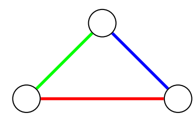
CSE421: Design and Analysis of Algorithms	April 13, 2022

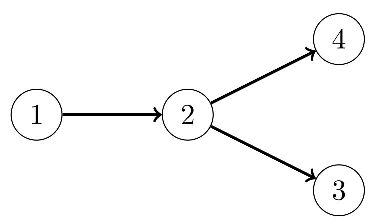
Homework 3

Shayan Oveis Gharan — Due: April 21, 2022 at 23:59 PM

P1) (20 points) Given a graph $G = (V, E)$ with n vertices such that the degree of every vertex of G is at most k. Design a polynomial time algorithm to color the edges of G with at most $2k - 1$ colors such that any pair of edges e, f which are incident to the same vertex have distinct colors. **Note that you don't necessarily have to use all of the $2k-1$ colors.** Your code can output for every edge its color, a number in the range $1, \ldots, 2k - 1$. For example, if G is a triangle, we have $k = 2$, and we can color edges of G with $2k - 1 = 3$ colors as follows:



P2) (20 points) An outward-rooted tree is a directed tree where the is a path from root to each vertex. Given a sequence $d_1, \ldots, d_n$ of integers design a polynomial time algorithm that construct a outward-rooted tree such that the out-degree of vertex i is d_i. If no such tree exists your algorithm must output "Impossible", otherwise output the edges of the tree. For example, given $1, 2, 0, 0$, we can construct the following tree:



Hint: Show that for every sequence $d_1, \ldots, d_n$ of integers there exists a outward-rooted tree where the out-degree of i is d_i if and only if $\sum_i d_i = n - 1$ and for all i, and we have $d_i \geq 0$ for all $1 \leq i \leq n$.

P3) (20 points) Prove or disprove: Every directed graph G has a source node if and only if it does not have a cycle. Note that to disprove the statement only one example is enough. But to prove the statement you have to prove both directions for every directed graph G.

P4) (20 points) 421 has m TAs. Suppose that the i-th TA takes exactly t_i seconds to grade a submission. We have n sheets that we need to grade. Design an algorithm that runs in time polynomial in $m, n, \max_i t_i$ and outputs the smallest number of seconds to grade all sheets. For example, if $m = 2$, $t_1 = 1, t_2 = 2$ and $n = 3$ then you should output 2.

P5) **Extra Credit:** Suppose G is a 3-colorable graph with n vertices, i.e., it is possible to color the vertices of G with 3 colors such that the endpoints of every edge have distinct colors. Design a polynomial time algorithm that colors vertices of G with $O(\sqrt{n})$ many colors.