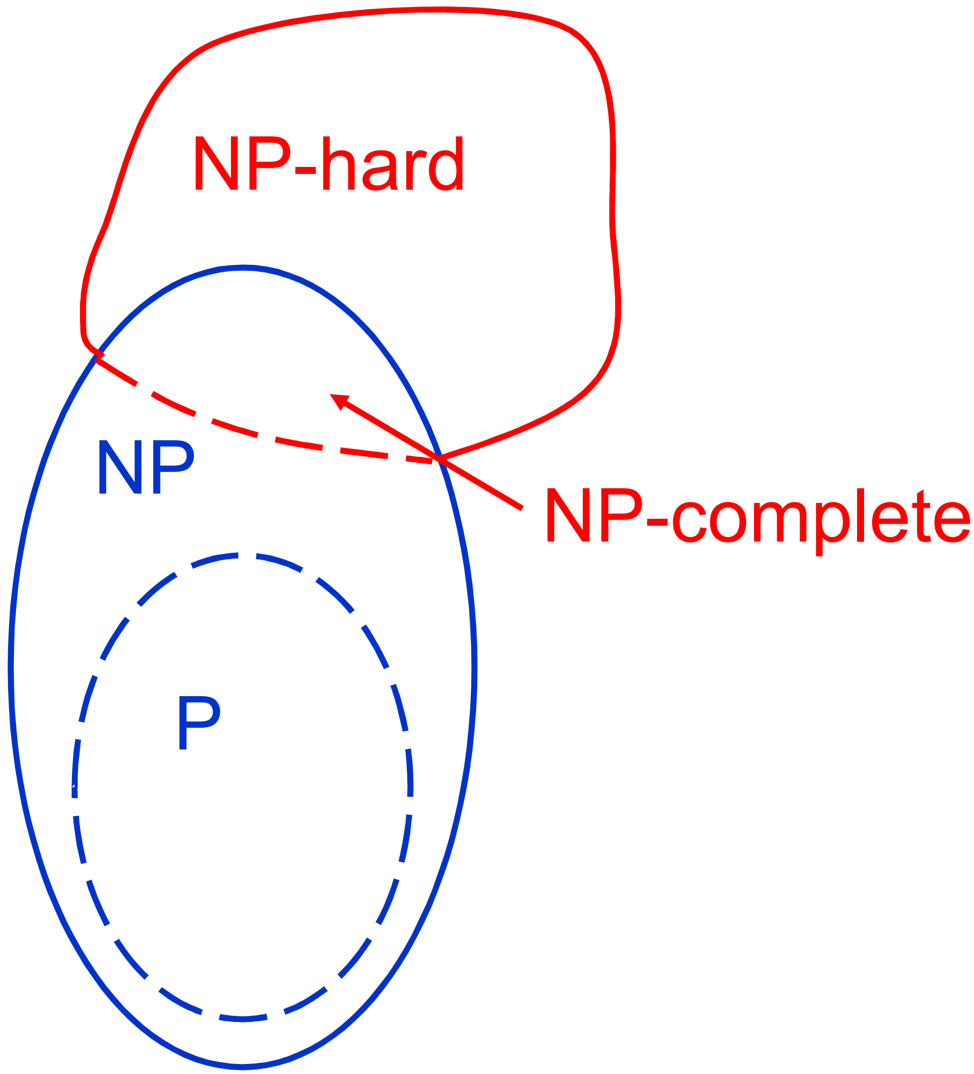
- Some Examples in **NP**
  - 3-SAT
  - Independent-Set
  - Clique
  - Vertex Cover
- All hard to solve; certificates seem to help on all
- Fast solution to *any* gives fast solution to *all!*

NP-hardness & NP-completeness

- Alternative approach to proving problems not in **P**
  - show that they are at least as hard as any problem in **NP**

- Rough definition:
  - A problem is NP-hard iff it is at least as hard as every problem in **NP**
  - A problem is NP-complete iff it is both
    - **NP**-hard
    - in **NP**

P and NP



NP-hardness & NP-completeness

- **Definition:** A problem **B** is NP-hard iff every problem **A**$\in$**NP** satisfies **A** $\leq_P$**B**

- **Definition:** A problem **B** is NP-complete iff **B** is **NP**-hard and **B** $\in$**NP**

- Even though we seem to have lots of hard problems in **NP** it is not obvious that such super-hard problems even exist!

Cook-Levin Theorem

- Theorem (Cook 1971, Levin 1973):

  3-SAT is **NP**-complete

- Recall
  - CNF formula
    - $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
  - If there is some assignment of **0**'s and **1**'s to the variables that makes it true then we say the formula is satisfiable
  - **3-SAT:** Given a 3-CNF formula F, is it satisfiable?

Implications of Cook-Levin Theorem?

- There is at least one interesting super-hard problem in **NP**
- Is that such a big deal?
- YES!
  - There are lots of other problems that can be solved if we had a polynomial-time algorithm for **3-SAT**
  - Many of these problems are exactly as hard as **3-SAT**

A useful property of polynomial-time reductions

- Theorem: If $A \leq_P B$ and $B \leq_P C$ then $A \leq_P C$

- Proof idea: (Using $\leq_P^1$)
  - Compose the reduction f from A to B with the reduction g from B to C to get a new reduction $h(x)=g(f(x))$ from A to C.
  - The general case is similar and uses the fact that the composition of two polynomials is also a polynomial

Cook-Levin Theorem & Implications

- Theorem (Cook 1971, Levin 1973):
  3-SAT is **NP**-complete

  For proof see CSE 431

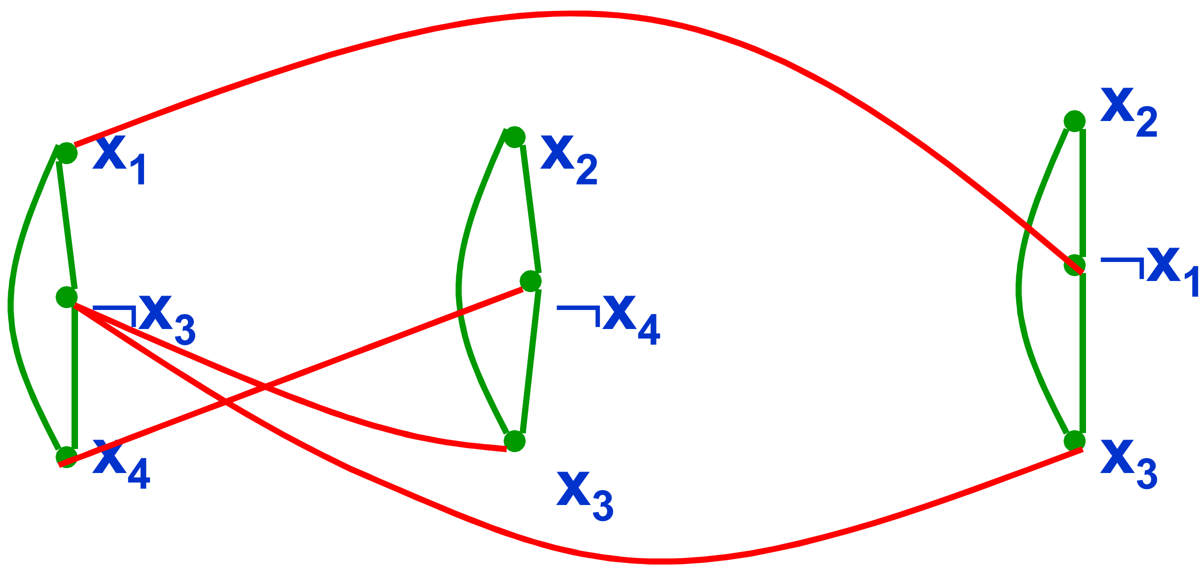
- Corollary: **B** is **NP**-hard $\Leftrightarrow$ **3-SAT** $\leq_P$**B**
  - (or **A** $\leq_P$**B** for any **NP**-complete problem **A**)
- Proof:
  - If **B** is **NP**-hard then every problem in **NP** polynomial-time reduces to **B**, in particular **3-SAT** does since it is in **NP**
  - For any problem **A** in **NP**, **A** $\leq_P$**3-SAT** and so if **3-SAT** $\leq_P$**B** we have **A** $\leq_P$ **B**.
    - therefore **B** is **NP**-hard if **3-SAT** $\leq_P$**B**

Another NP-complete problem: 3-SAT $\leq_P$ Independent-Set

- A Tricky Reduction:
  - mapping CNF formula **F** to a pair <**G**,**k**>
  - Let **m** be the number of clauses of **F**
  - Create a vertex in **G** for each literal in **F**
  - Join two vertices **u**, **v** in **G** by an edge iff
    - **u** and **v** correspond to literals in the same clause of **F**, (green edges) or
    - **u** and **v** correspond to literals **x** and ¬**x** (or vice versa) for some variable **x**. (red edges).
  - Set **k**=**m**
  - Clearly polynomial-time

3-SAT $\leq_P$ Independent-Set

F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$

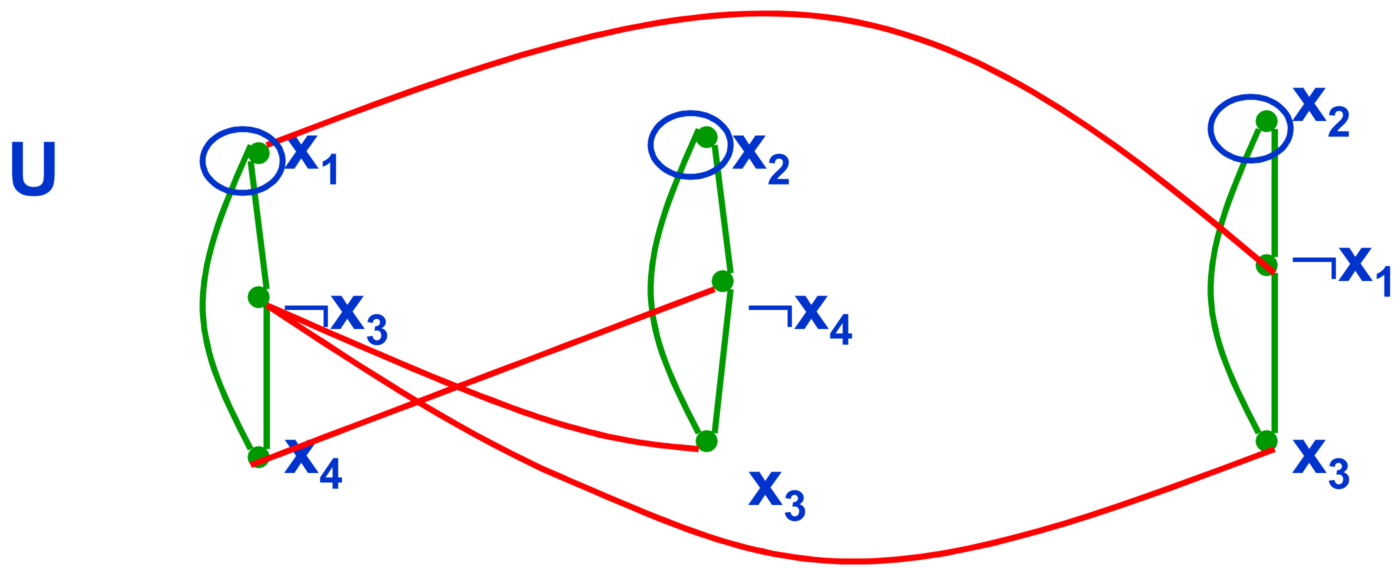


3-SAT $\leq_P$ Independent-Set

- Correctness:
  - If **F** is **satisfiable** then there is some assignment that satisfies at least one literal in each clause.
  - Consider the set **U** in **G** corresponding to the **first satisfied literal in each clause**.
    - $|\mathbf{U}|=\mathbf{m}$
    - Since **U** has only one vertex per clause, no two vertices in **U** are joined by green edges
    - Since a truth assignment never satisfies both **x** and **¬x**, **U** doesn't contain vertices labeled both **x** and **¬x** and so no vertices in **U** are joined by red edges
    - Therefore **G** has an independent set, **U**, of size at least **m**
  - Therefore (**G**,**m**) is a **YES** for independent set.

3-SAT $\leq_P$ Independent-Set

$$\begin{array}{cccccccccc} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ \mathbf{F:} & (x_1 \vee & \neg x_3 \vee & x_4) \wedge & (x_2 \vee & \neg x_4 \vee & x_3) \wedge & (x_2 \vee & \neg x_1 \vee & x_3) \end{array}$$



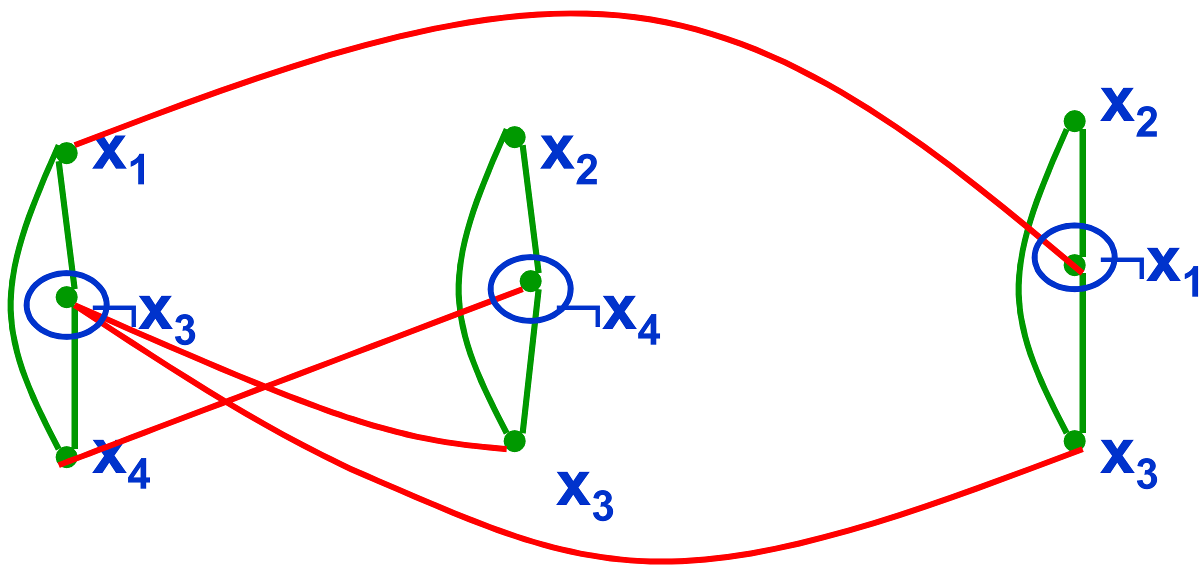
Given assignment $x_1=x_2=x_3=x_4=1$,
U is as circled

3-SAT $\leq_P$ Independent-Set

- Correctness continued:
  - If **(G,m)** is a **YES** for Independent-Set then there is a set **U** of **m** vertices in **G** containing no edge.
    - Therefore **U** has precisely one vertex per clause because of the green edges in **G**.
    - Because of the red edges in **G**, **U** does not contain vertices labeled both **x** and **¬x**
    - Build a truth assignment **A** that makes all literals labeling vertices in **U** true and for any variable not labeling a vertex in **U**, assigns its truth value arbitrarily.
    - By construction, **A** satisfies **F**
  - Therefore **F** is a **YES** for 3-SAT.

3-SAT $\leq_P$ Independent-Set

$$\begin{array}{l} \quad\;\; 0 \quad\;\; 1 \quad\;\; 0 \qquad\quad ? \quad\;\; 1 \quad\;\; 0 \qquad\quad ? \quad\;\; 1 \quad\;\; 0 \\ F: \;(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3) \end{array}$$



Given **U**, satisfying assignment is $x_1 = x_3 = x_4 = 0$, $x_2 = 0$ or 1

Independent-Set is NP-complete

- We just showed that **Independent-Set** is NP-hard and we already knew **Independent-Set** is in **NP**.

- Corollary: **Clique** is **NP**-complete
  - We showed already that **Independent-Set** $\leq_P$ **Clique** and **Clique** is in **NP**.

Problems we already know are NP-complete

- 3-SAT
- Independent-Set
- Clique
- Vertex-Cover
- Set-Cover

- There are 1000's of practical problems that are NP-complete, e.g. scheduling, optimal VLSI layout etc.

Steps to Proving Problem B is NP-complete

- Show **B** is **NP**-hard:
  - State:"Reduction is from **NP**-hard Problem **A**"
  - Show what the map **f** is
  - Argue that **f** is polynomial time
  - Argue correctness: two directions Yes for **A** implies Yes for **B** and vice versa.
- Show **B** is in **NP**
  - State what hint/certificate is and why it works
  - Argue that it is polynomial-time to check.

Some other NP-complete examples you should know

- **Hamiltonian-Cycle** Given a directed graph G is there a cycle in G that visits each vertex in G exactly once?
- **Hamiltonian-Path** Given a directed graph G is there a path in G that visits each vertex in G exactly once?
  - Both are also **NP**-complete when G is an undirected graph
- Note that deciding the similar questions for **Eulerian-Cycle** and **Eulerian-Path** (which require that each edge be visited exactly once rather than each vertex) can be done in polynomial time.
  - How?

Travelling-Salesman Problem (TSP)

- Given a set of **n** cities $v_1,\ldots,v_n$ and distances between each pair of cities $d(v_i,v_j)$ what is the shortest tour that visits all the cities?
  - Not a decision problem

- **DecisionTSP**:
  - Given a set of distances given by **d** for each pair of cities in $v_1,\ldots,v_n$ and an integer **D**, does there exist a tour that visits all cities having total weight at most **D**?

Hamiltonian-Cycle$\leq_P$DecisionTSP

- Define the reduction
  - Vertices **V** of **G=(V,E)** become cities
  - Set $\mathbf{d}(\mathbf{v_i},\mathbf{v_j})$ to **1** if $(\mathbf{v_i},\mathbf{v_j})\in\mathbf{E}$

    2 if not
  - Set $\mathbf{D}=|\mathbf{V}|$

- Claim: There is a Hamiltonian cycle in **G** iff there is a tour of length $|\mathbf{V}|$

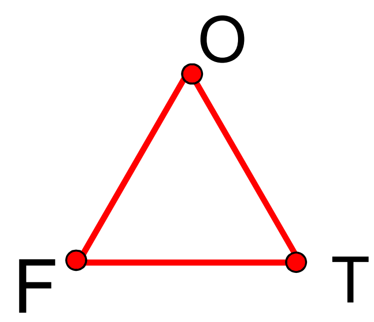
Graph Colorability

- Defn: Given a graph G=(V,E), and an integer k, a k-coloring of G is
  - an assignment of up to k different colors to the vertices of G so that the endpoints of each edge have different colors.
- 3-Color: Given a graph G=(V,E), does G have a 3-coloring?
- Claim: 3-Color is NP-complete
- Proof: 3-Color is in NP:
  - Hint is an assignment of red,green,blue to the vertices of G
  - Easy to check that each edge is colored correctly

3-SAT $\leq_P$ 3-Color

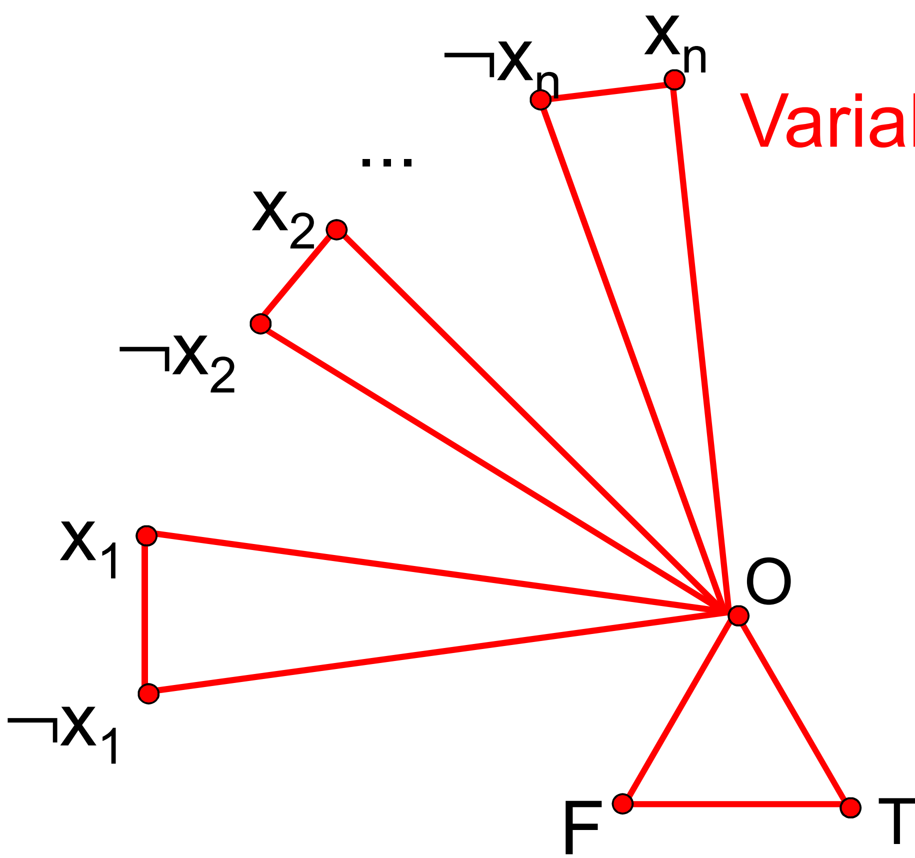
- Reduction:
  - We want to map a 3-CNF formula **F** to a graph **G** so that
    - **G** is 3-colorable iff **F** is satisfiable

3-SAT $\leq_P$ 3-Color



Base Triangle

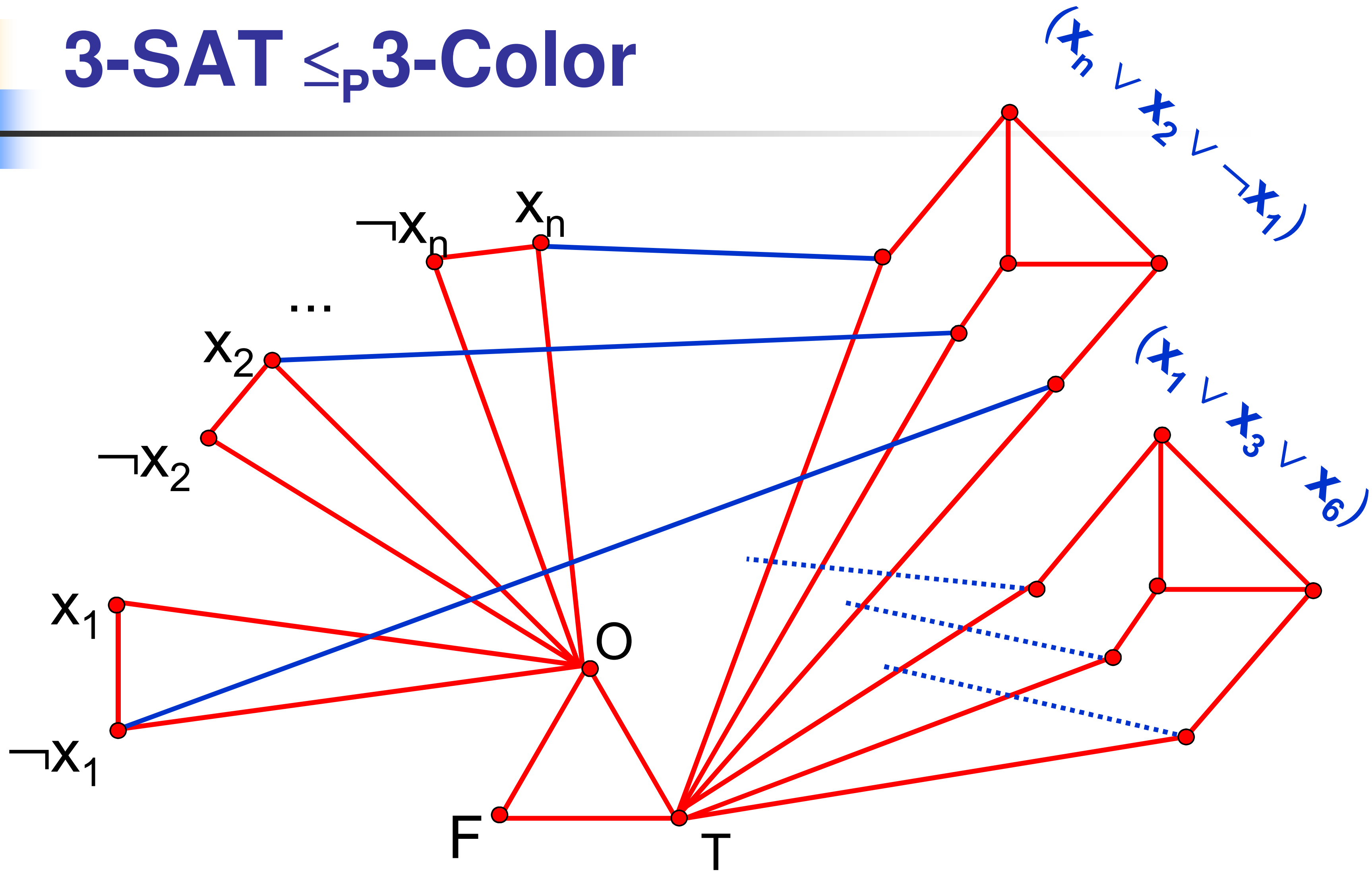
3-SAT $\leq_P$ 3-Color



Variable Part:

in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)

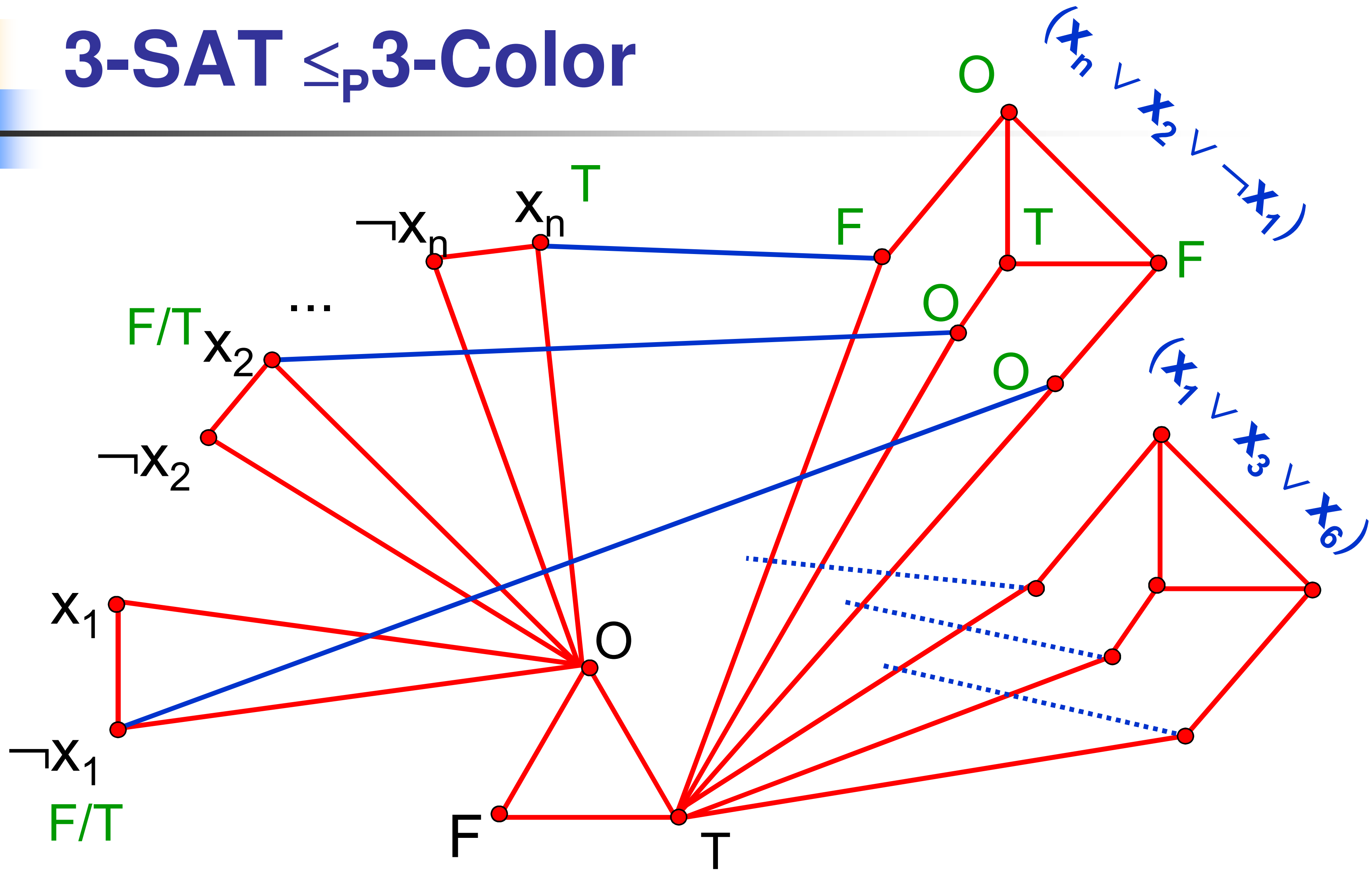
3-SAT $\leq_P$ 3-Color



Clause Part:

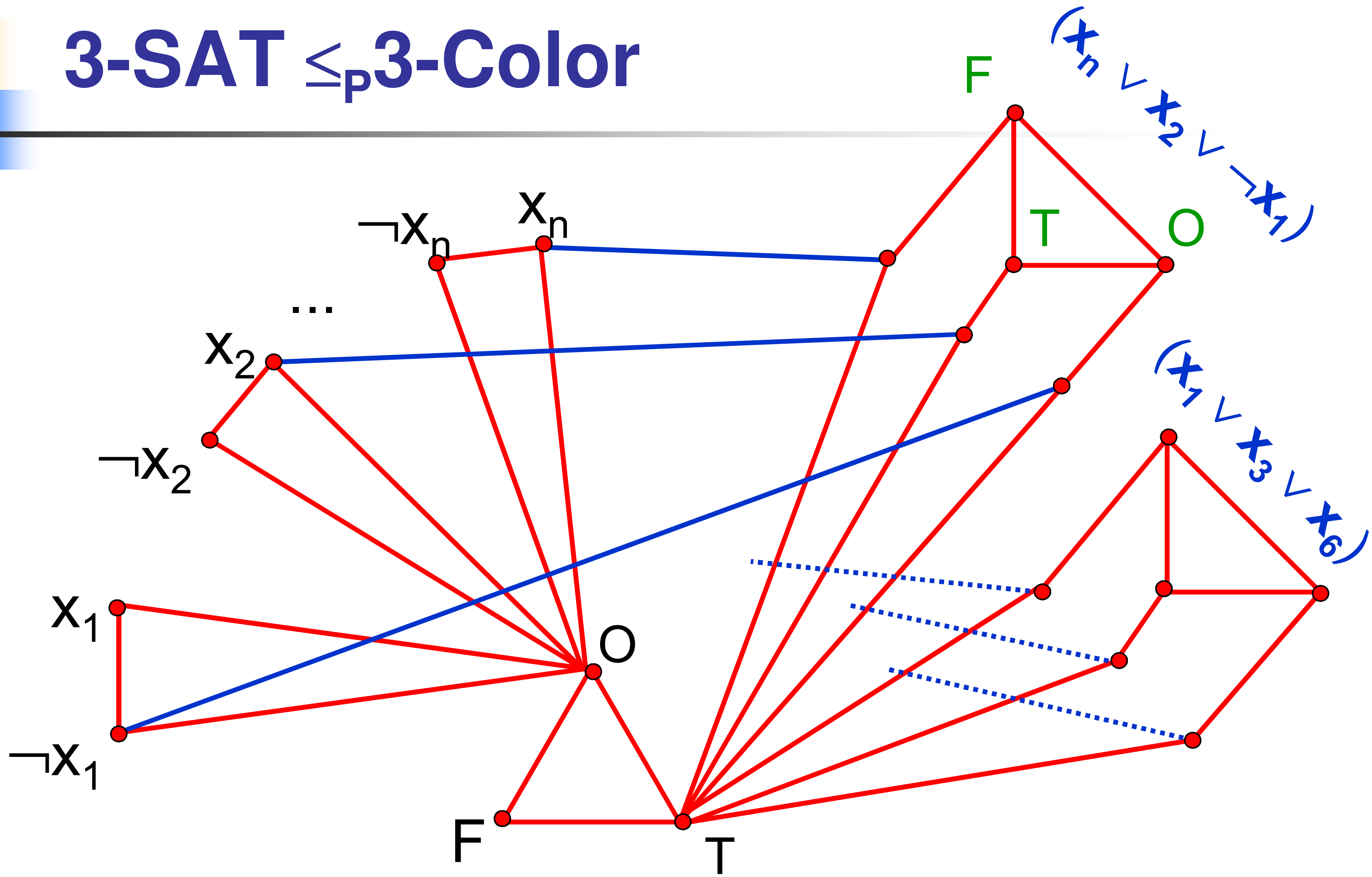
Add one 6 vertex gadget per clause connecting its 'outer vertices' to the literals in the clause

3-SAT $\leq_P$ 3-Color



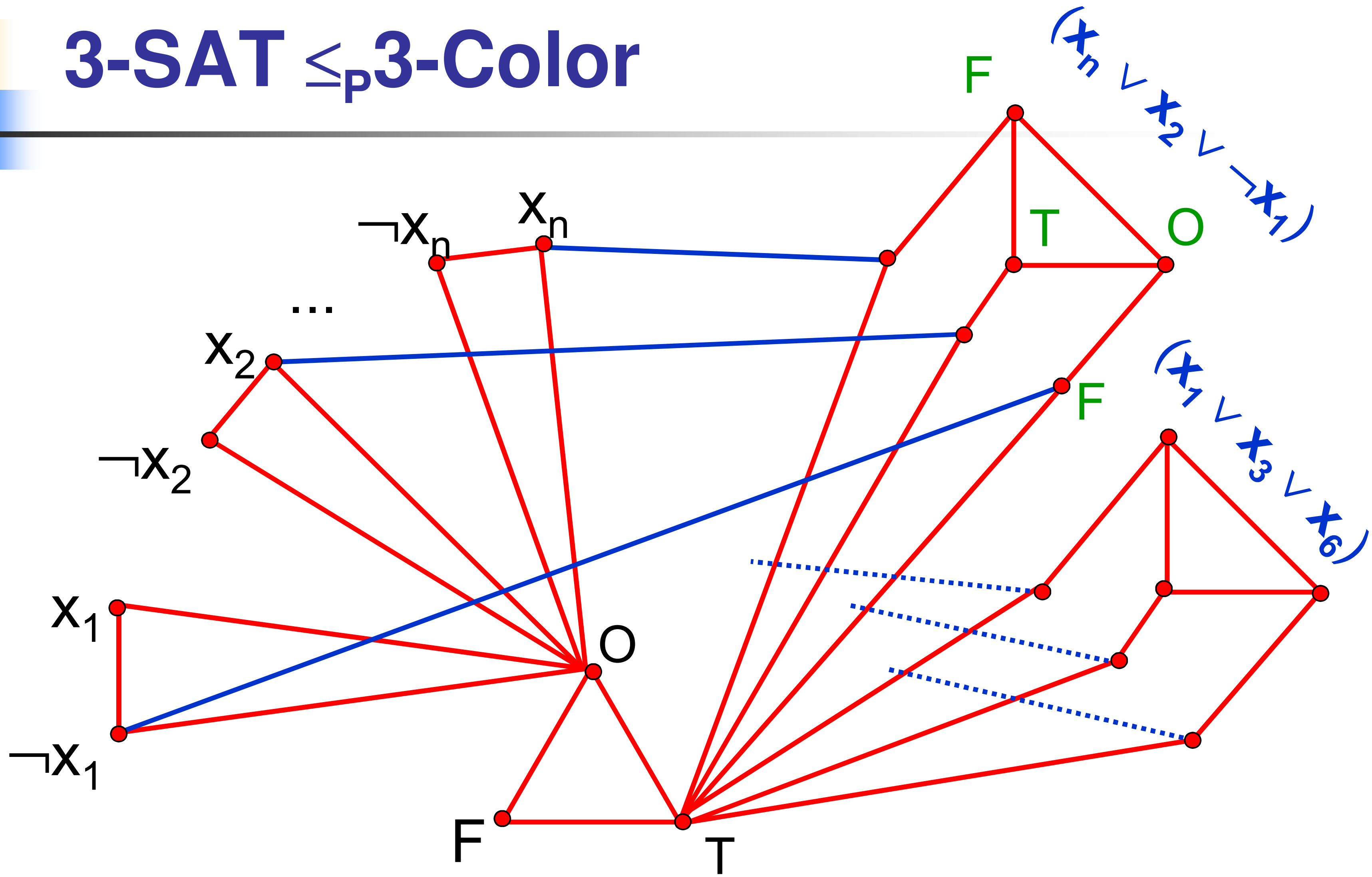
Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph

3-SAT $\leq_P$ 3-Color



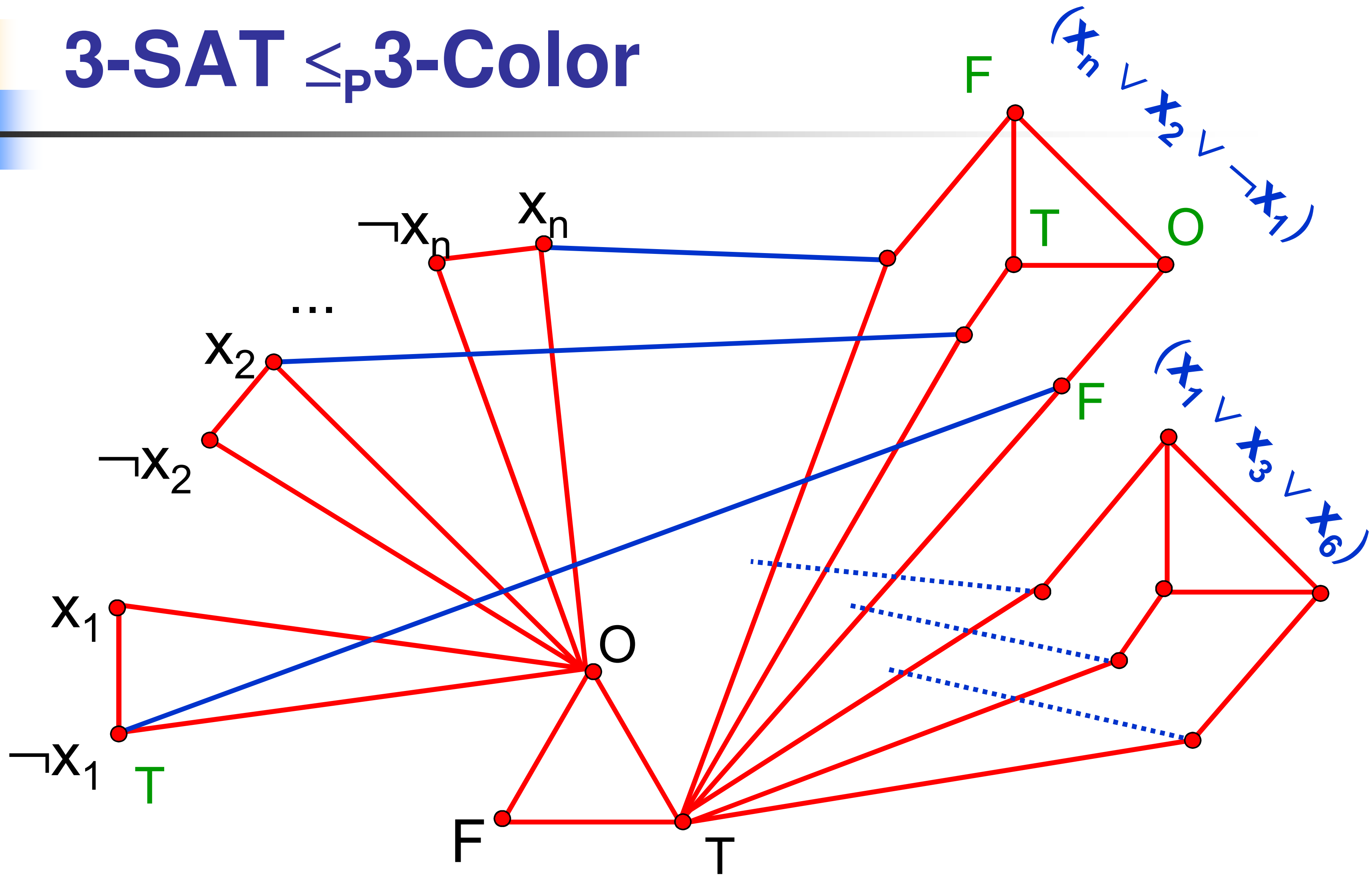
Any 3-coloring of the graph colors each gadget triangle using each color

3-SAT $\leq_P$ 3-Color



Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget

3-SAT $\leq_P$ 3-Color



Any 3-coloring of the graph has T at the other end of the blue edge connected to the F

More NP-completeness

- Subset-Sum problem
  (Decision version of Knapsack)
  - Given n integers $w_1,...,w_n$ and integer W
  - Is there a subset of the n input integers that adds up to exactly W?

- $O(nW)$ solution from dynamic programming but if W and each w_i can be n bits long then this is exponential time

3-SAT $\leq_P$ Subset-Sum

- Given a 3-CNF formula with **m** clauses and **n** variables
- Will create 2**m**+2**n** numbers that are **m+n** digits long
  - Two numbers for each variable x_i
    - t_i and f_i (corresponding to x_i being true or x_i being false)
  - Two extra numbers for each clause
    - u_j and v_j (filler variables to handle number of false literals in clause C_j)

3-SAT $\leq_P$ Subset-Sum

				i						j		
	1	2	3	4	…	n	1	2	3	4	…	m
t_1	1	0	0	0	…	0	0	0	1	0	…	1
f_1	1	0	0	0	…	0	1	0	0	1	…	0
t_2	0	1	0	0	…	0	0	1	0	0	…	1
f_2	0	1	0	0	…	0	0	0	1	1	…	0
					…			….				
$u_1=v_1$	0	0	0	0	…	0	1	0	0	0	…	0
$u_2=v_2$	0	0	0	0	…	0	0	1	0	0	…	0
					…				….			
W	1	1	1	1	…	1	3	3	3	3	…	3

$C_3=(x_1 \vee \neg x_2 \vee x_5)$

Matching Problems

- **Perfect Bipartite Matching**
  - Given a bipartite graph $\mathbf{G}=(\mathbf{V},\mathbf{E})$ where $\mathbf{V}=\mathbf{X}\cup\mathbf{Y}$ and $\mathbf{E} \subseteq \mathbf{X} \times \mathbf{Y}$, is there a set $\mathbf{M}$ in $\mathbf{E}$ such that every vertex in $\mathbf{V}$ is in precisely one edge of $\mathbf{M}$?

- In **P**
  - Network Flow gives $\mathbf{O}(\mathbf{nm})$ algorithm where $\mathbf{n}=|\mathbf{V}|$, $\mathbf{m}=|\mathbf{E}|$.

3-Dimensional Matching

- **Perfect Bipartite Matching** is in **P**
  - Given a bipartite graph $\mathbf{G}=(\mathbf{V},\mathbf{E})$ where $\mathbf{V}=\mathbf{X}\cup\mathbf{Y}$ and $\mathbf{E} \subseteq \mathbf{X} \times \mathbf{Y}$, is there a subset **M** in **E** such that every vertex in **V** is in precisely one edge of **M** ?

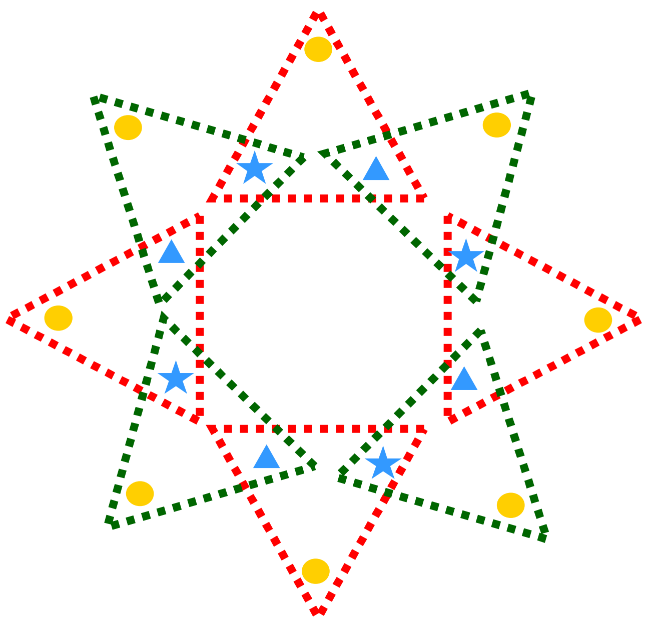
- **3-Dimensional Matching**
  - Given a tripartite hypergraph $\mathbf{G}=(\mathbf{V},\mathbf{E})$ where $\mathbf{V}=\mathbf{X}\cup\mathbf{Y}\cup\mathbf{Z}$ and $\mathbf{E} \subseteq \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$, is there a subset **M** in **E** such that every vertex in **V** is in precisely one hyperedge of **M** ?
    - is in **NP:** Certificate is the set **M**

3-Dimensional Matching

- Theorem: **3-Dimensional Matching** is **NP**-complete
- Proof:
  - We've already seen that it is in **NP**
  - **3-Dimensional Matching** is **NP**-hard:
    - Reduction from **3-SAT**
    - Given a 3-CNF formula **F** we create a tripartite hypergraph ("hyperedges" are triangles) **G** based on **F** as follows

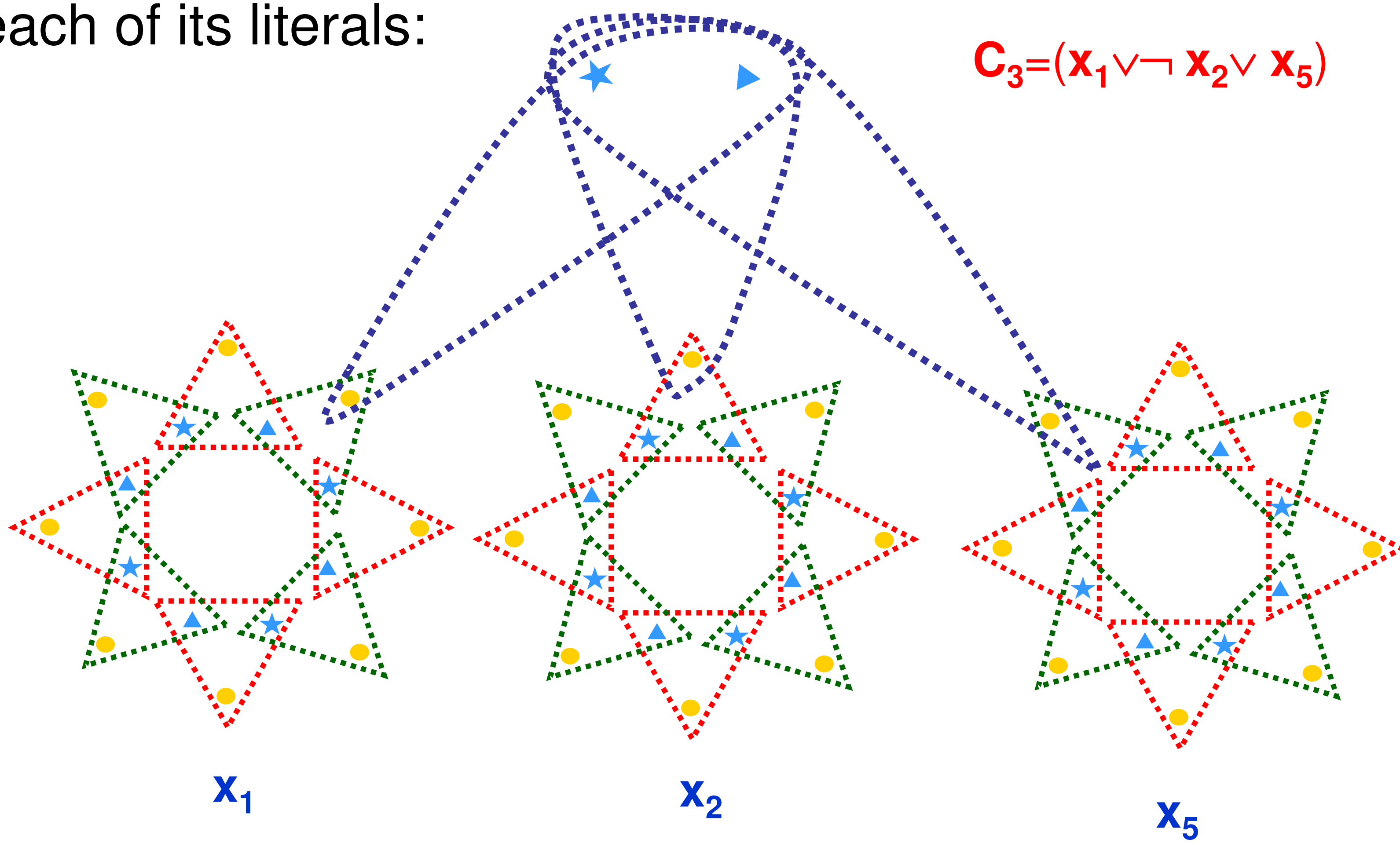
3-SAT $\leq_P$ 3-Dimensional Matching

- Variable part:
  - If variable x_i occurs r_i times in F create r_i red and r_i green triangles linked in a circle, one pair per occurrence
    - Perfect matching M must either use all the green edges leaving red tips uncovered (x_i is assigned false) or all the red edges leaving all green tips uncovered (x_i is assigned true)



3-SAT $\leq_P$ 3-Dimensional Matching

- Clause part: Two new nodes per clause joined to each of its literals:

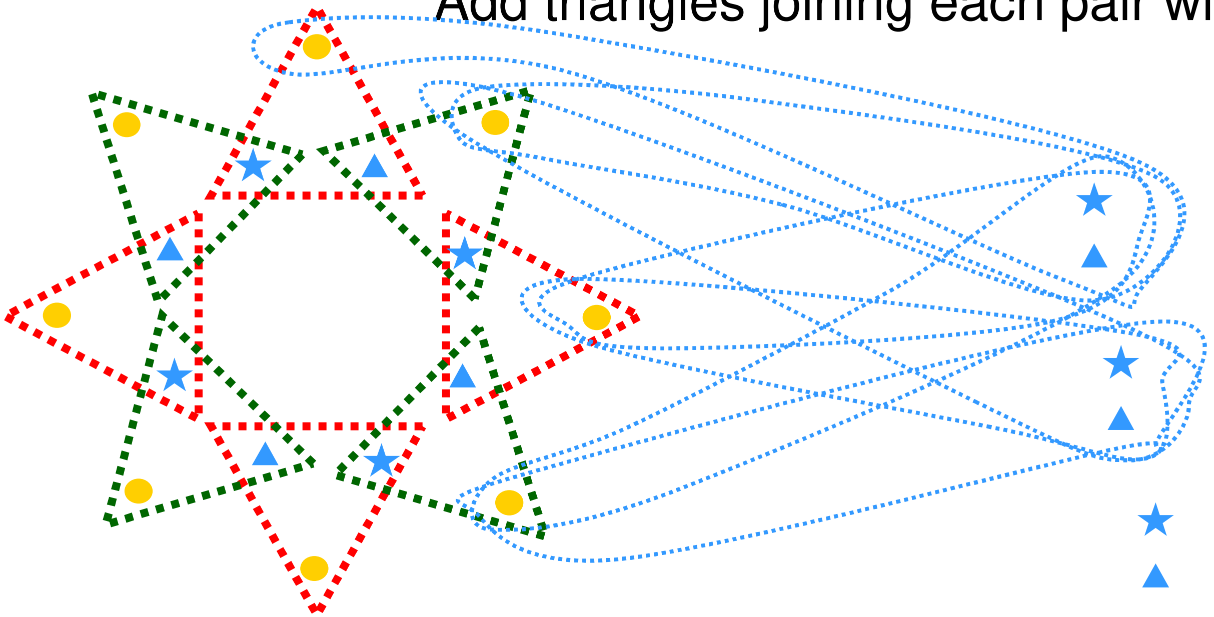


3-SAT $\leq_P$ 3-Dimensional Matching

- Slack: If there are **m** clauses then there are **3m** variable occurrences. That means **3m** total tips are not covered by whichever of red or green triangles not chosen. Of these, **m** are covered if each clause is satisfied. Need to cover the remaining **2m** tips.

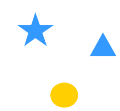
Solution: Add **2m** pairs of slack vertices

Add triangles joining each pair with every tip!



3-SAT $\leq_P$ 3-Dimensional Matching

- Well-formed: Each triangle has one of each type of node:



- Correctness:
  - If **F** has a satisfying assignment then choose the following triangles which form a perfect 3-dimensional matching in **G**:
    - Either the red or the green triangles in the cycle for $\mathbf{x_i}$ - the opposite of the assignment to $\mathbf{x_i}$
    - The triangle containing the first true literal for each clause and the two clause nodes
    - **2m** slack triangles one per new pair of nodes to cover all the remaining tips

3-SAT $\leq_P$ 3-Dimensional Matching

- Correctness continued:
  - If **G** has a perfect 3-dimensional matching then:
    - Each blue node in the cycle for each $\mathbf{x_i}$ is contained in exactly two triangles, exactly one of which much be in **M**. If one triangle in the cycle is in **M**, the others must be the same color. We use the color not used to define the truth assignment to $\mathbf{x_i}$
    - The two nodes for any clause must be contained in an edge which must also contain a third node that corresponds to a literal made true by the truth assignment. Therefore the truth assignment satisfies **F** so it is satisfiable.

P vs NP

- Theory
  - **P = NP?**
  - Open Problem!
  - Bet against it
  - Online commerce could never be secure if it were
    - e.g., A polynomial time algorithm could figure out any password efficiently
- Practice
  - Many interesting, useful, natural, well-studied problems in many fields known to be **NP**-complete
  - No one always succeeds in finding exact solutions to large, arbitrary instances
    - But sometimes guaranteed to find approximate solutions quite well
    - Algorithms that mostly fail can also work when we need them

Is NP as bad as it gets?

- NO! **NP**-complete problems are frequently encountered, but there are worse:
  - Some problems provably require exponential time.
    - Ex: Does **M** halt on input **x** in $2^{|x|}$ steps?
  - Some require 2^n, 2^{2^n}, $2^{2^{2^n}}$, ... steps
  - And some are just plain uncomputable