1 Asymptotics

Some properties of asymptotics:

- If $f \leq O(g)$ and $g \leq O(h)$ then $f \leq O(h)$.
- If $f \geq \Omega(g)$ and $g \geq \Omega(h)$ then $f \geq \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.
- If $f = O(h)$, $g = O(h)$ then $f + g = O(h)$.

Some common running times:

- Polynomial: $O(n^d)$. Exponential $2^{O(n)}$, Logarithmic $O(\log n)$.
- For every positive ϵ (no matter how small), $\log n \leq O(n^\epsilon)$. For every positive d (no matter how large), $n^d \leq O(2^n)$.

2 In class exercise

Arrange in increasing order of asymptotic growth. All logs are in base 2.

a) $n^{5/3} \log^2 n$

b) $2^{\sqrt{\log n}}$

c) $\sqrt{n^n}$

d) $\frac{n^2}{\log n}$

e) 2^n.

Hint: Recall rules of logarithm

- $\log(a \cdot b) = \log a + \log b$,
- $\log(a/b) = \log a - \log b$.
- $\log a^b = b \log a$.

Always keep in mind $n = 2^{\log_2 n}$. For example, $n^{1.5} = 2^{1.5 \log_2 n}$. Also recall and that $(2^a)^b = 2^{a \cdot b}$. Furthermore, $2^{(a^b)} \neq (2^a)^b$.

3 Solution

In this part I will discuss the solution to the exercise. In many cases it might be difficult to directly compare two function $f(n), g(n)$ asymptotically. An idea that usually helps out is to compare $\log f(n)$ with $\log g(n)$. Recall that logarithm is an increasing function, so if $f(n) > g(n)$ for some $n > N$ then $\log f(n) > \log g(n)$ for $n > N$. Here are two important rules when comparing the logs.

When comparing logarithms ignore additive constants: We said in class that $n, 3n$ are asymptotically the same. If you take the log, then you are comparing $\log n$ with $\log n + \log 3$. So, you can ignore the additive $\log 3$.

When comparing logarithms, multiplicative constants matter: Consider the two functions n, n^2. Obviously n^2 grows asymptotically faster. When comparing the log, we have $\log n, 2\log n$. So, the 2 multiplicative constant matters and shows that n^2 grows faster.

Having said these, I will write down the solution to exercise. First, we calculate the logs:

a) $\log n^{5/3} \log^2 n = \frac{5}{3}\log n + 2\log\log n$.

b) $\log 2^{\sqrt{\log n}} = \sqrt{\log n}\log 2 = \sqrt{\log n}$.

c) $\log\sqrt{n^n} = \frac{1}{2}\log n^n = \frac{n}{2}\log n$.

d) $\log\frac{n^2}{\log n} = \log n^2 - \log\log n = 2\log n - \log\log n$.

e) $\log 2^n = n\log 2 = n$.

Therefore, $\sqrt{\log n} < \frac{5}{3}\log n < 2\log n - \log\log n < n < \frac{n}{2}\log n$. Note that when comparing $\frac{5}{3}\log n + 2\log\log n$ and $2\log n - \log\log n$, $\log\log n$ is a lower order term. So, first we compare the dominating terms $5/3\log n$ and $2\log n$. In this case the latter is bigger. If the dominating term were the same then we would have compared to lower order terms.