CSE421: Design and Analysis of Algorithms April 15, 2022

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1 In Class Exercise

Theorem 1. *Let G be a graph with n vertices such that the degree of every vertex of G is at most k. Prove that we can color vertices of G with $k+1$ colors such that the endpoints of every edge get two distinct colors.*

Proof This problem is a bit more complex because there are two parameters that we can induct on: n and k. In this case, we let k as a fixed number in the entire proof and we will prove the statement by induction on n.

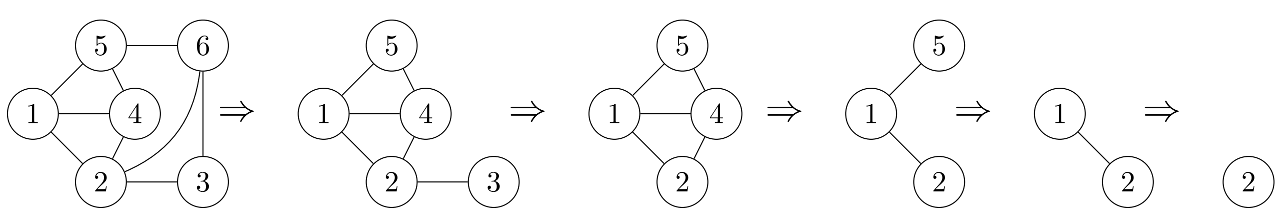
We prove by induction on n. First define $P(n)$ be "every graph with n vertices such that the degree of every vertex is at most k can be colored with $k+1$ colors such that the endpoints of every edge have two distinct colors".

Base Case: $n = 1$. In this case we color the single vertex with a color. We can do so because $k \geq 0$.

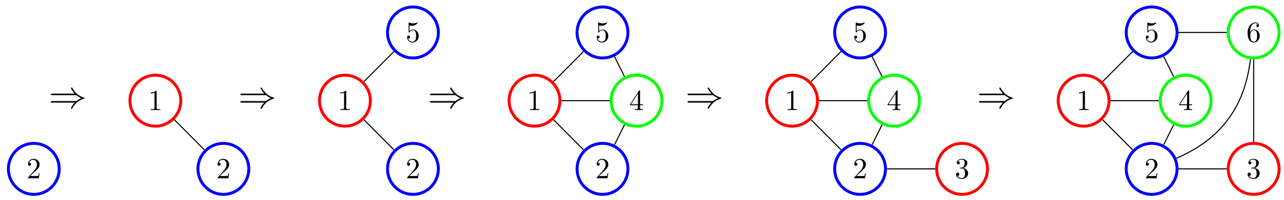
IH: Suppose $P(n-1)$ holds.

IS: We need to prove $P(n)$. Let G be an arbitrary graph with n vertices such that the degree of every vertex of G is at most k. Let v be an arbitrary vertex of G. Let $G' = G - v$ (we also remove all edges incident to v. Now, by removing v (and edges of v) we can only reduce degree of the rest of the vertices. Therefore, every vertex of G' also has degree at most k. Since G' has $n-1$ vertices by IH we can color vertices of G' with $k+1$ colors such that endpoints of every edge have distinct colors. Now, we color G. We color every vertex of G (except v) with the same color in G'. Now, to color v, note that it has at most k neighbors. Since we have $k+1$ colors there is a color that is not used in any of the neighbors of v. We color v with that color. ■

Note that this proof also gives an algorithm to color such a graph. Here is a sample execution of such an algorithm. Say $k = 3$, so we have 4 colors available. Say we remove vertices in the following order $6, 3, 4, 5, 1$.



Now, we ca color. First, we color the last vertex 2 with blue. Then, we add back the removed vertices and each time we use a color not used on the neighbors: Note that to color the last vertex 6 we got lucky. Even though it had 3 neighbors, two of them were color blue. So, we could color 6 with green and this way totally we used only 3 colors (of 4 available colors). We also had the option of coloring 6 with orange and that would also be a valid coloring.



2 Coloring Planar graphs

Theorem 2. *The vertices of any planar graph can be colored with* 6 *colors in such a way that every edge gets exactly two distinct colors.*

In order to prove the theorem first prove the following claim:

Claim 3. *In any planar graph there exists a vertex v with* $\deg(v) \leq 5$.

Proof of Claim 3: **Hint:** Feel free to use the following fact without proof:

Fact 4. *For any planar graph with n vertices and m edges we have* $3n - 4 \geq m$.

First, recall that for any graph G

$$\sum_v \deg(v) = 2m.$$

But since by claim assumption, $2m \leq 6n - 8$, we have $\sum_v \deg(v) \leq 6n - 8$.

We prove by contradiction that there exists a vertex v with $\deg(v) \leq 5$. If for all v, $\deg(v) \geq 6$, then

$$6n - 4 \geq \sum_v \deg(v) \geq 6n$$

which is a contradiction. ■

Proof of Thm 2:
Base Case: A planar graph with 1 vertex can be colored with 6 colors obviously.
IH: Every planar graph with $n - 1$ vertices can be colored with 6 colors.
IS: We want to show that every planar graph with n vertices can be colored with 6 colors. Let G be a planar graph with n vertices. We show that G can be colored with 6 colors. By claim G has a vertex v with $\deg(v) \leq 5$. Let $H = G - \{v\}$.

We claim that H is also planar. Because if we can draw G on the plane with no crossing, when we remove v and its edges, we still have a drawing of the remaining graph (i.e., H) with no crossing. Therefore, H is a planar graph with $n - 1$ vertices. So, by IH, H can be colored with 6 colors.

Now, let's add vertex v (and its edges) back in. We need to find a consistent color vertex v and this would complete the proof. By definition, v has at most 5 neighbors. Since we have 6 colors, there exists a color which is not used in any of the neighbors of v. We color v with that color and we obtain a consistent coloring. ■