1 Triangles in Graphs (Optional)

Theorem 1. *If a graph on $2n$ vertices has n^2+1 edges, then it has a triangle.*

Proof We prove it by induction on n. $P(n) =$"Any graph $G=(V,E)$ with $2n$ vertices and $m \geq n^2+1$ edges has a triangle.

Base Case: $P(1)$: When $n=1$, $P(1)$ holds since the number of edges is at most $1 < n^2+1$.

IH: $P(n)$ holds for some $n \geq 1$.

IS: We prove $P(n+1)$. Let G be an arbitrary graph with $2(n+1)$ vertices and at least $m \geq (n+1)^2+1$ edges. Let $\{x,y\}$ be an arbitrary edge in the graph. Consider the graph $G' = G - x - y$ on $2n$ vertices obtained by **deleting** x, y (and all of their incident edges) from the original graph. If G' has at least n^2+1 edges, then by IH it has a triangle, and we are done.

Otherwise, G' has at most n^2 edges. Since G has at least $(n+1)^2+1$ edges, by removing x, y we have deleted $(n+1)^2+1-n^2 = 2n+2$ edges from G. Since $\{x,y\}$ is also an edge, there are at least $2n+1$ edges that connect x, y to the vertices of G'. Thus by the pigeonhole principle, there is some vertex z so that $\{x,z\},\{y,z\}$ are both edges of G. But, then x, y, z form a triangle in G. ■

The above theorem is tight. Consider the graph with n vertices on the left and n vertices on the right and every vertex on the left is connected to every vertex on the right. This graph has no triangles but n^2 edges.

Also, note the importance of deletion in the induction. Here, we crucially used that the x, y pair deleted from G were **neighbors** in G.