CSE 421

Approximation Alg

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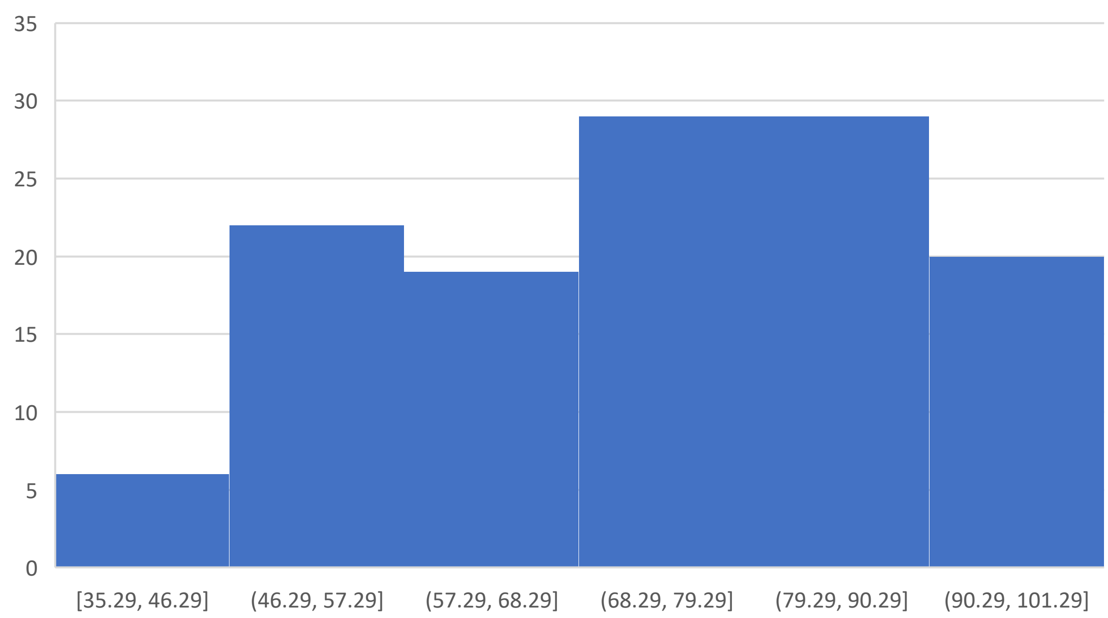
Midterm

Congratulations! You did great in the midterm
Median ~ 74.5%

- I did very well in the midterm; so I'll get a 4.0, Yaay! (not really) Final is harder and has a significant impact on your final gpa
- I did terrible in midterm, can I still get 3.9 or 4.0? Yes!
- If you are way below median below 50% try harder

Final will be harder



Q/A

- HW problems are too hard for me
  - We have resources to prepare for HW
    - problem solving section, OH, …
    - Exercises in the book.
    - USA Olympiad training website: https://train.usaco.org
  - Difficult HW problems prepare you for real world algorithm problems
- Grading rules are too strict
  - Every week I spent hours to train TAs how to grade. The well-defined rubric is my effort to have a systematic grading guidelines that all TAs can follow. Without it everybody grades arbitrarily.
  - Everything is not about grade! We are here to learn.
- TAs have not responded to my re-grade requests
  - TAs are also humans; give them sometime.
  - Send me an email or come to OH, I'll look into your request
- What is the point of this course after all? Why do you have to prove correctness of an algorithm?
  - Often algorithms that we design are incorrect.

Approximation Algorithms

Approximation Algorithm

Polynomial-time Algorithms with a guaranteed approximation ratio.

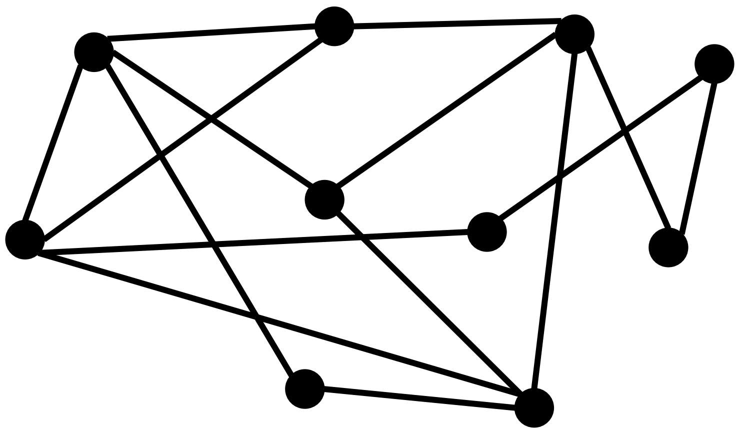
$$\alpha = \frac{\text{Cost of computed solution}}{\text{Cost of the optimum}}$$

worst case over all instances.

Goal: For each NP-hard problem find an approximation algorithm with the best possible approximation ratio.

Vertex Cover

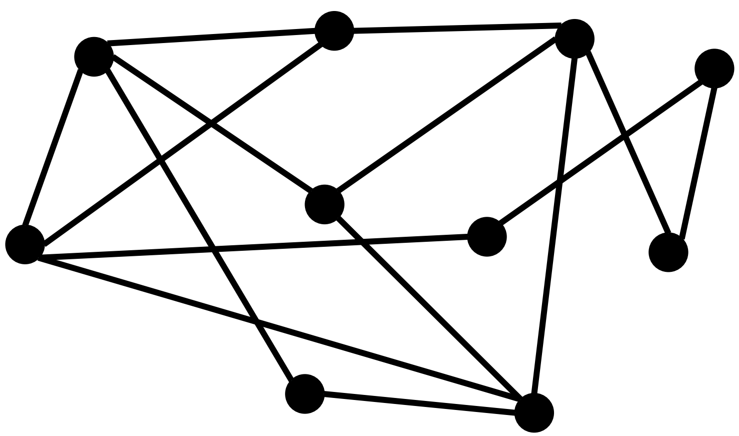
Given a graph G=(V,E), Find smallest set of vertices touching every edge



A Different Greedy Rule

Greedy 2: Iteratively, pick both endpoints of an uncovered edge.

Vertex cover = 6



Greedy (2) gives 2-approximation

Thm: Size of greedy (2) vertex cover is at most twice as big as size of optimal cover

Pf: Suppose Greedy (2) picks endpoints of edges $e_1, \dots, e_k$.
Since these edges do not touch, every valid cover must pick one vertex from each of these edges!

i.e., $OPT \geq k$.

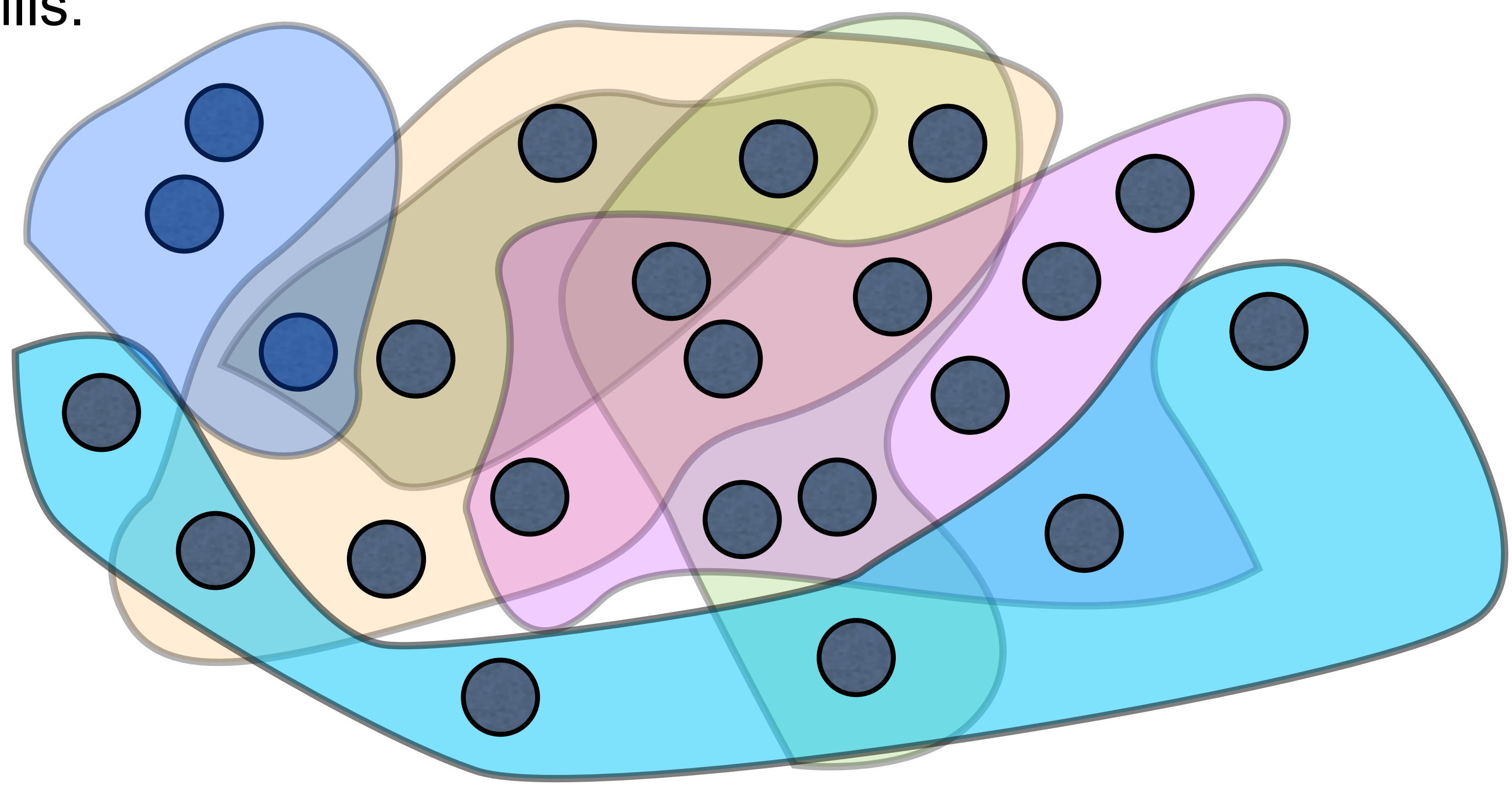
But the size of greedy cover is 2k. So, Greedy is a 2-approximation.

Set Cover

Given a number of sets on a ground set of elements,

Goal: choose minimum number of sets that cover all.

e.g., a company wants to hire employees with certain skills.

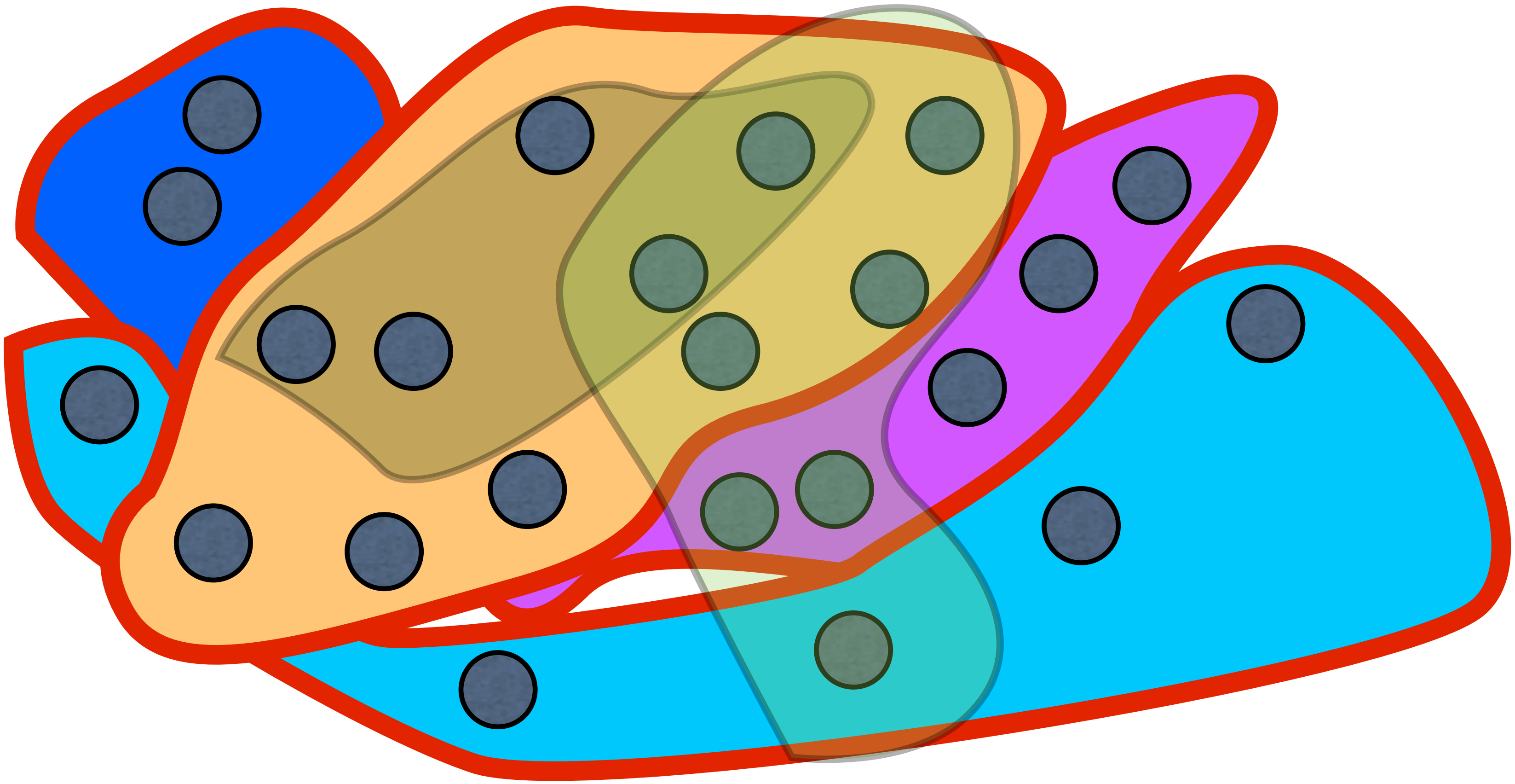


Set Cover

Given a number of sets on a ground set of elements,

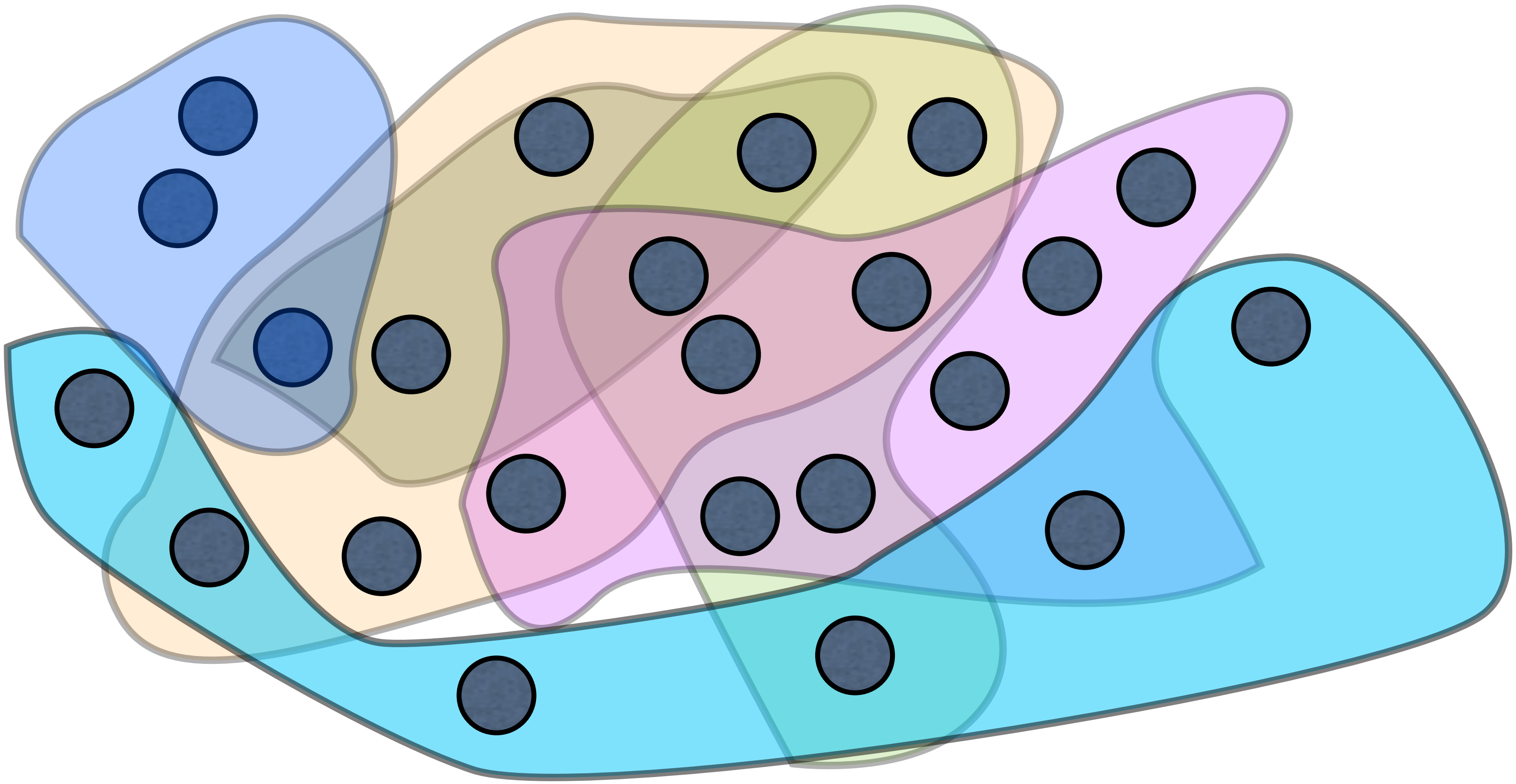
Goal: choose minimum number of sets that cover all.

Set cover = 4



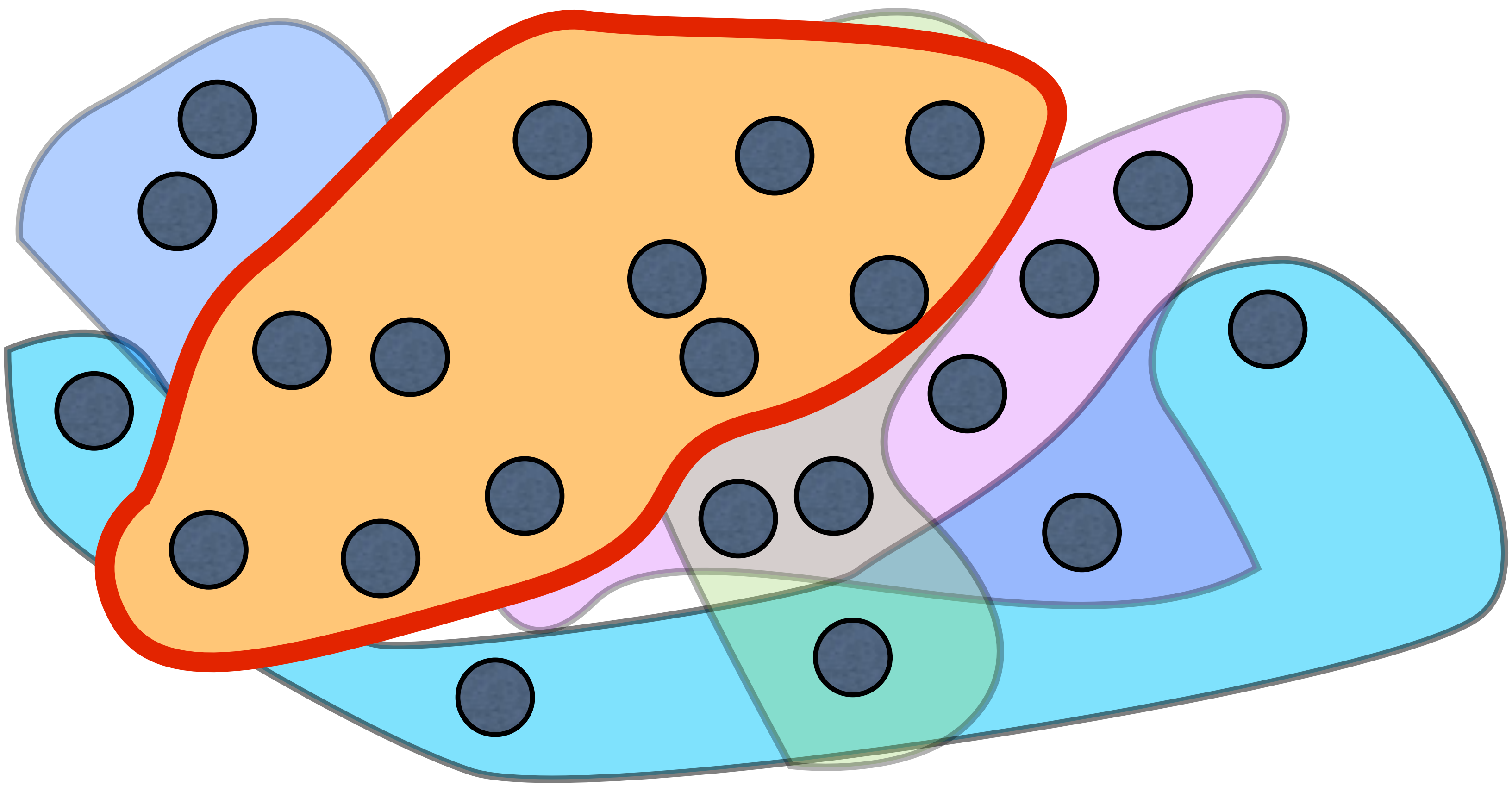
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



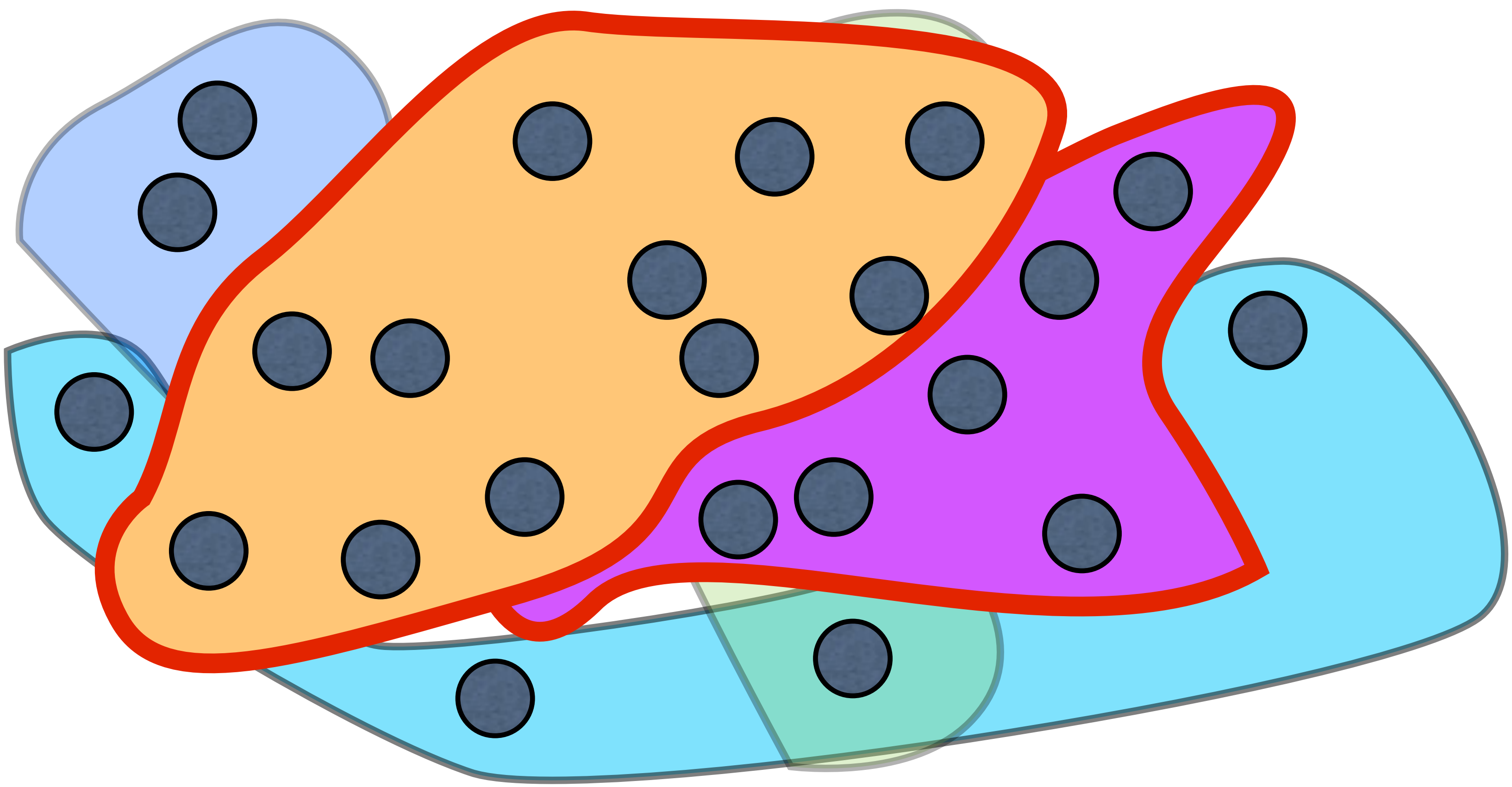
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



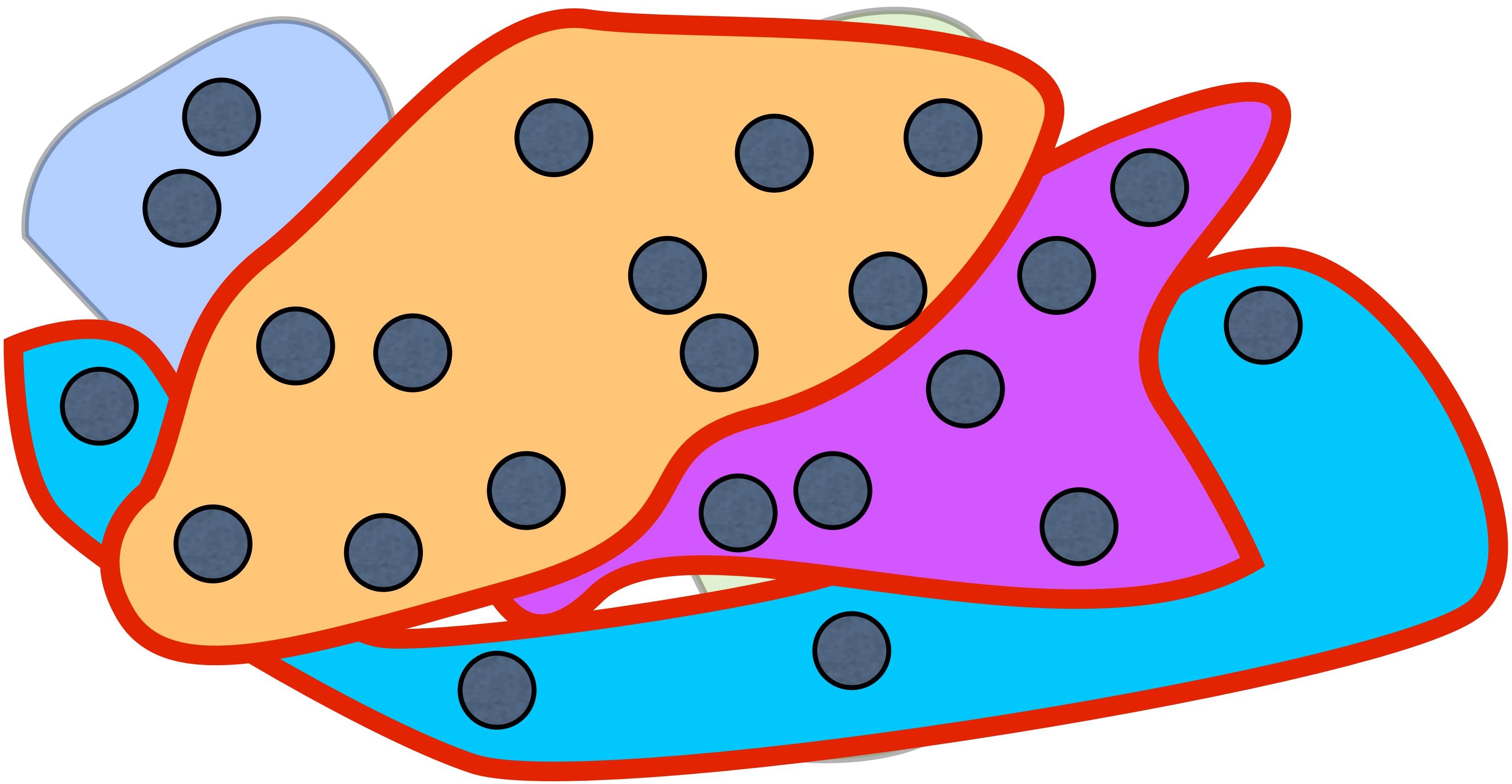
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered



A Greedy Algorithm

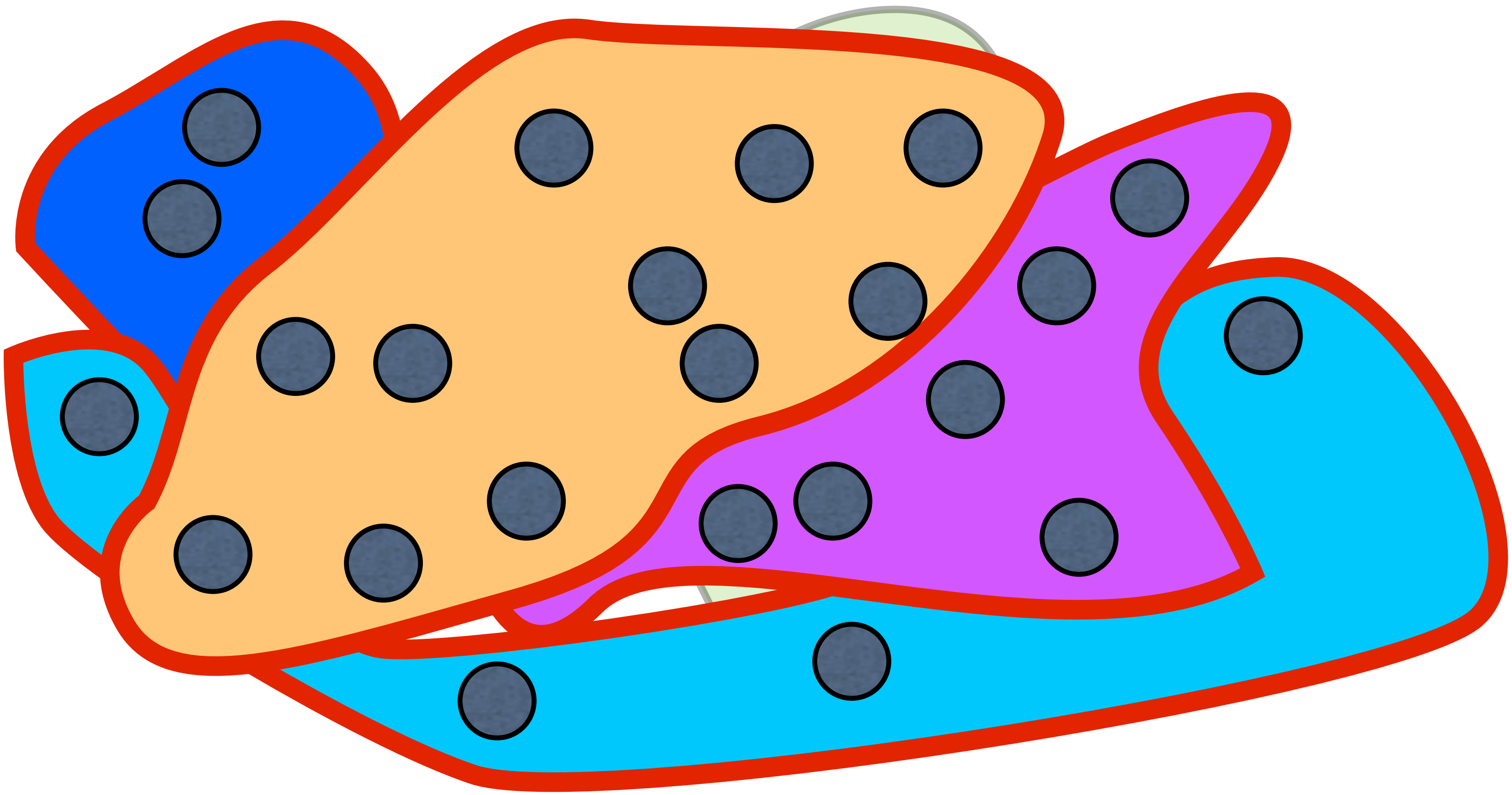
Strategy: Pick the set that maximizes # new elements covered



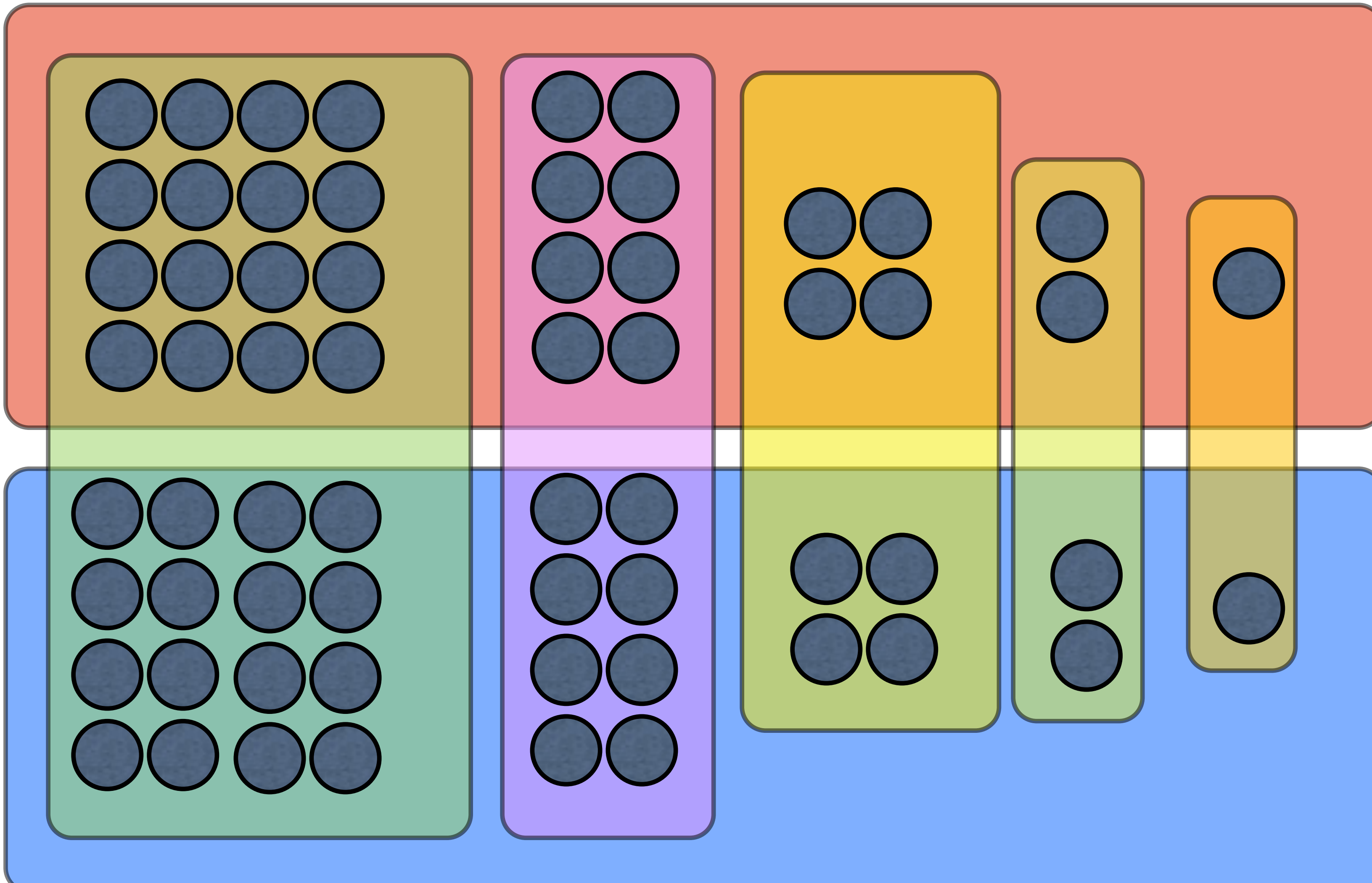
A Greedy Algorithm

Strategy: Pick the set that maximizes # new elements covered

Thm: Greedy has ln n approximation ratio



A Tight Example for Greedy



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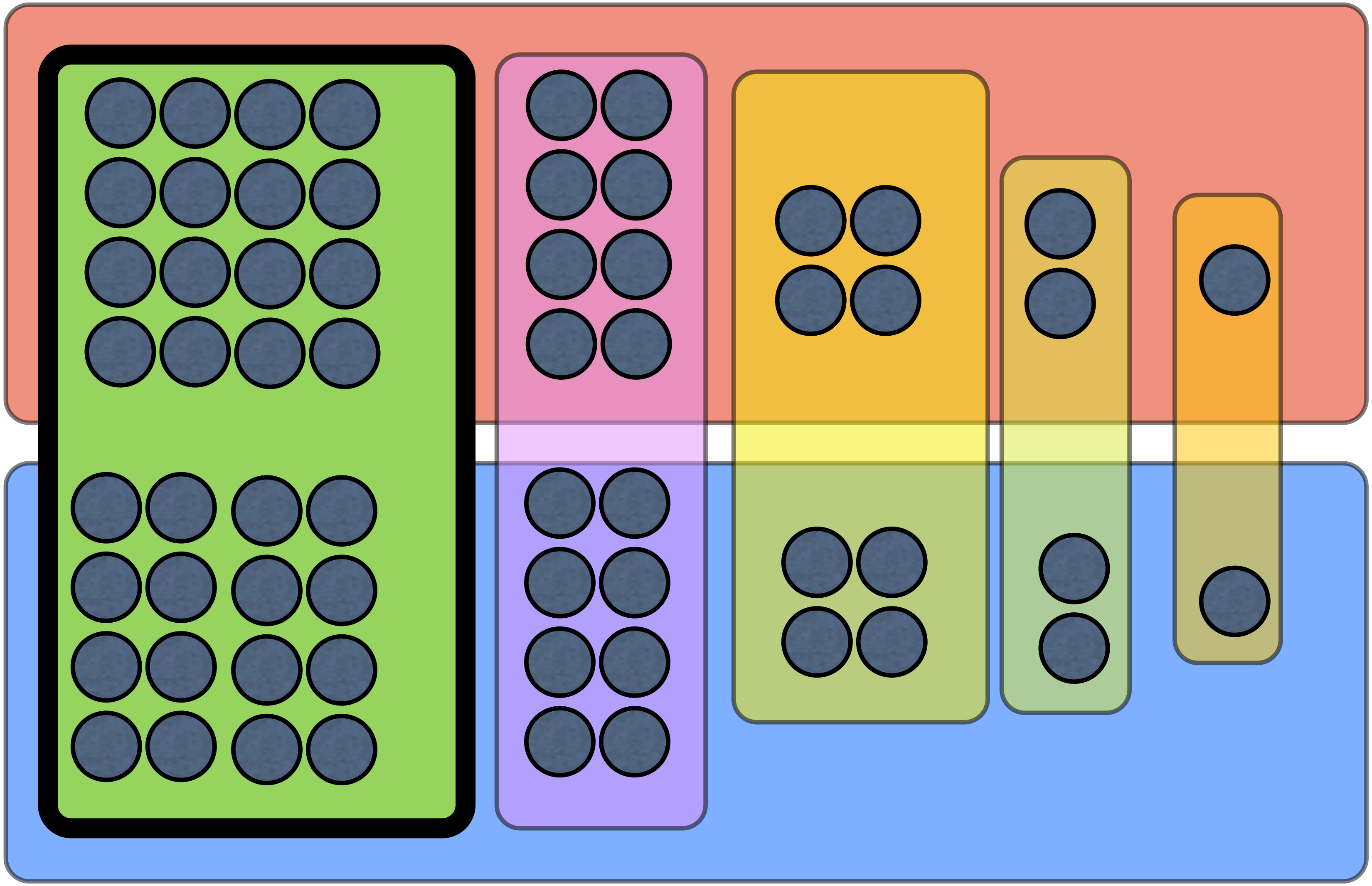
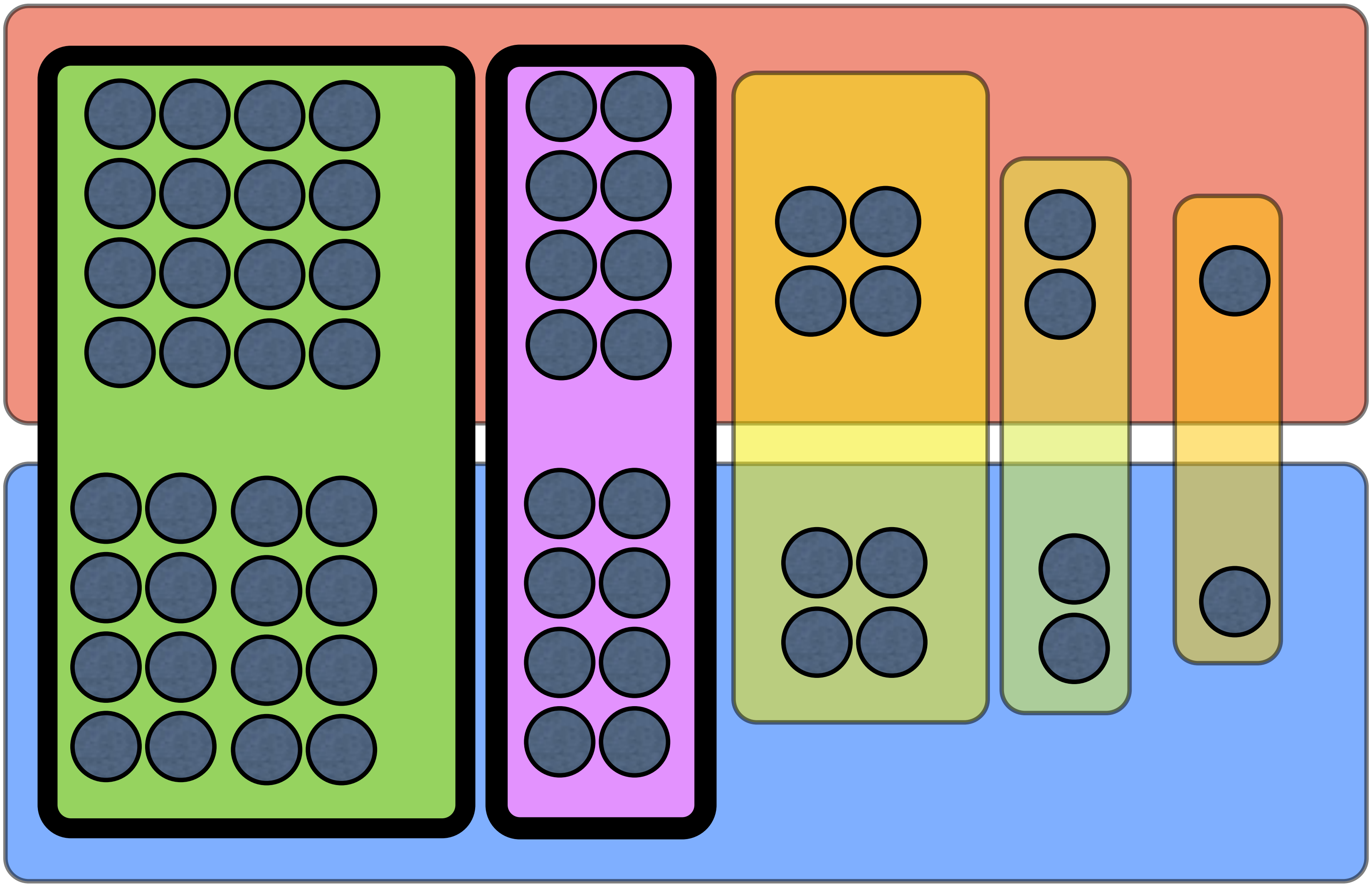


A Tight Example for Greedy



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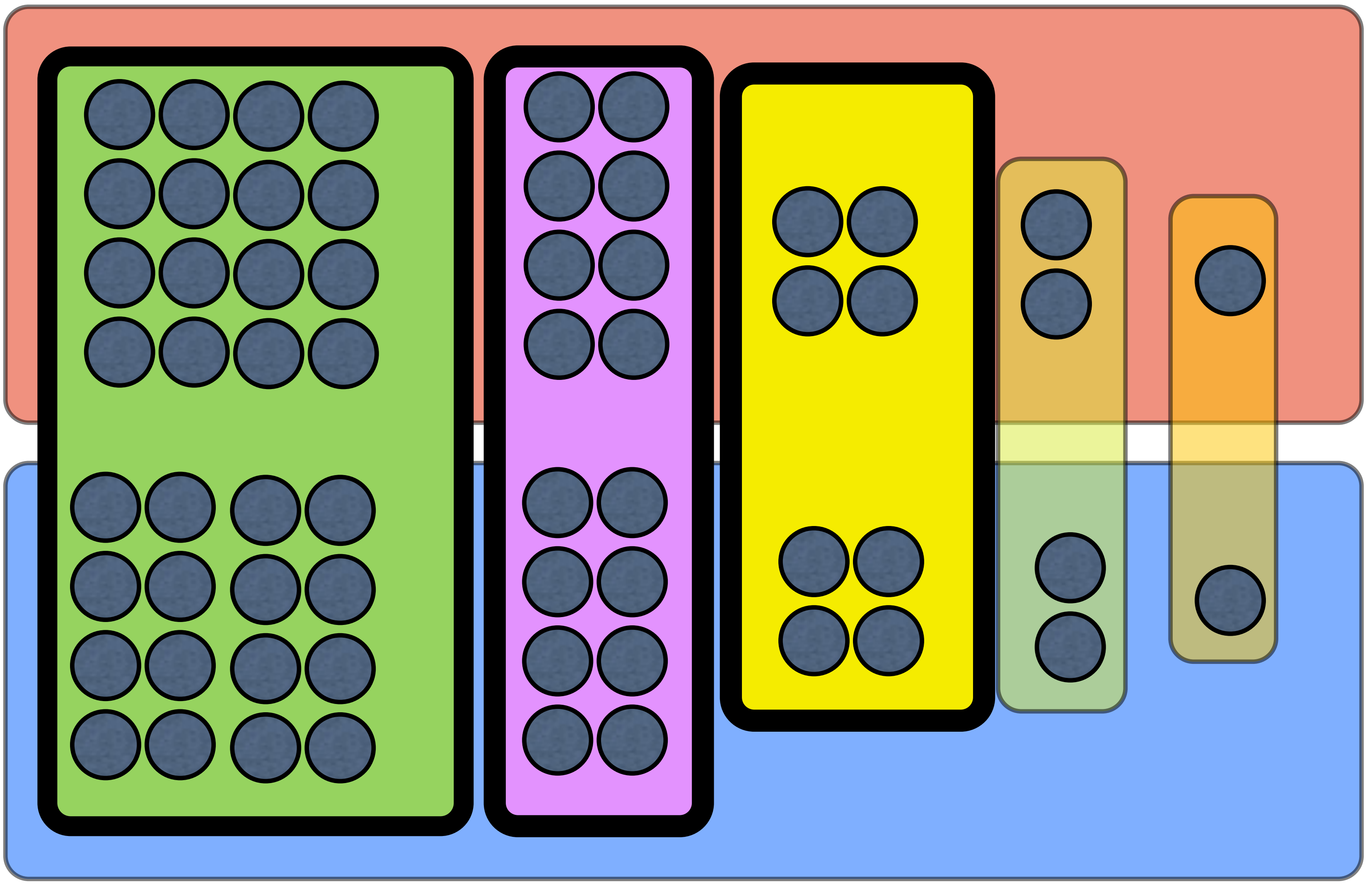
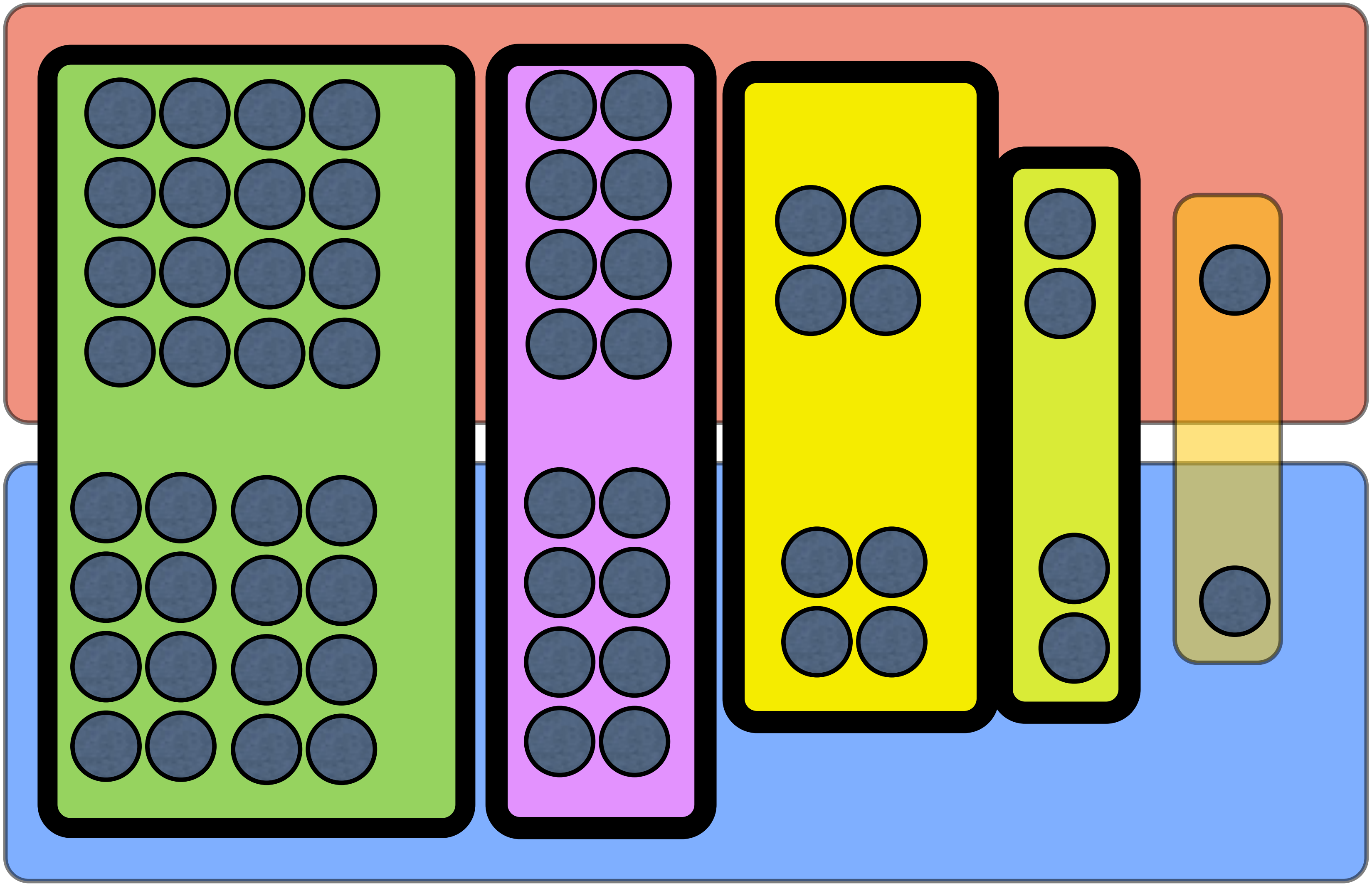
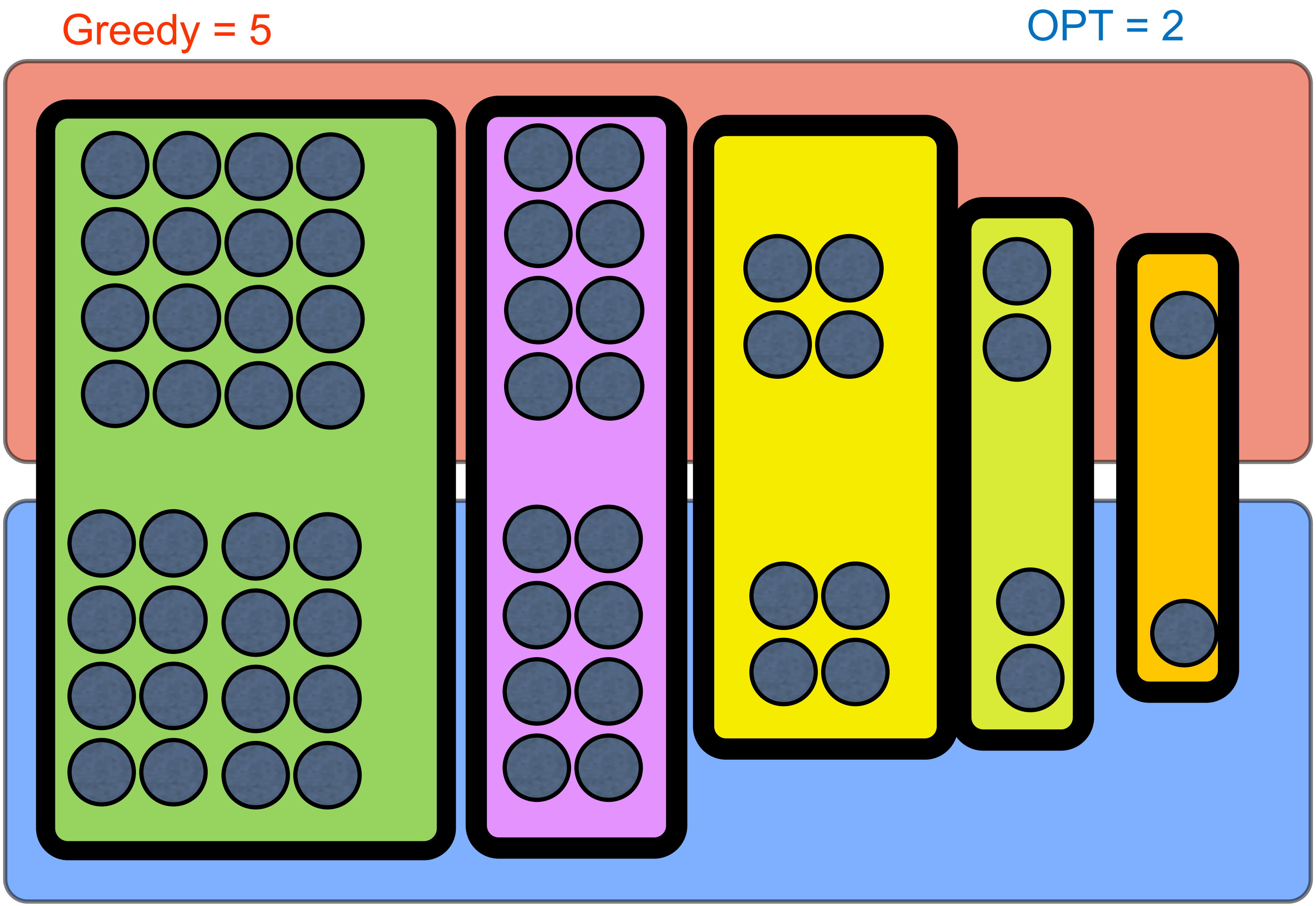


A Tight Example for Greedy



A Tight Example for Greedy



Greedy Gives O(log(n)) approximation

Thm: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf: Suppose OPT=k

There is a set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements.

So in each step, algorithm will cover 1/k fraction of remaining elements.

#elements uncovered after t steps

$$\leq n\left(1-\frac{1}{k}\right)^t \leq ne^{-\frac{t}{k}}$$

So after $t = k \ln n$ steps, # uncovered elements < 1.

Approximation Alg Summary

- To design approximation Alg, always find a way to lower bound OPT

- The best known approximation Alg for vertex cover is the greedy.
  - It has been open for 50 years to obtain a polynomial time algorithm with approximation ratio better than 2

- The best known approximation Alg for set cover is the greedy.
  - It is NP-Complete to obtain better than ln n approximation ratio for set cover.

Dynamic Programming

Algorithmic Paradigm

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems. Memorize the answers to obtain polynomial time ALG.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

Dynamic programming = planning over time.

Secretary of Defense was hostile to mathematical research.

Bellman sought an impressive name to avoid confrontation.

- "it's impossible to use dynamic in a pejorative sense"
- "something not even a Congressman could object to"

Dynamic Programming Applications

Areas:

- Bioinformatics
- Control Theory
- Information Theory
- Operations Research
- Computer Science: Theory, Graphics, AI, …

Some famous DP algorithms

- Viterbi for hidden Markov Model
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Dynamic Programming

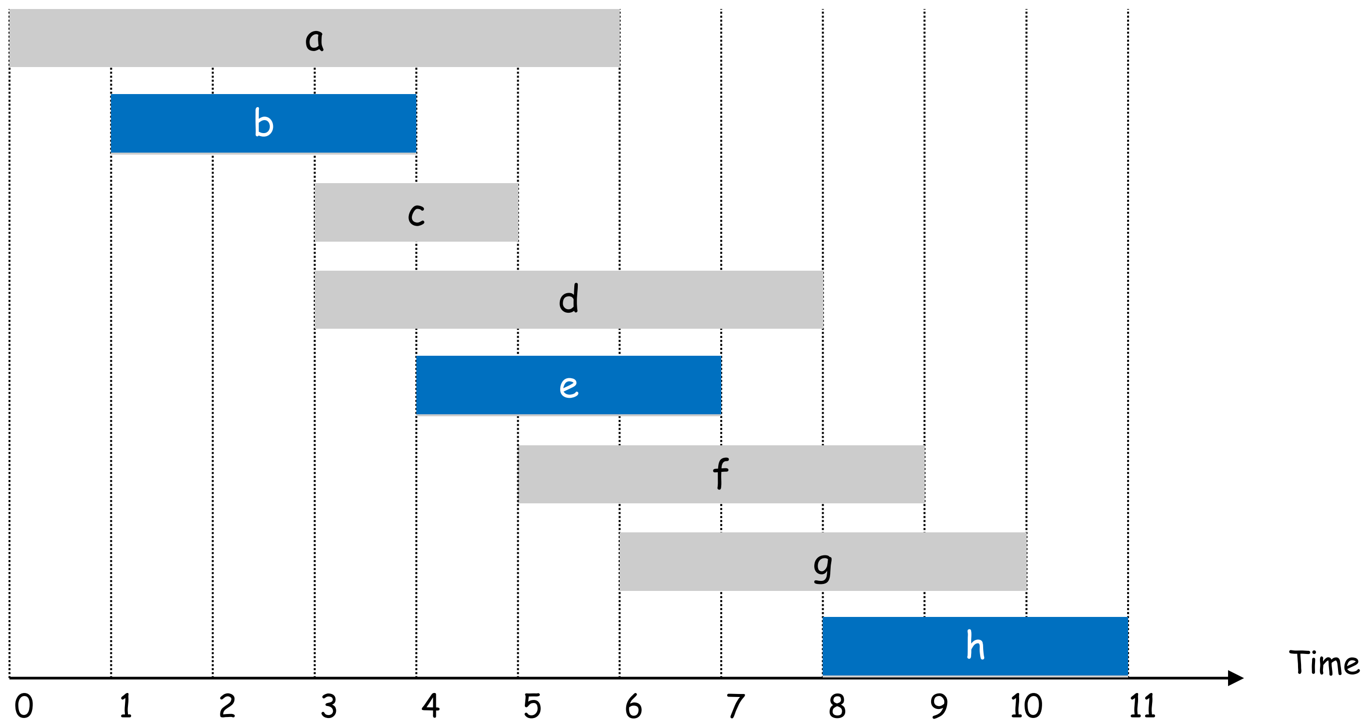
Dynamic programming is nothing but algorithm design by induction!

We just ”remember” the subproblems that we have solved so far to avoid re-solving the same sub-problem many times.

Weighted Interval Scheduling

Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$ and has weight w_j
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

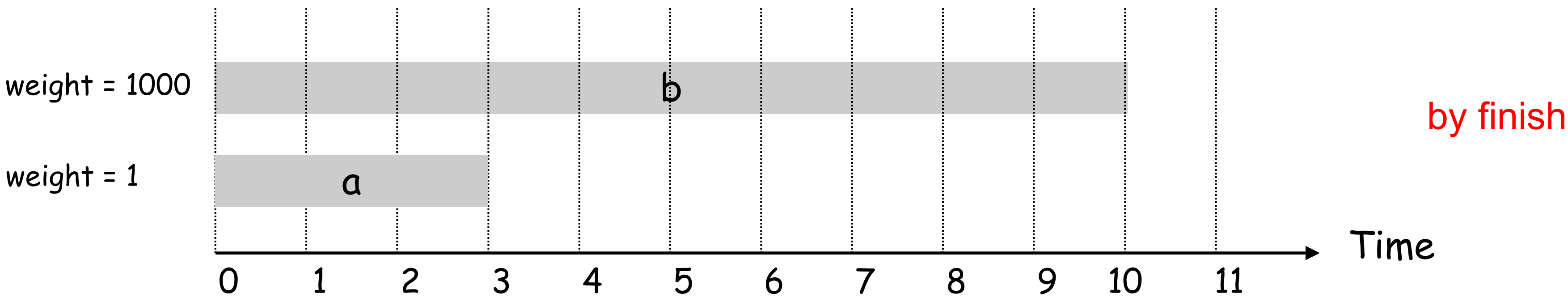


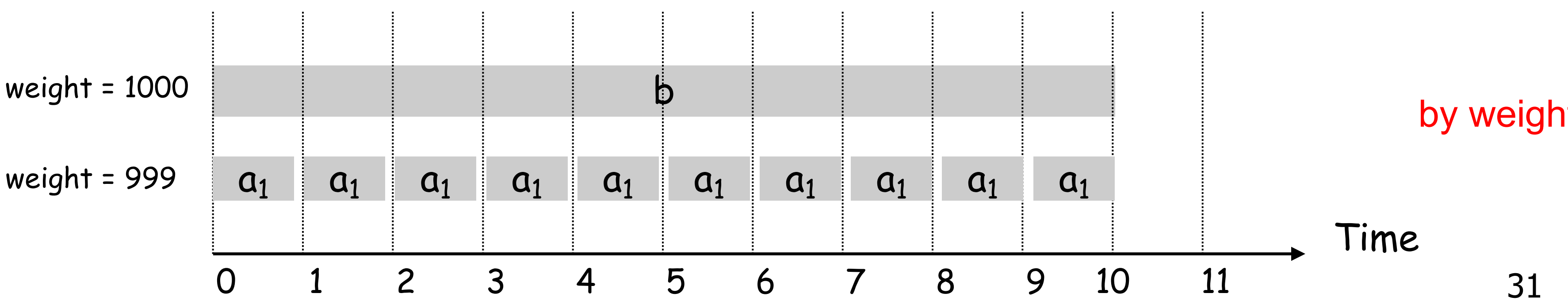
Unweighted Interval Scheduling: Review

Recall: Greedy algorithm works if all weights are 1:

- Consider jobs in ascending order of finishing time
- Add job to a subset if it is compatible with prev added jobs.

OBS: Greedy ALG fails spectacularly (no approximation ratio) if arbitrary weights are allowed:





Weighted Job Scheduling by Induction

Suppose $1, \ldots, n$ are all jobs. Let us use induction:

IH (strong ind): Suppose we can compute the optimum job scheduling for $< n$ jobs.

IS: Goal: For any n jobs we can compute OPT.

Case 1: Job n is not in OPT.

-- Then, just return OPT of $1, \ldots, n-1$.

Take best of the two

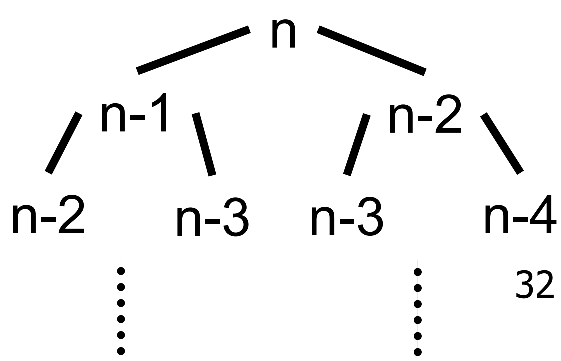
Case 2: Job n is in OPT.

-- Then, delete all jobs not compatible with n and recurse.

Q: Are we done?

A: No, How many subproblems are there?

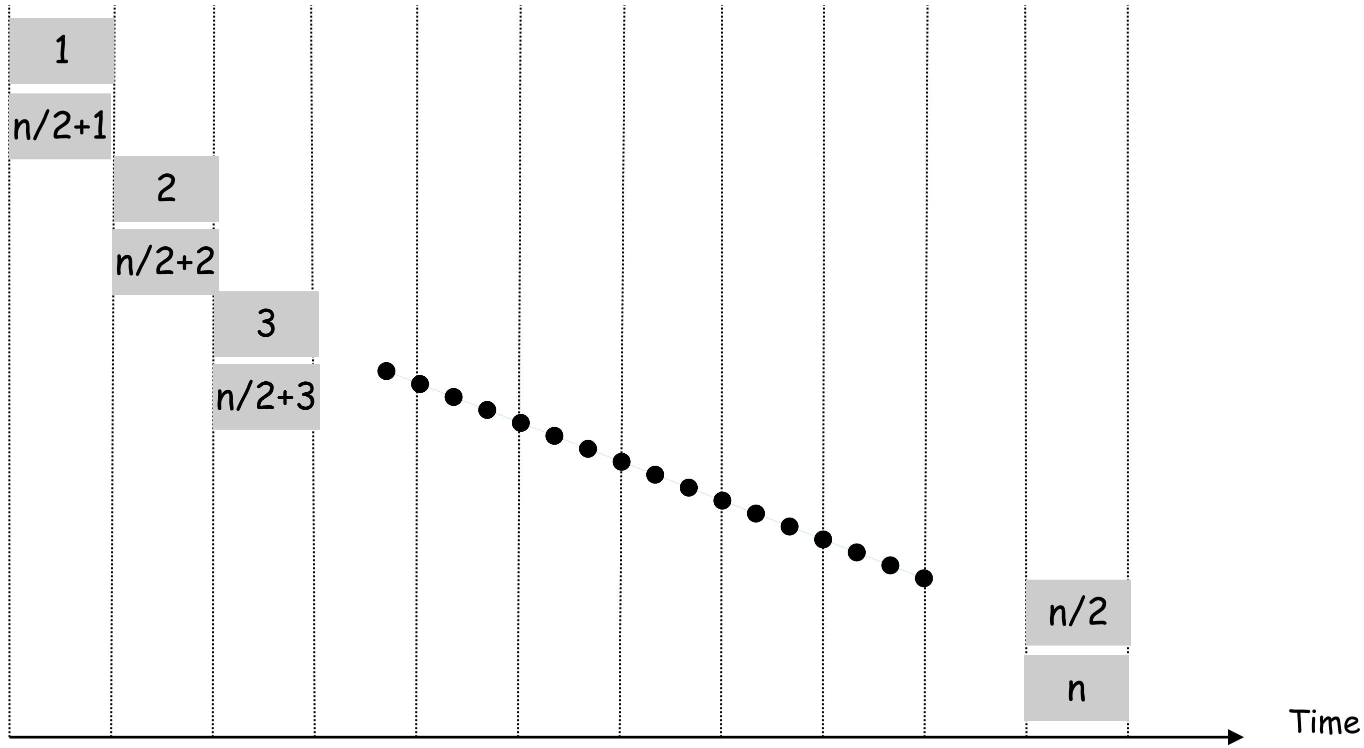
Potentially 2^n all possible subsets of jobs.



A Bad Example

Consider jobs n/2+1,…,n. These decisions have no impact on one another.

How many subproblems do we get?



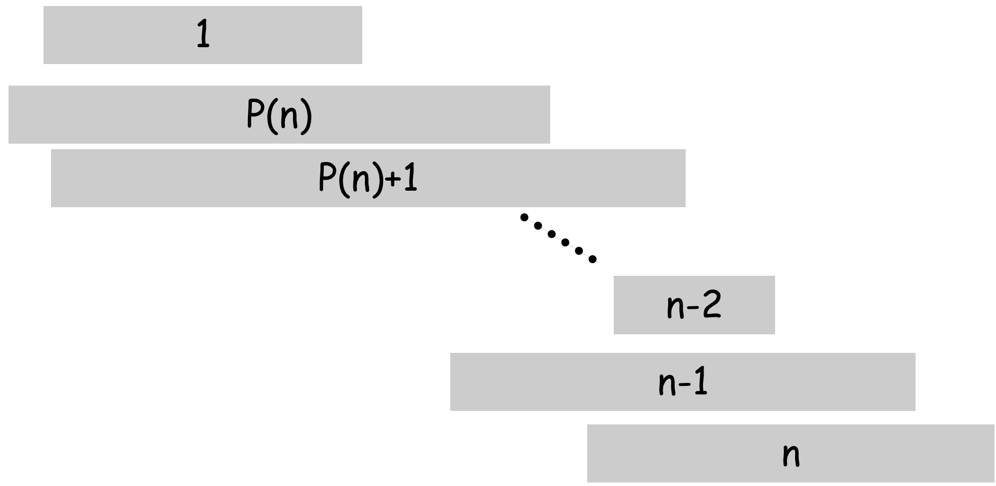
Sorting to Reduce Subproblems

IS: For jobs 1,…,n we want to compute OPT

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let p(n) = largest index i < n such that job i is compatible with n.
- Then, we just need to find OPT of $1, \ldots, p(n)$



Sorting to reduce Subproblems

IS: For jobs 1,…,n we want to compute OPT

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let p(n) = l… patible with n.
- Then, …

This is how we differentiate from solving Maximum Independent Set Problem

Case 2: OPT d…

- Then, OPT is just the optimum $1, \ldots, n-1$

Take best of the two

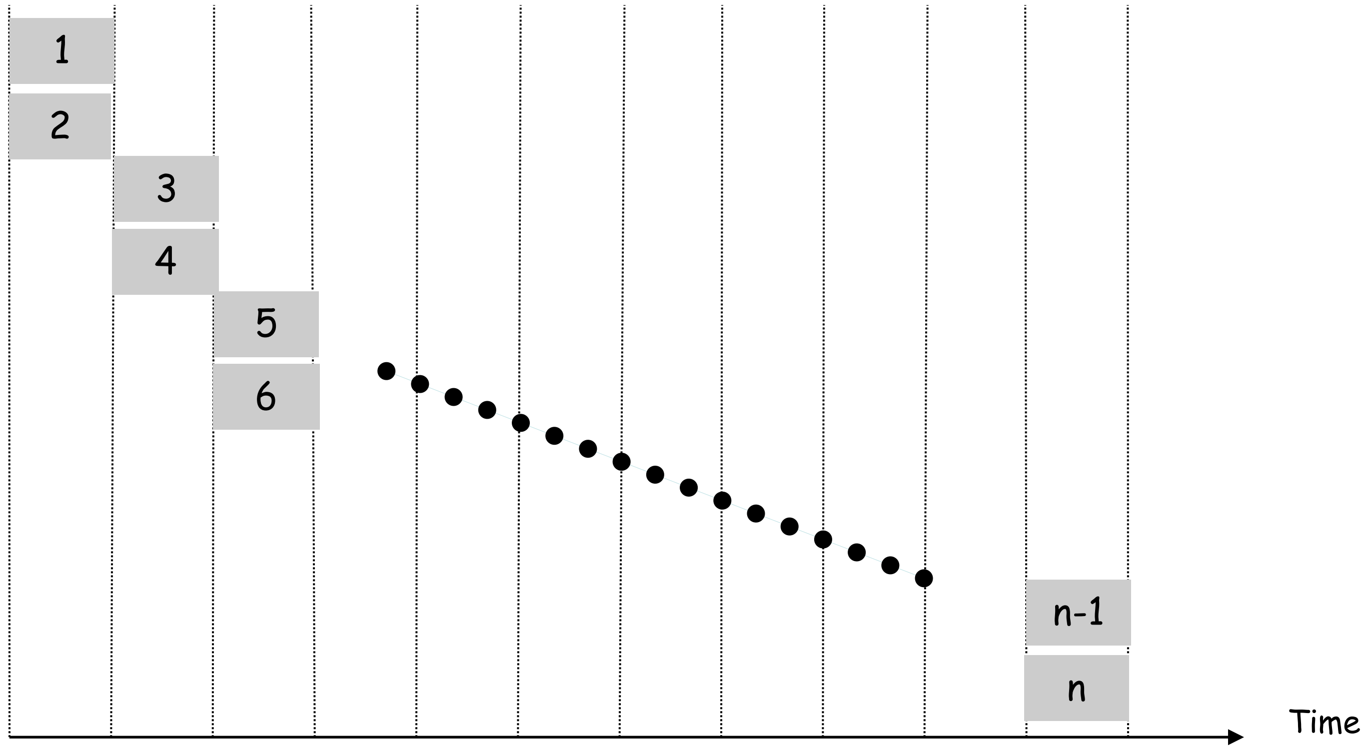
Q: Have we made any progress (still reducing to two subproblems)?

A: Yes! This time every subproblem is of the form $1, \ldots, i$ for some i

So, at most n possible subproblems.

Bad Example Review

How many subproblems do we get in this sorted order?



Weighted Job Scheduling by Induction

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

Let OPT(j) denote the OPT solution of $1, \ldots, j$

> This is the most important step in design DP algorithms

To solve OPT(j):

Case 1: OPT(j) has job j,

- So, all jobs i that are n
- Let p(j) = largest index i j.
- So $OPT(j) = OPT\big(p(j)\big) \cup \{j\}$.

Case 2: OPT(j) does not select job j.

- Then, $OPT(j) = OPT(j-1)$

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\Big(w_j + OPT\big(p(j)\big), OPT(j-1)\Big) & \text{o.w.} \end{cases}$$

Algorithm

Input: n, $s(1), \ldots, s(n)$ and $f(1), \ldots, f(n)$ and $w_1, \ldots, w_n$.

Sort jobs by finish times so that $f(1) \leq f(2) \leq \cdots f(n)$.

Compute $p(1), p(2), \ldots, p(n)$

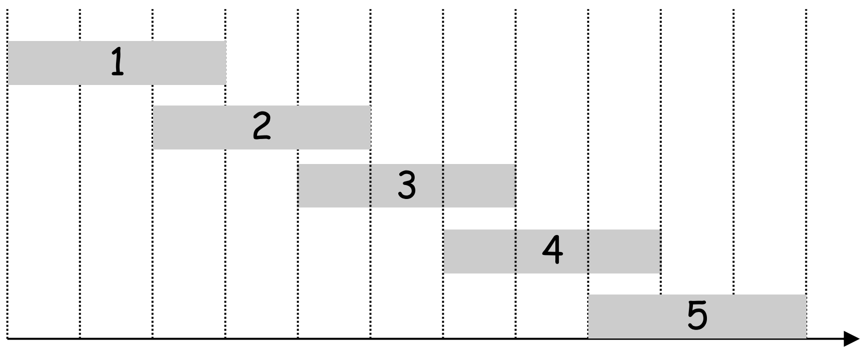
```
Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(w_j + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Recursive Algorithm Fails

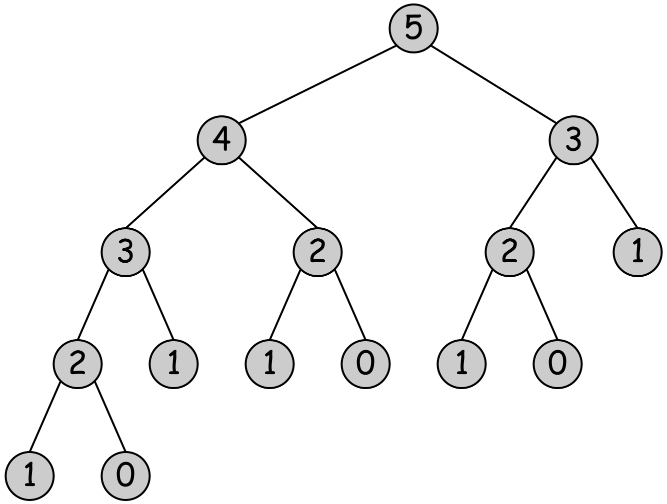
Even though we have only n subproblems, we do not store the solution to the subproblems

- ➢ So, we may re-solve the same problem many many times.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence



p(1) = 0, p(j) = j-2



Algorithm with Memoization

Memoization. Compute and Store the solution of each sub-problem in a cache the first time that you face it. lookup as needed.

```
Input: n, s(1),…,s(n)  and f(1),…,f(n) and w_1,…,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),…,p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(w_j + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Bottom up Dynamic Programming

You can also avoid recusion

- recursion may be easier conceptually when you use induction

Input: n, $s(1), \ldots, s(n)$ and $f(1), \ldots, f(n)$ and $w_1, \ldots, w_n$.

Sort jobs by finish times so that $f(1) \le f(2) \le \cdots f(n)$.

Compute $p(1), p(2), \ldots, p(n)$

```
Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(w_j + M[p(j)], M[j-1])
}

Output M[n]
```

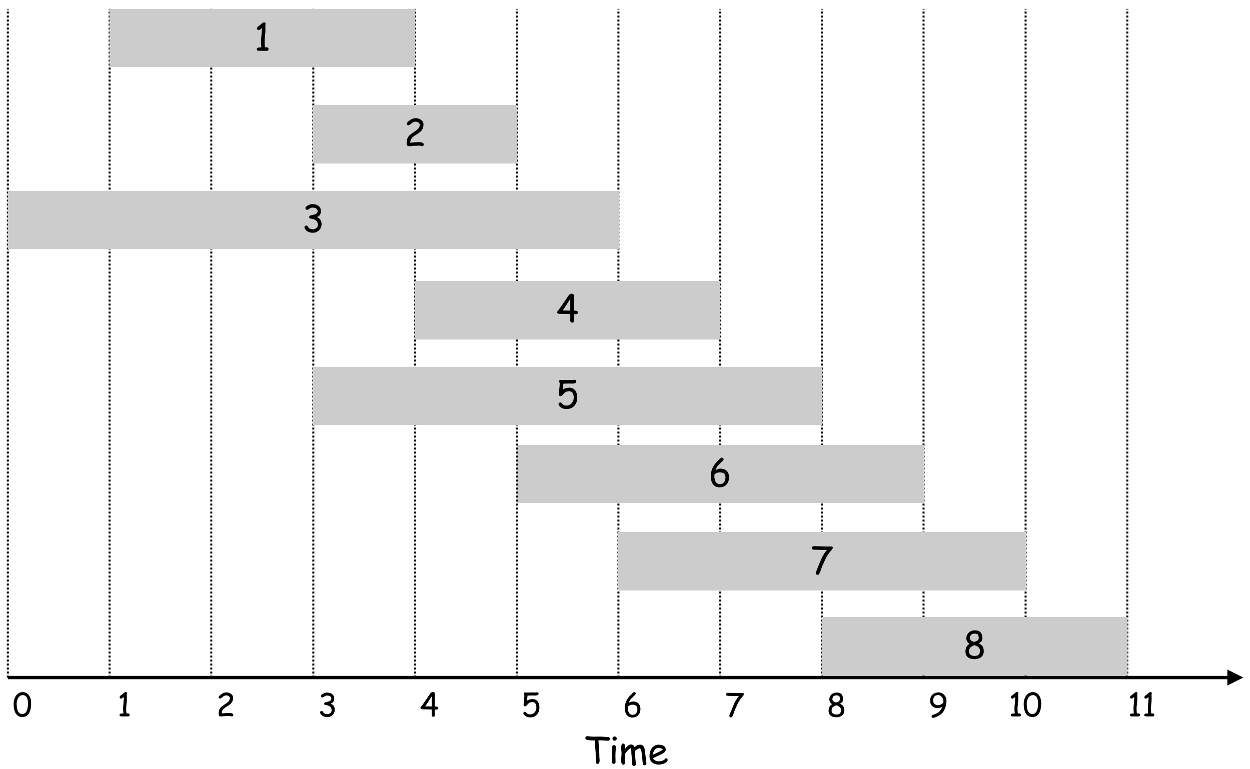
Claim: M[j] is value of OPT(j)

Timing: Easy. Main loop is O(n); sorting is O(n log n)

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

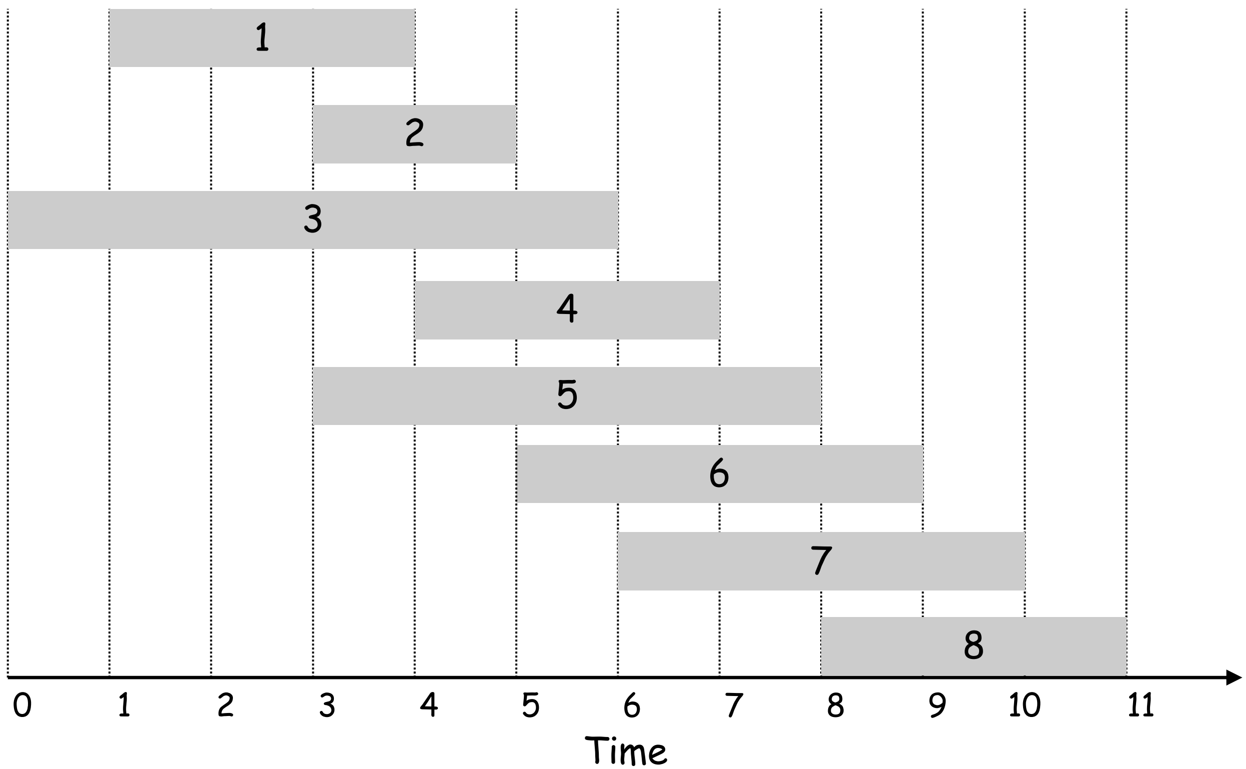


j	w_j	p(j)	OPT(j
0			0
1	3	0	
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

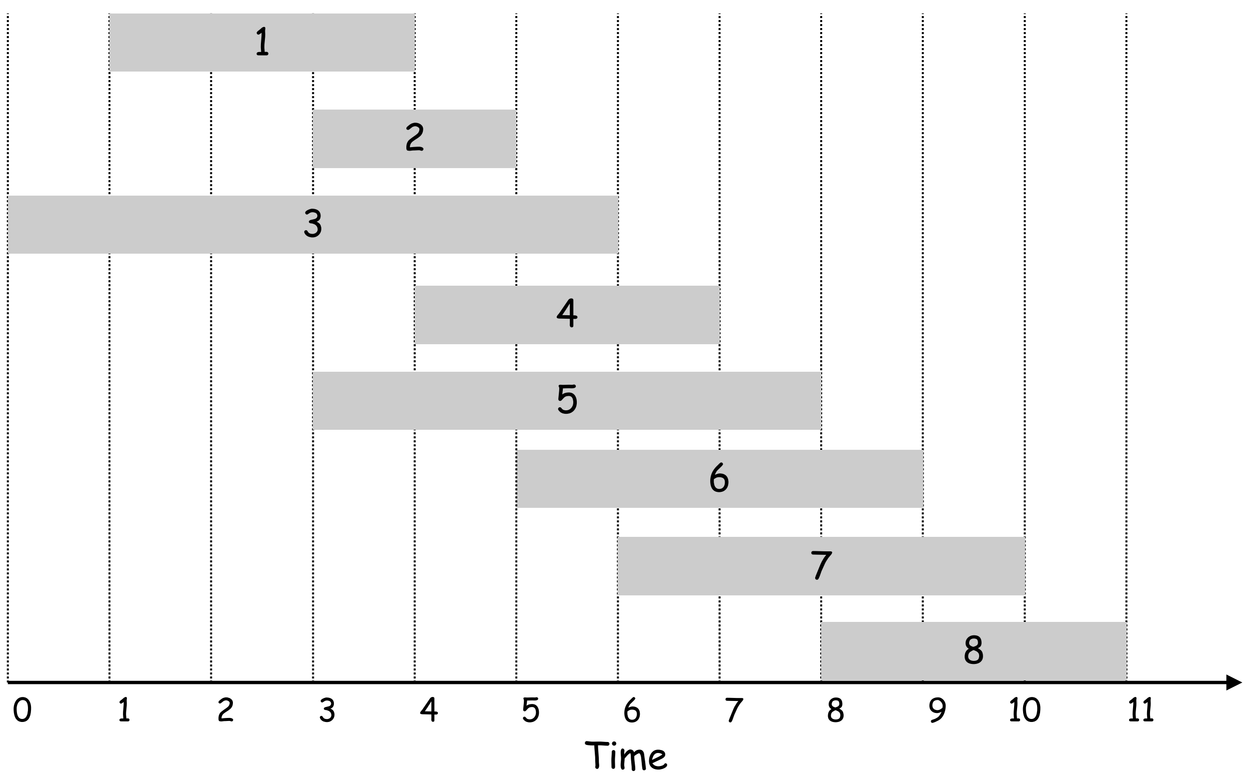


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

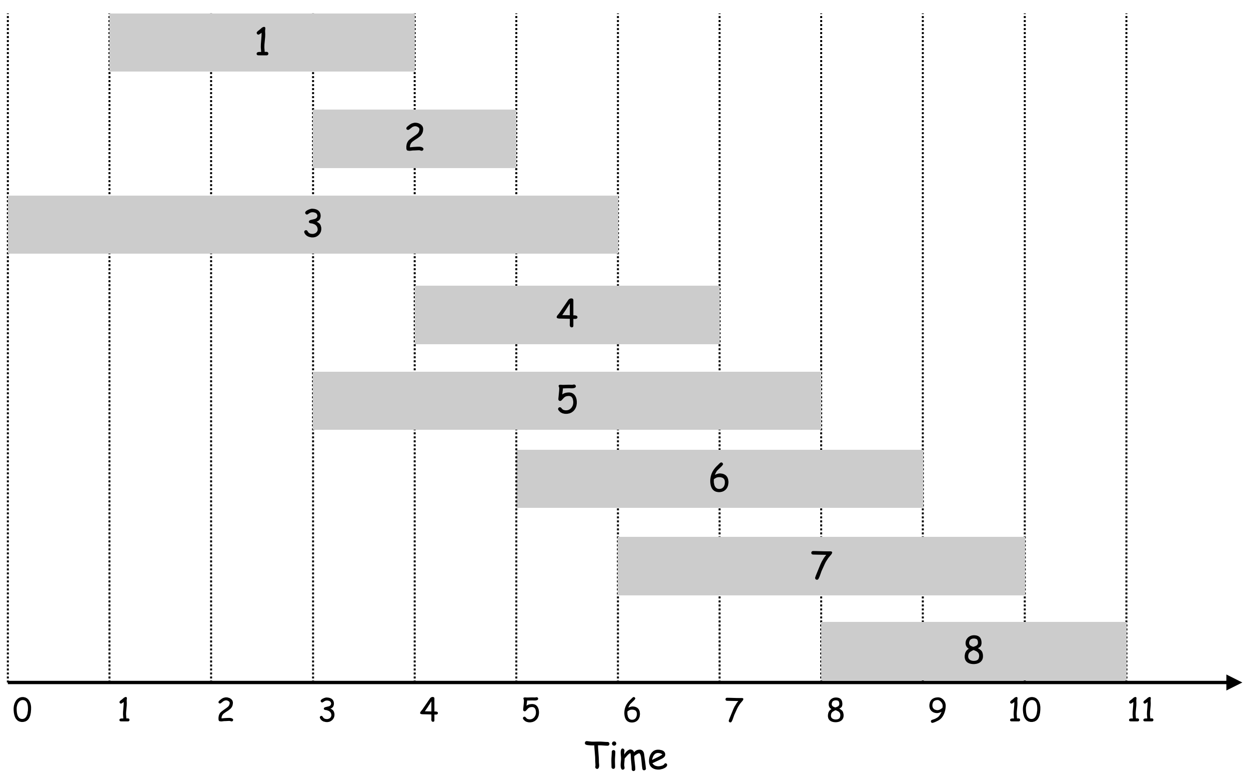


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	**4**
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

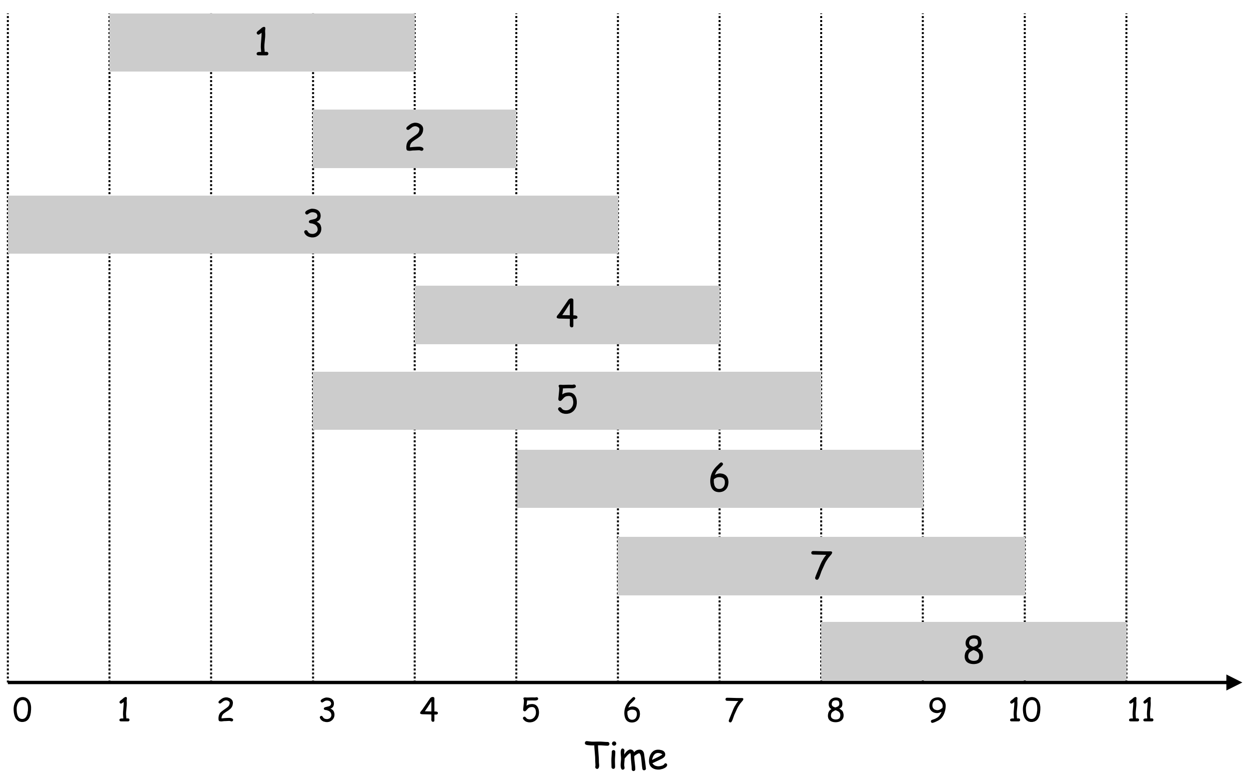


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

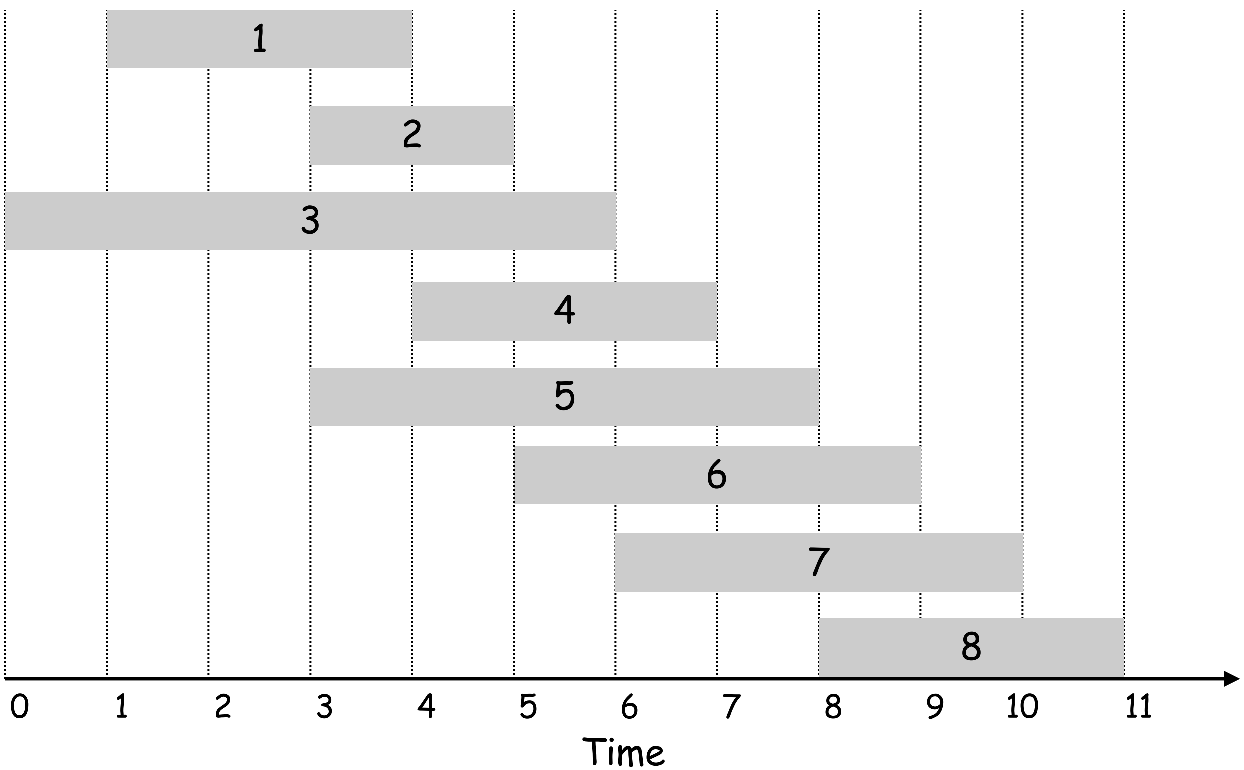


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

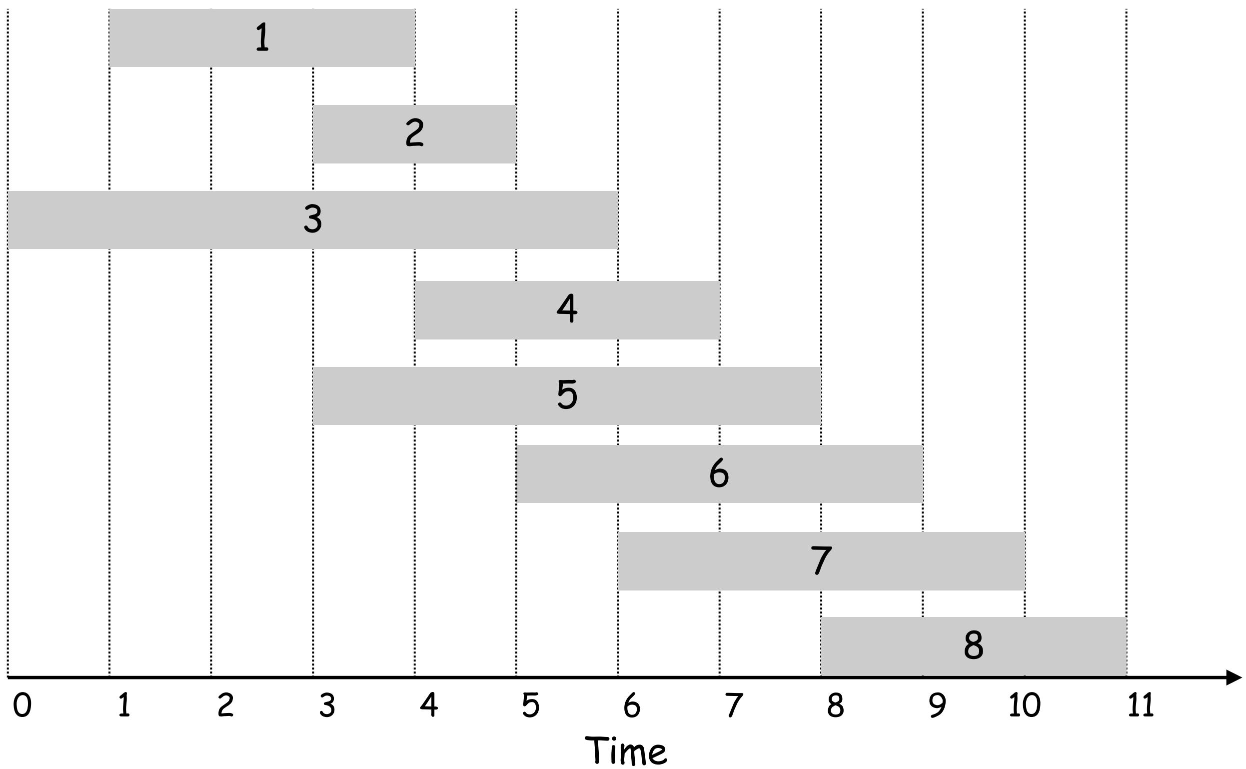


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

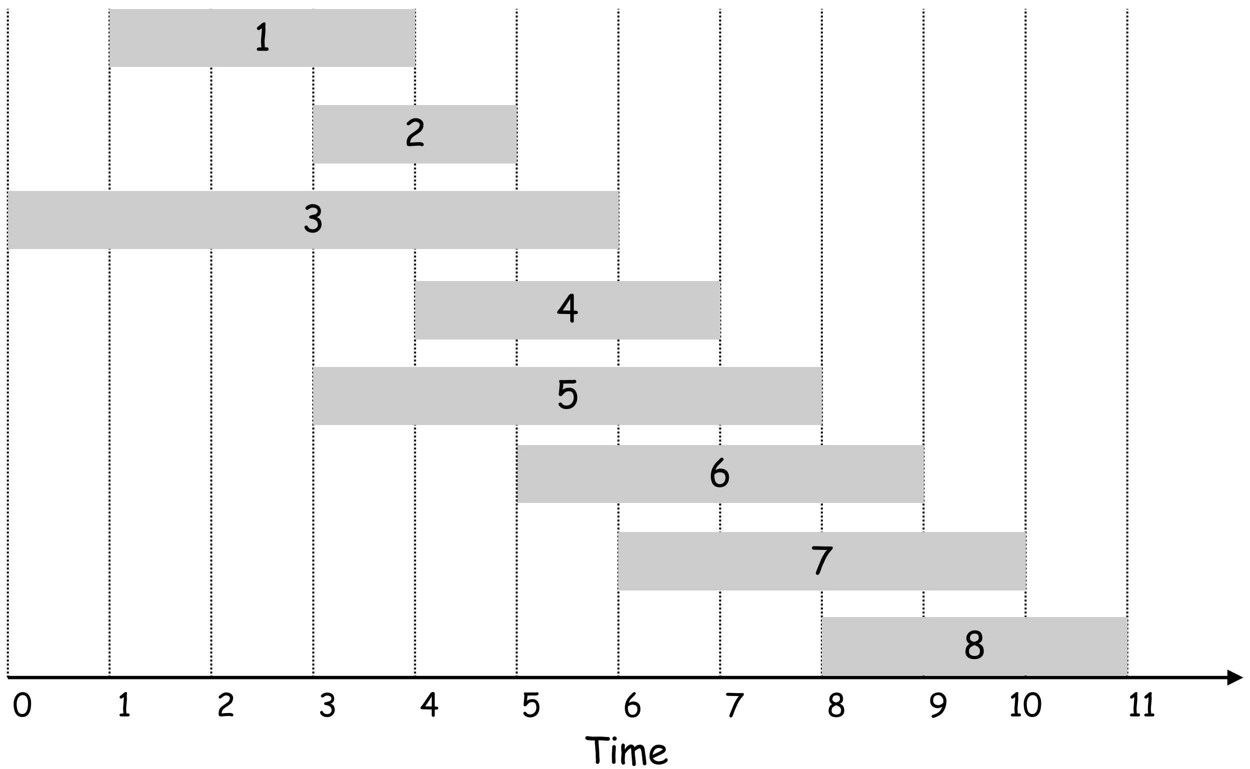


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

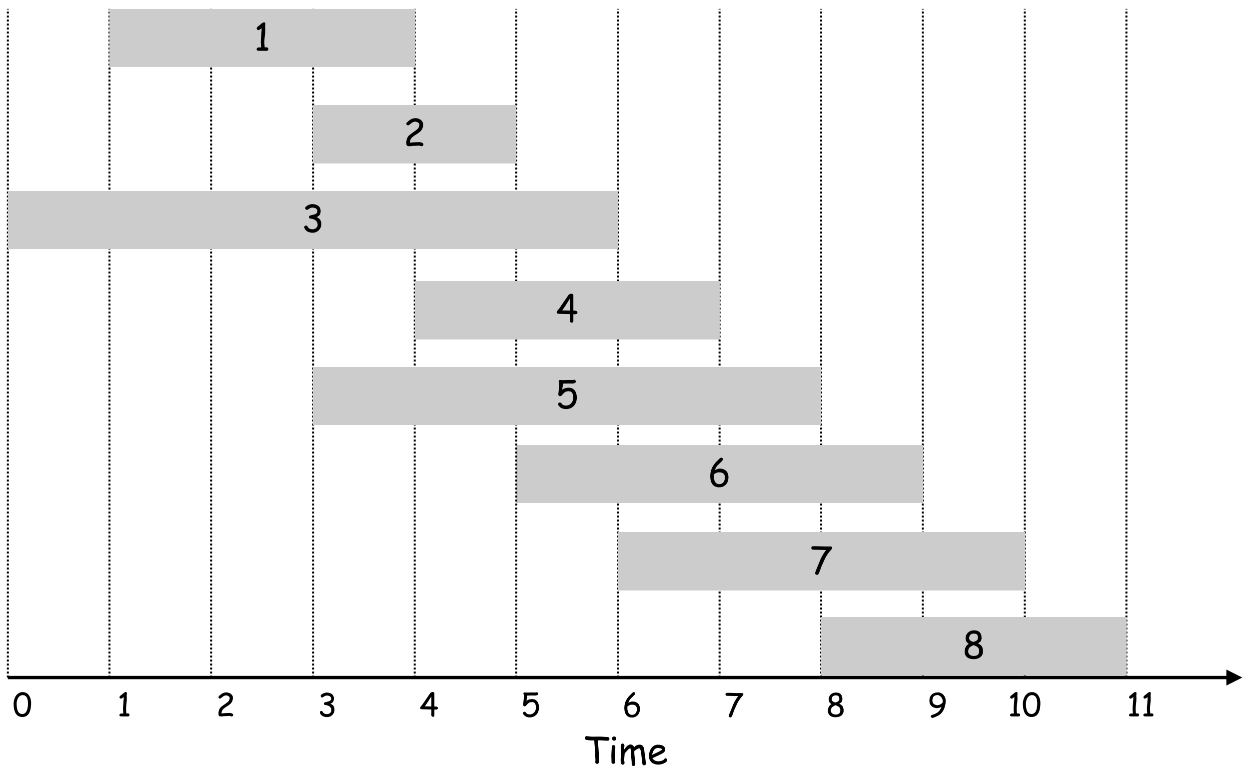


j	w_j	p(j)	OPT(j
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.

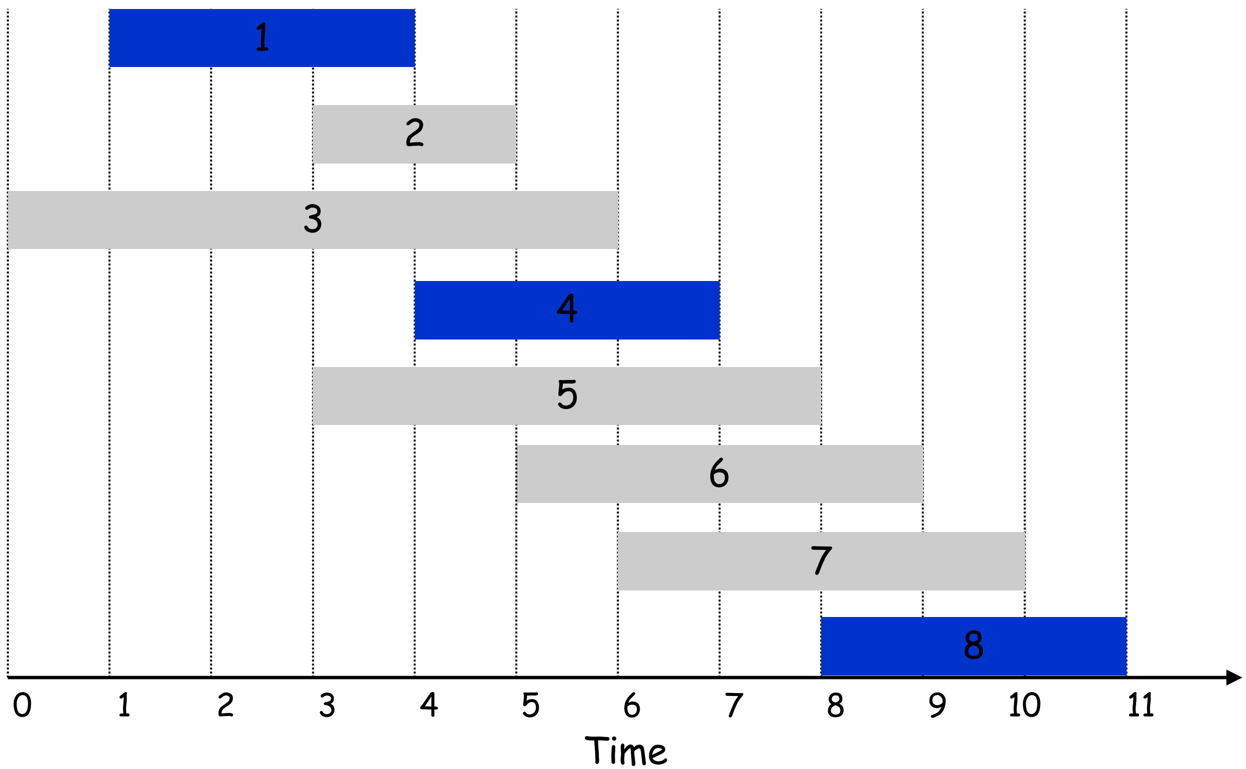


j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**

Example

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

p(j) = largest index i < j such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**