P1) Given a connected graph $G = (V, E)$ with $n$ vertices and $m$ edges where every edge has a positive weight $w_e > 0$, for any pair of vertices $u, v$ define $d(u, v)$ to denote the length of the shortest path from $u$ to $v$ in $G$.

a) Prove that $d(.,.)$ is a metric, namely it satisfies the following three properties: (i) $d(u, v) \geq 0$ for all $u, v$ and $d(u, v) = 0$ only when $u = v$. (ii) $d(u, v) = d(v, u)$ for all vertices $u, v \in V$. (iii) $d(u, v) + d(v, w) \geq d(u, w)$ for all $u, v, w \in V$.

b) Let $d^* := \max_{u,v \in V} d(u, v)$ denote the longest shortest path in $G$. Design an $O(m \log(n))$ time algorithm that gives a 2-approximation to $d^*$, i.e., your algorithm should output a number $\tilde{d}^*$ such that
$$\tilde{d}^* \leq d^* \leq 2\tilde{d}^*.$$

P2) Suppose you are given $n$ coins with value $v_1, \ldots, v_n$ dollars, and you want to change $S$ dollars. You can assume $v_i \neq v_j$ for all $i \neq j$. Design a polynomial time algorithm that outputs the number of ways to change $S$ dollars with the given $n$ coins. For example, if for values $1, 2, 3, 4$ we can change 6 with in 2 ways as follows:

$$2 + 4, 1 + 2 + 3$$