CSE 421

Dynamic Programming

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Announcement

- No Homework due this week.
- Office hour is both on Zoom and in person this and next week.
    - (As requested by some student.)
- My OH is on Monday. (Sorry that it was not clear in the website before)
- We haven’t graded the midterm. It will be done this week.

Quiz

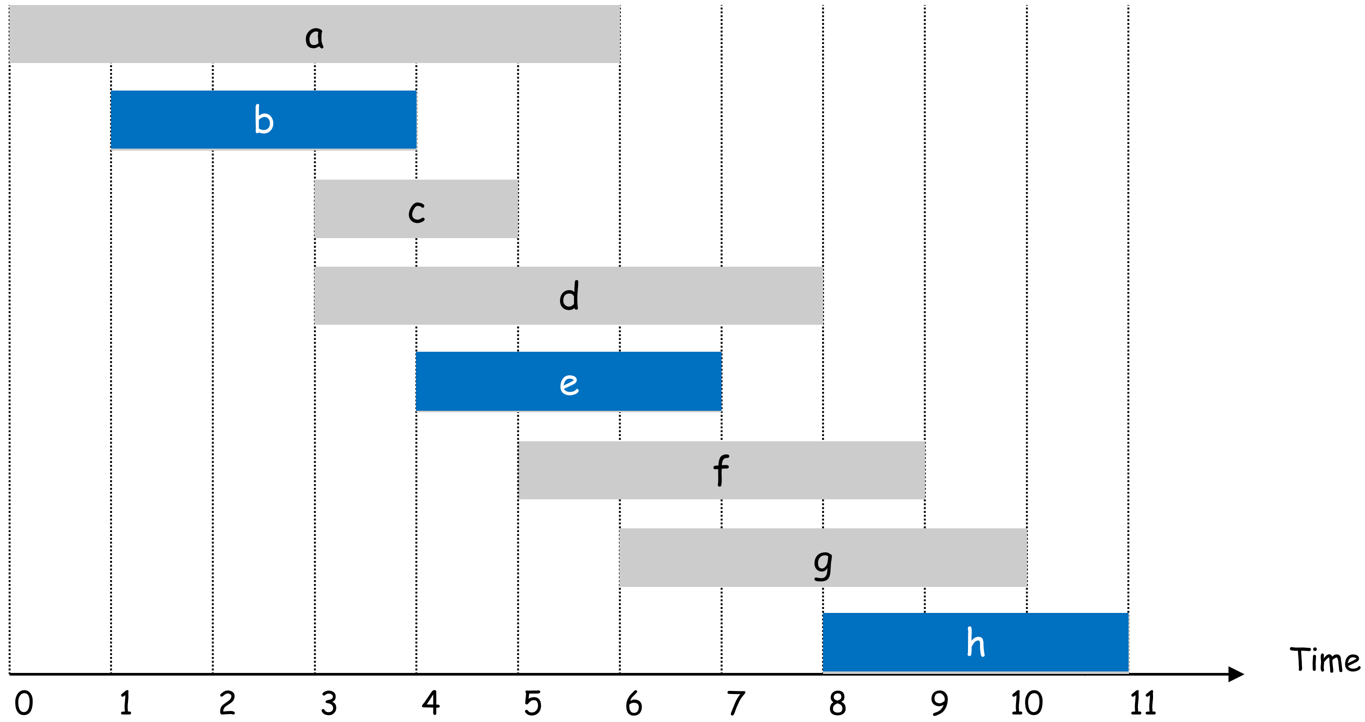
Jeremy Lin has created a time machine. Now, he knows exactly the price of $GME for the next n days, which are $p_1, p_2, \cdots, p_n$.

Give an algorithm for Jeremy to finds the best days to buy and sell the stocks.

Weighted Interval Scheduling

Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$ and has weight w_j
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

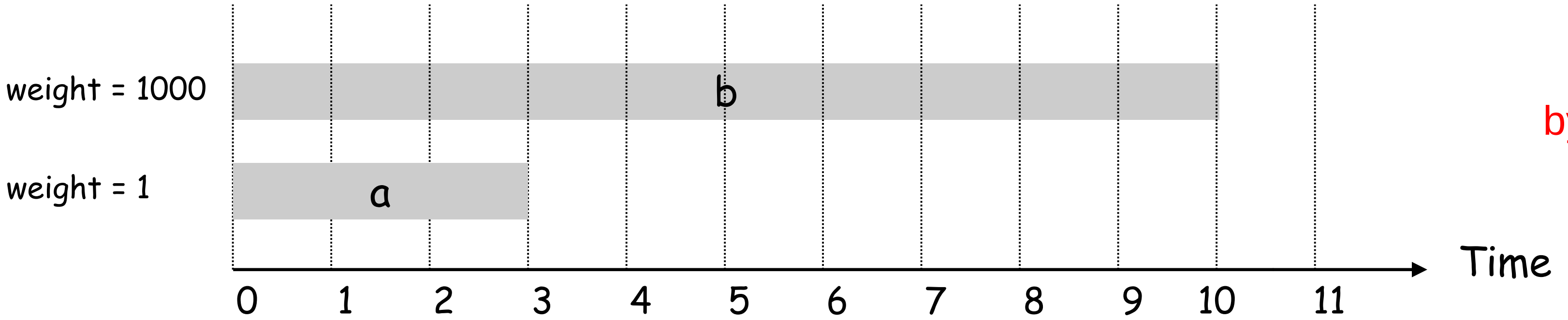


Unweighted Interval Scheduling: Review

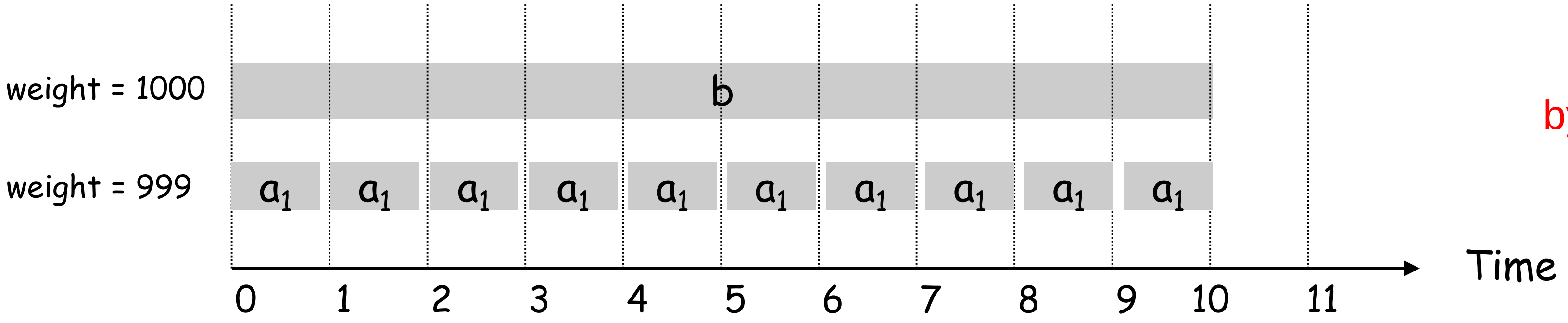
Recall: Greedy algorithm works if all weights are 1:

- Consider jobs in ascending order of finishing time
- Add job to a subset if it is compatible with prev added jobs.

Observation: Greedy ALG fails spectacularly if arbitrary weights are allowed:



by finish



by weight

Weighted Job Scheduling by Induction

Suppose $1, \dots, n$ are all jobs. Let us use induction:

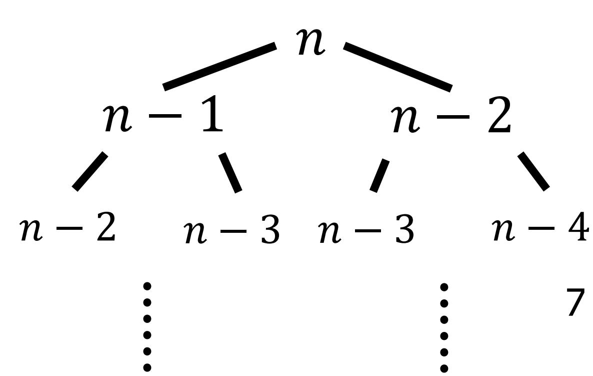
IH: Suppose we can compute the optimum job scheduling for $< n$ jobs.

IS: Goal: For any n jobs we can compute OPT.
Case 1: Job n is not in OPT.
-- Then, just return OPT of $1, \dots, n-1$.

Take best of the two

Case 2: Job n is in OPT.
-- Then, delete all jobs not compatible with n and recurse.

Q: Are we done?
A: No, How many subproblems are there?
Potentially 2^n all possible subsets of jobs.



Sorting to Reduce Subproblems

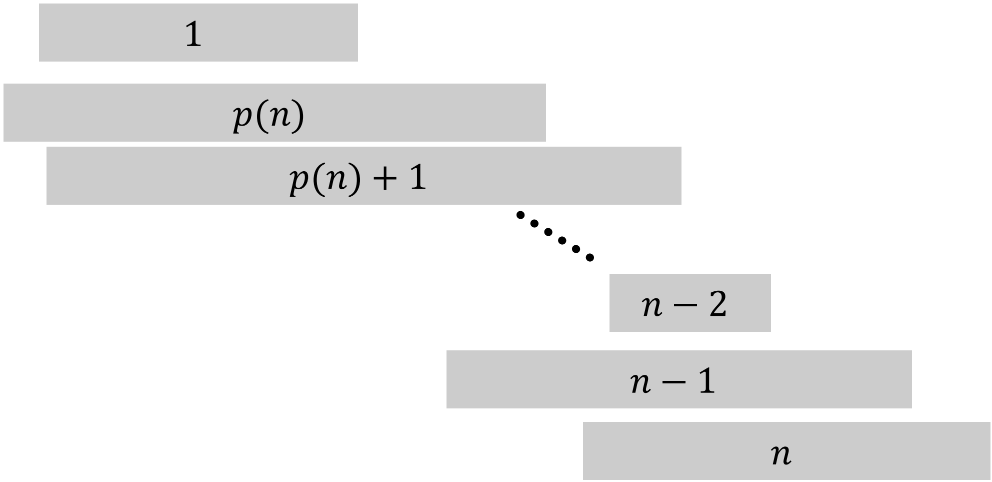
Why can’t we order by start time?

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$

IS: For jobs $1, \ldots, n$ we want to compute OPT

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not OPT
- Let $p(n)$ = largest index $i < n$ such that job i is compatible with n.
- Then, we just need to find optimal schedule for jobs $1, \ldots, p(n)$



Sorting to Reduce Subproblems

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$
IS: For jobs $1, \dots, n$ we want to compute OPT

Case 1: Suppose OPT has job n.

- So, all jobs i that are not compatible with n are not in OPT
- Let $p(n) =$ largest index $i < n$ such that job i is compatible with n.
- Then, we just need to find OPT of $1, \dots, p(n)$

Case 2: OPT does not select job n.

- Then, OPT is just the OPT of $1, \dots, n-1$

Take best of the two

Q: Have we made any progress (still reducing to two subproblems)?
A: Yes! This time every subproblem is of the form $1, \dots, i$ for some i
So, at most n possible subproblems.

Weighted Job Scheduling by Induction

Sorting Idea: Label jobs by finishing time $f(1) \leq \cdots \leq f(n)$
Def $OPT(j)$ denote the weight of OPT solution of $1, \ldots, j$

The most important part of a correct DP; It fixes IH

To solve $OPT(j)$:
Case 1: $OPT(j)$ has job j.
- So, all jobs that are not compatible with j are not in $OPT(j)$.
- Let $p(j) =$ largest index $i < j$ such that job i is compatible with j.
- So $OPT(j) = OPT\big(p(j)\big) + w_j$.

Case 2: $OPT(j)$ does not select job j.
- Then, $OPT(j) = OPT(j-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\Big(w_j + OPT\big(p(j)\big), OPT(j-1)\Big) & \text{o.w.} \end{cases}$$

Algorithm

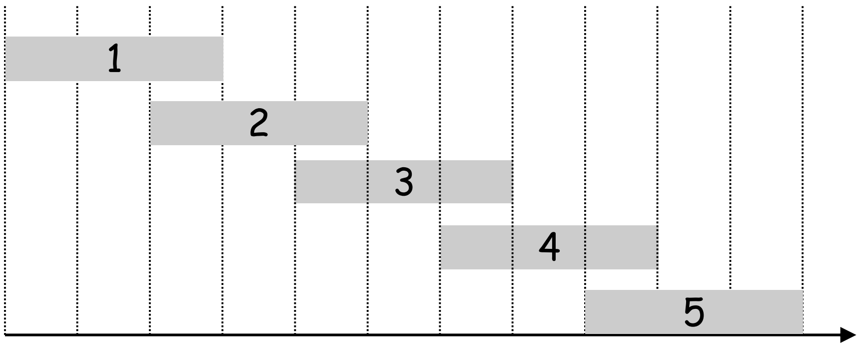
```
Input: n, s(1),...,s(n) and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),...,p(n)

OPT(j) {
   if ( j = 0 )
      return 0
   else
      return max (w_j + OPT(p(j)), OPT(j − 1)).
}
```

Recursive Algorithm Fails

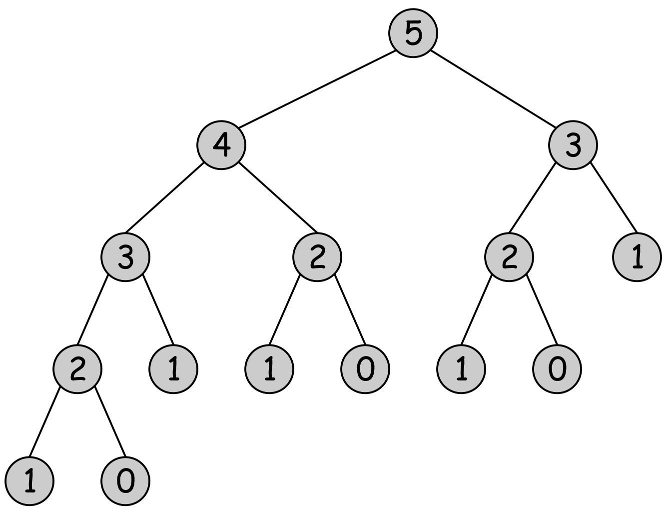
Even though we have only n subproblems, if we do not store the solution to the subproblems

- ➢ we may re-solve the same problem many many times.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence



$p(1) = 0, p(j) = j - 2$



Algorithm with Memoization

Memorization. Compute and Store the solution of each sub-problem in a cache the first time that you face it. lookup as needed.

```
Input: n, s(1),...,s(n)  and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),...,p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0

OPT(j) {
   if (M[j] is empty)
      M[j] = max(w_j + OPT(p(j)), OPT(j − 1)).
   return M[j]
}
```

In practice, you may get stackoverflow if $n \gg 10^6$ (depends on the language).

Bottom up Dynamic Programming

You can also avoid recursion

- recursion may be easier conceptually when you use induction

```
Input: n, s(1),...,s(n)  and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),...,p(n)

OPT(j) {
   M[0] = 0
   for j = 1 to n
      M[j] = max (w_j + M[p(j)], M[j − 1]).
}

Output M[n]
```

Claim: $M[j]$ is value of $OPT(j)$

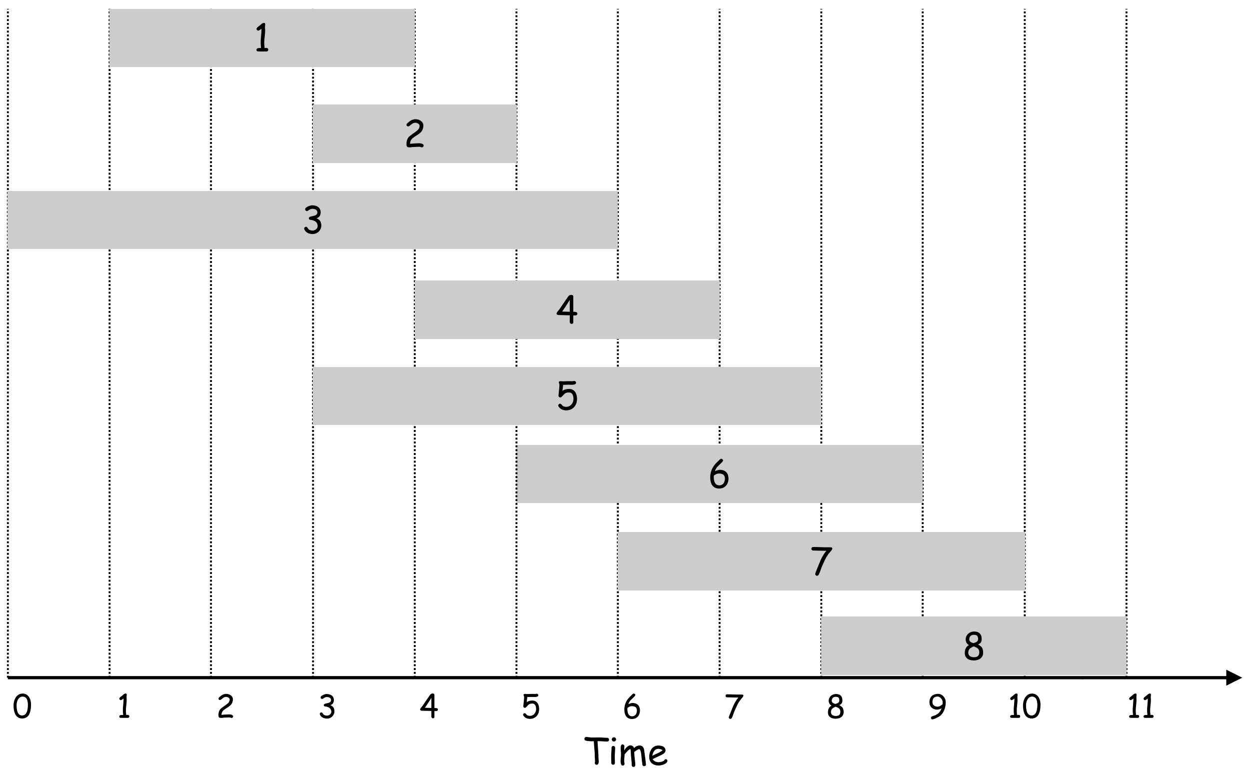
Timing: Easy. Main loop is $O(n)$; sorting is $O(n \log n)$.

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT\left(p(j)\right), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

$p(j)$ = largest index $i < j$ such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT\left(p(j)\right), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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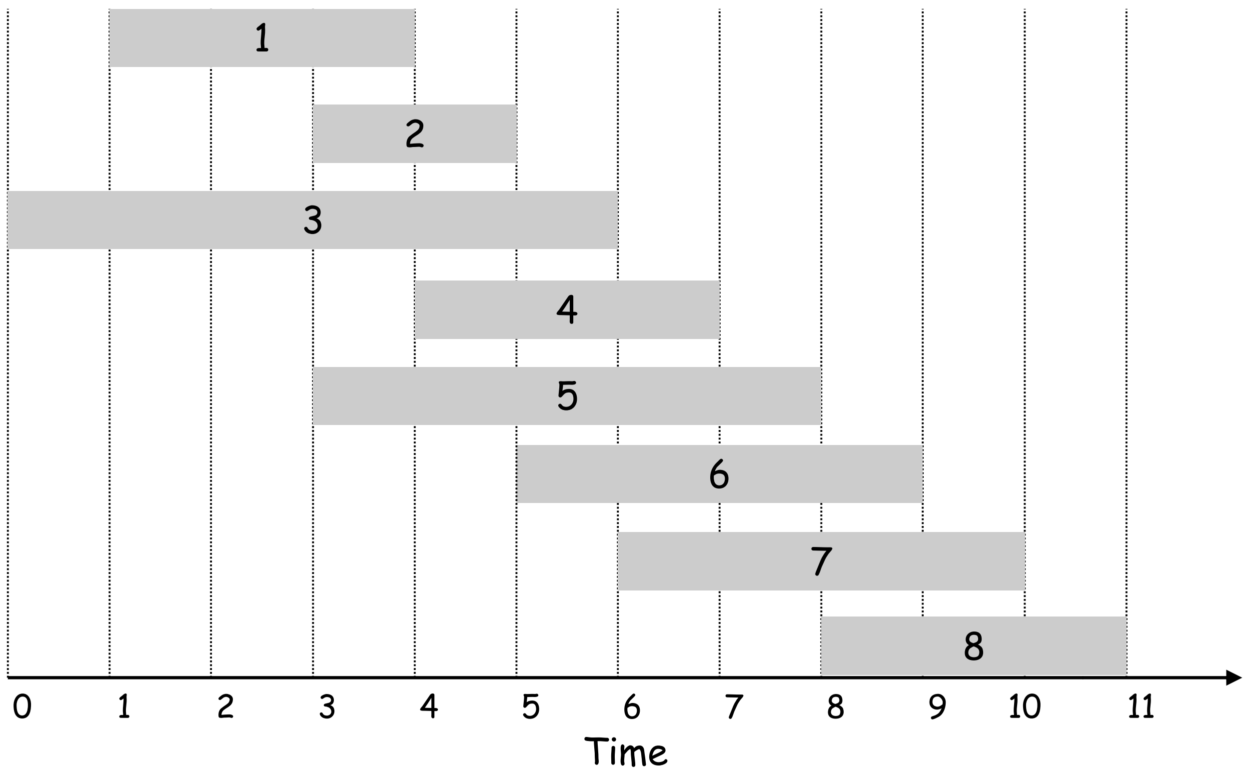

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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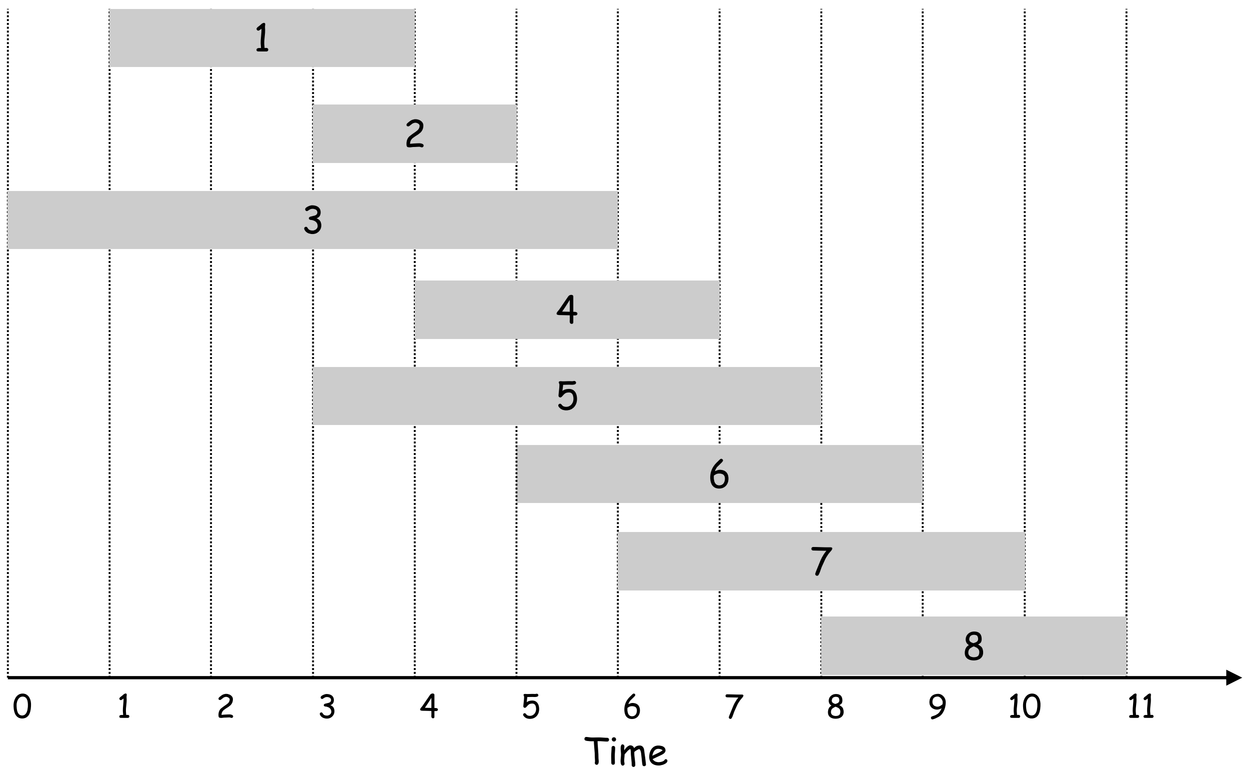
$p(j) =$ largest index $i < j$ such that job i is compatible with j.



j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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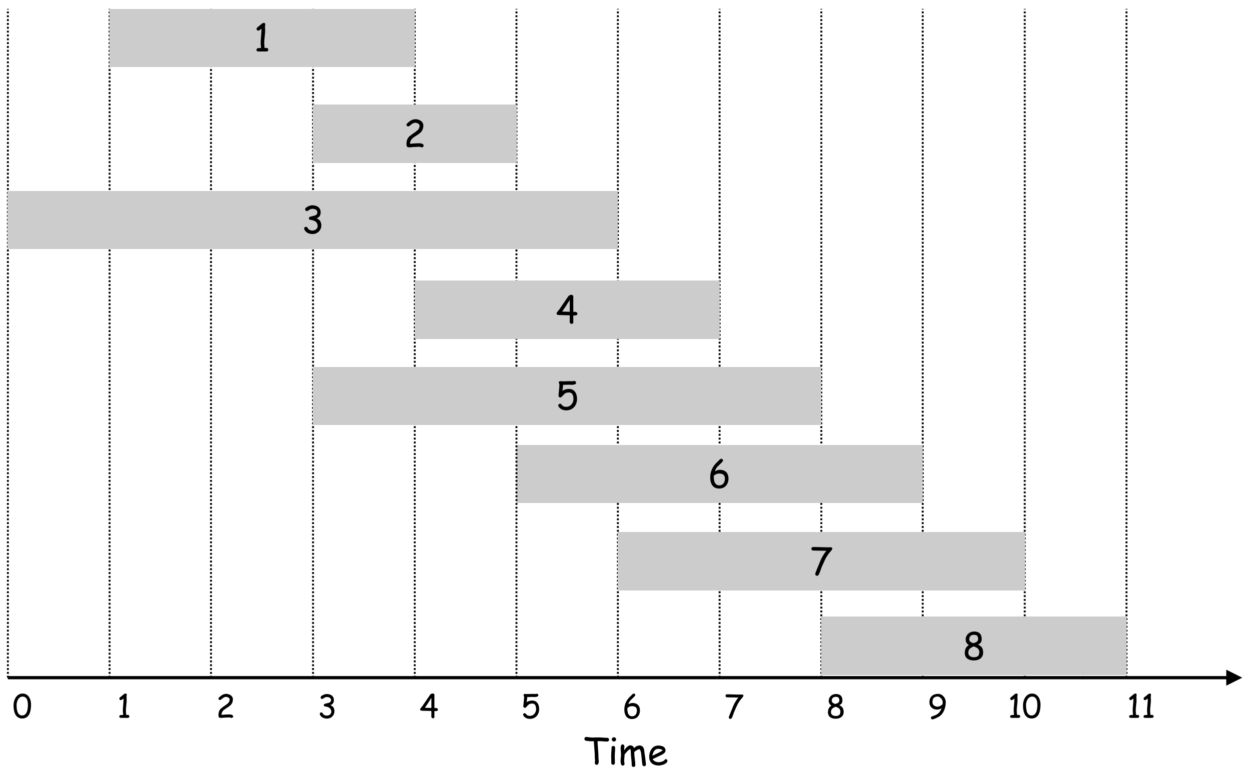

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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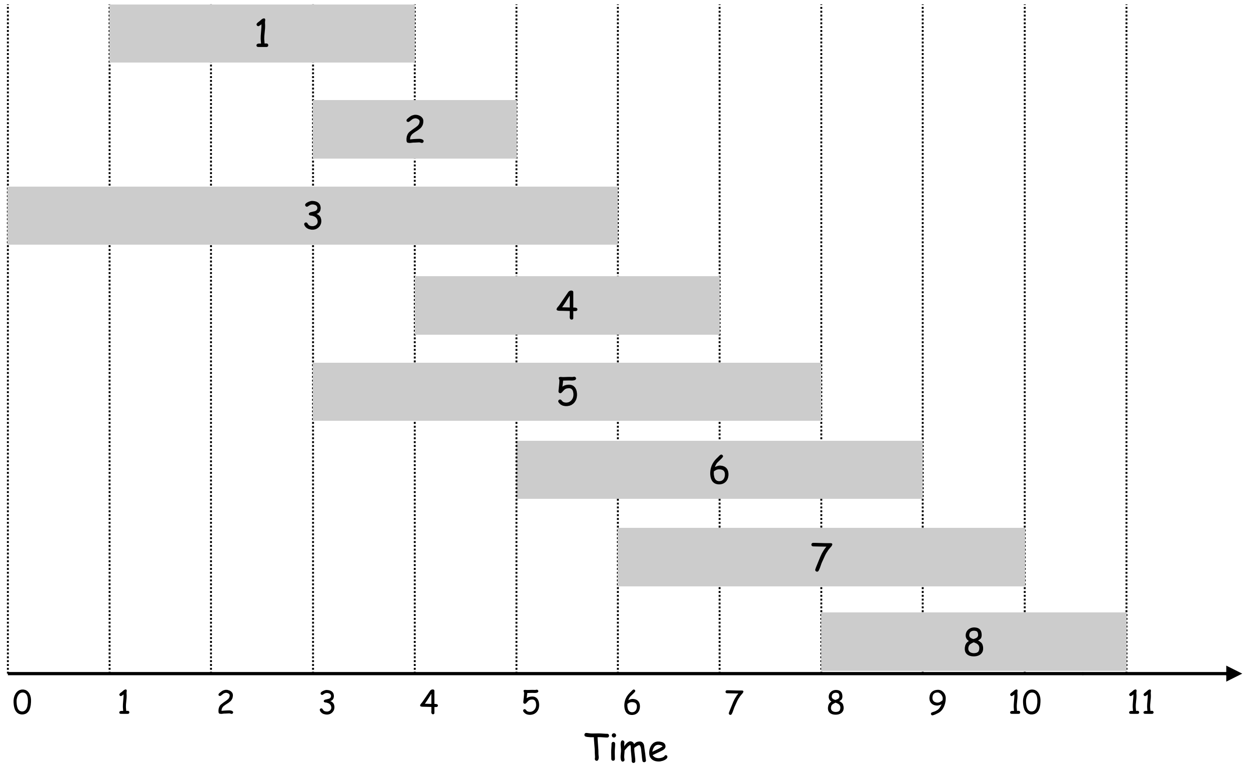

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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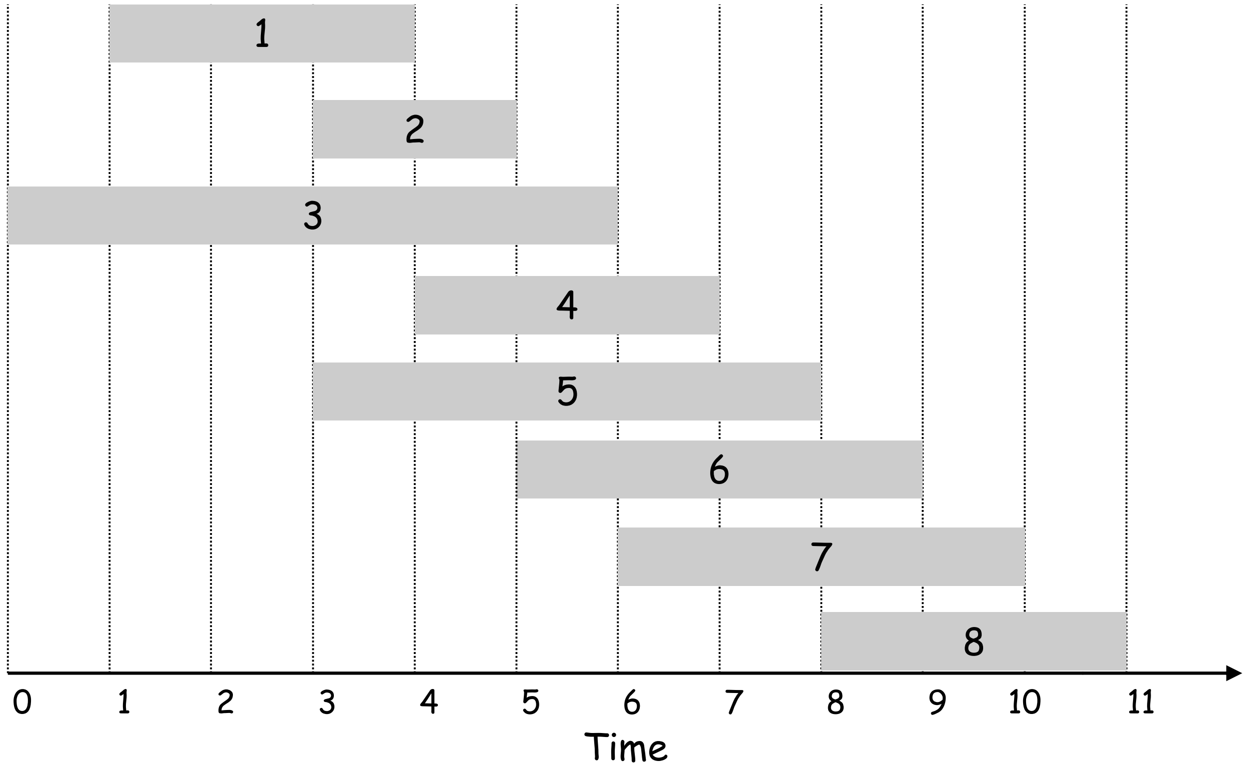

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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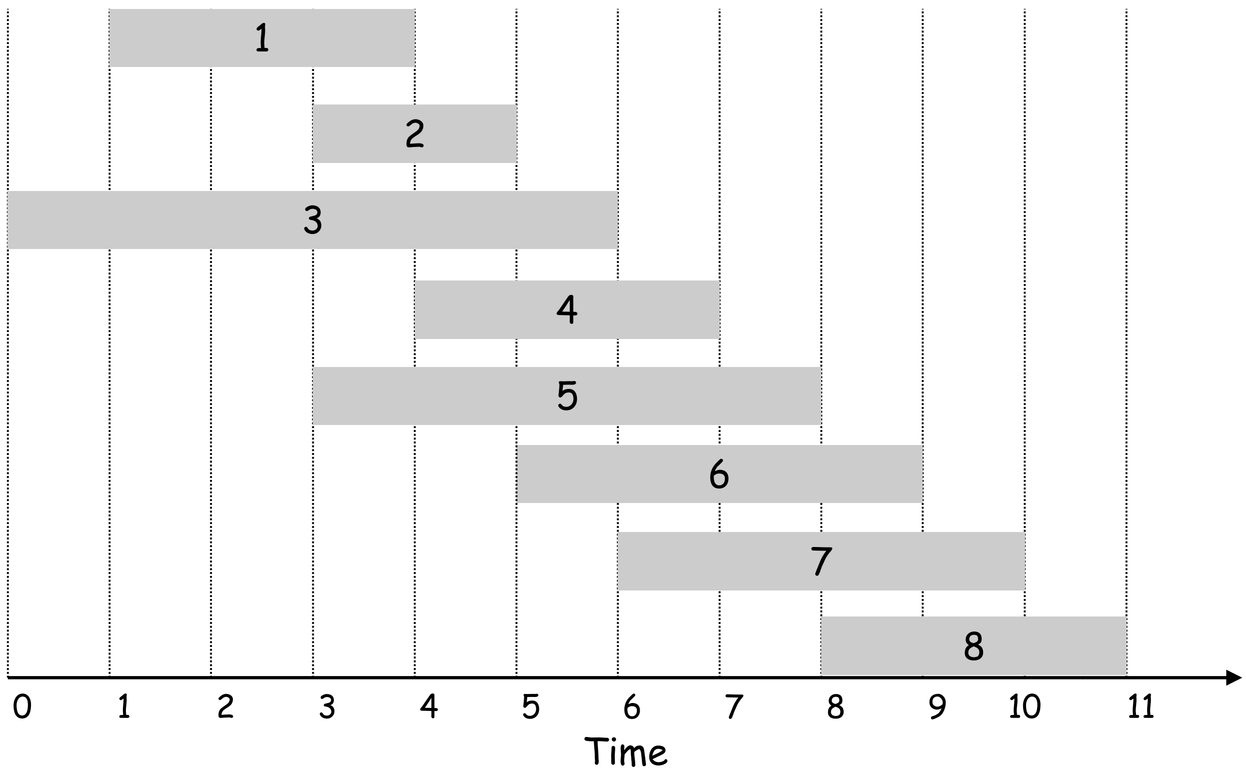

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

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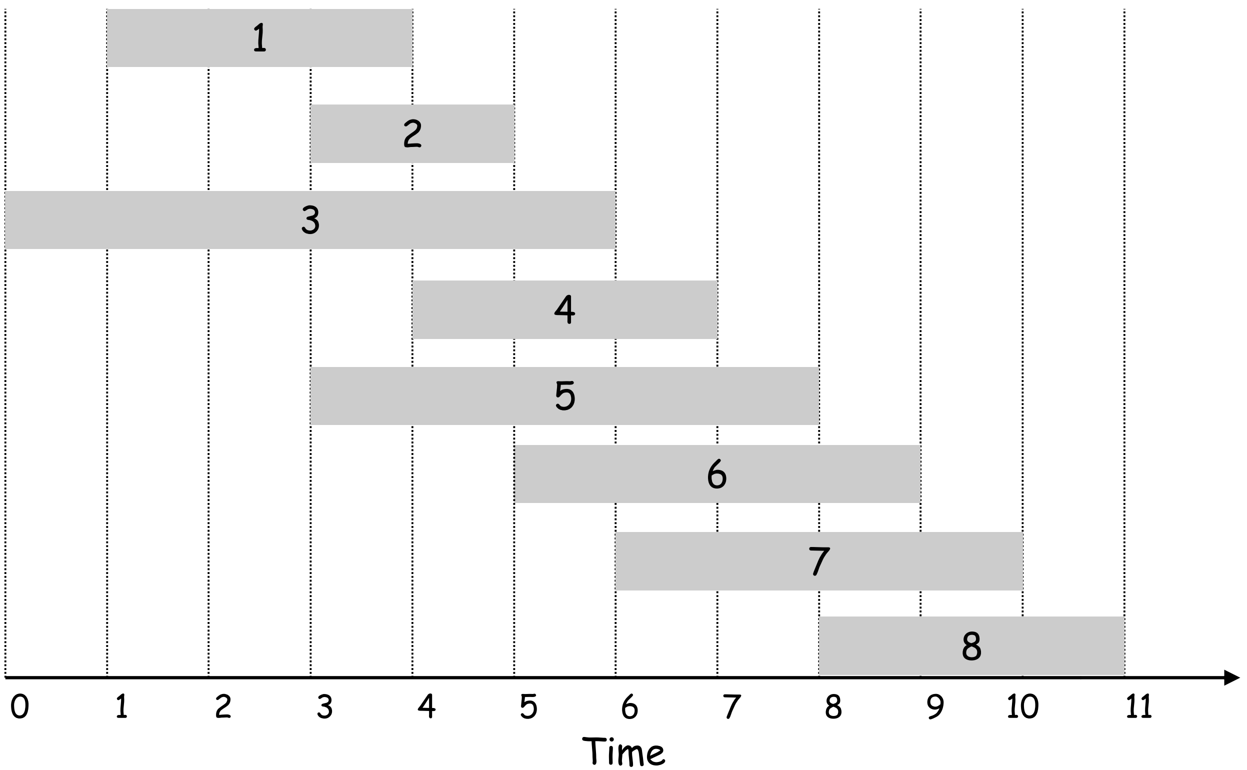

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \leq \cdots \leq f(n)$.

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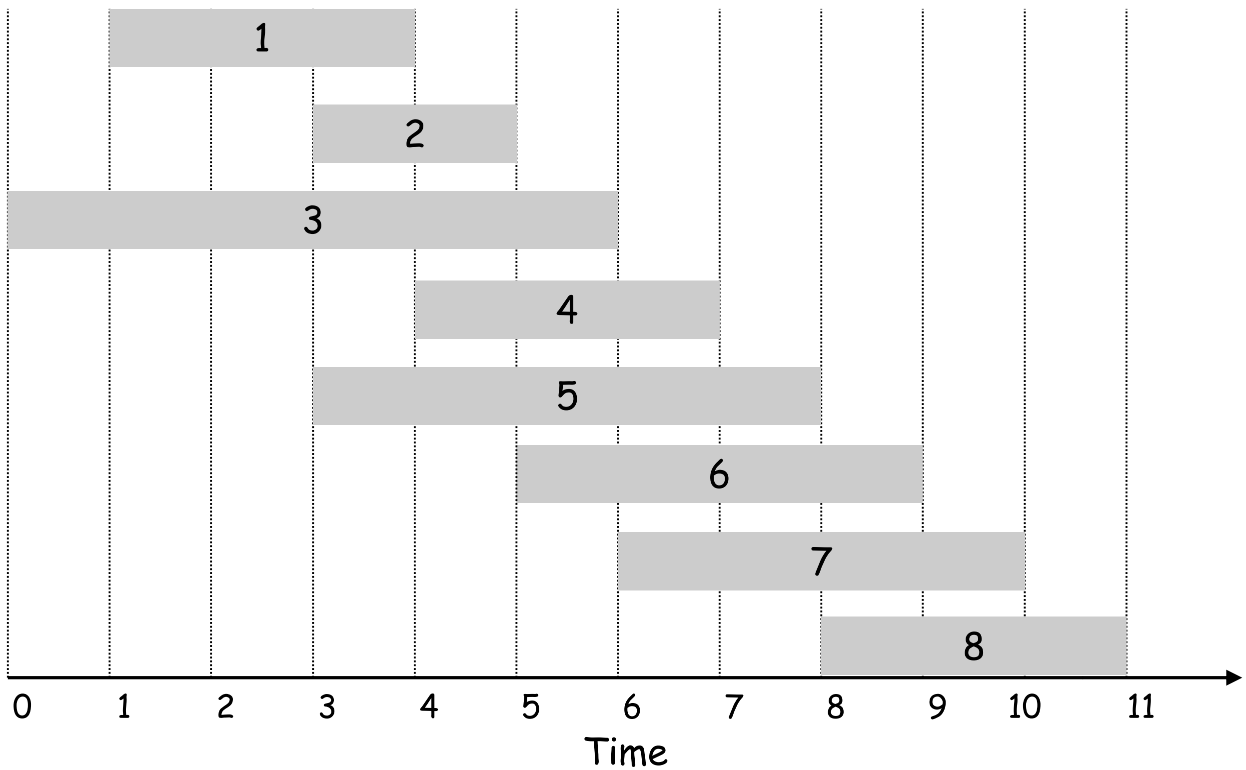

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	**10**

Example

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left(w_j + OPT(p(j)), OPT(j-1)\right) & \text{o.w.} \end{cases}$$

Label jobs by finishing time: $f(1) \le \cdots \le f(n)$.

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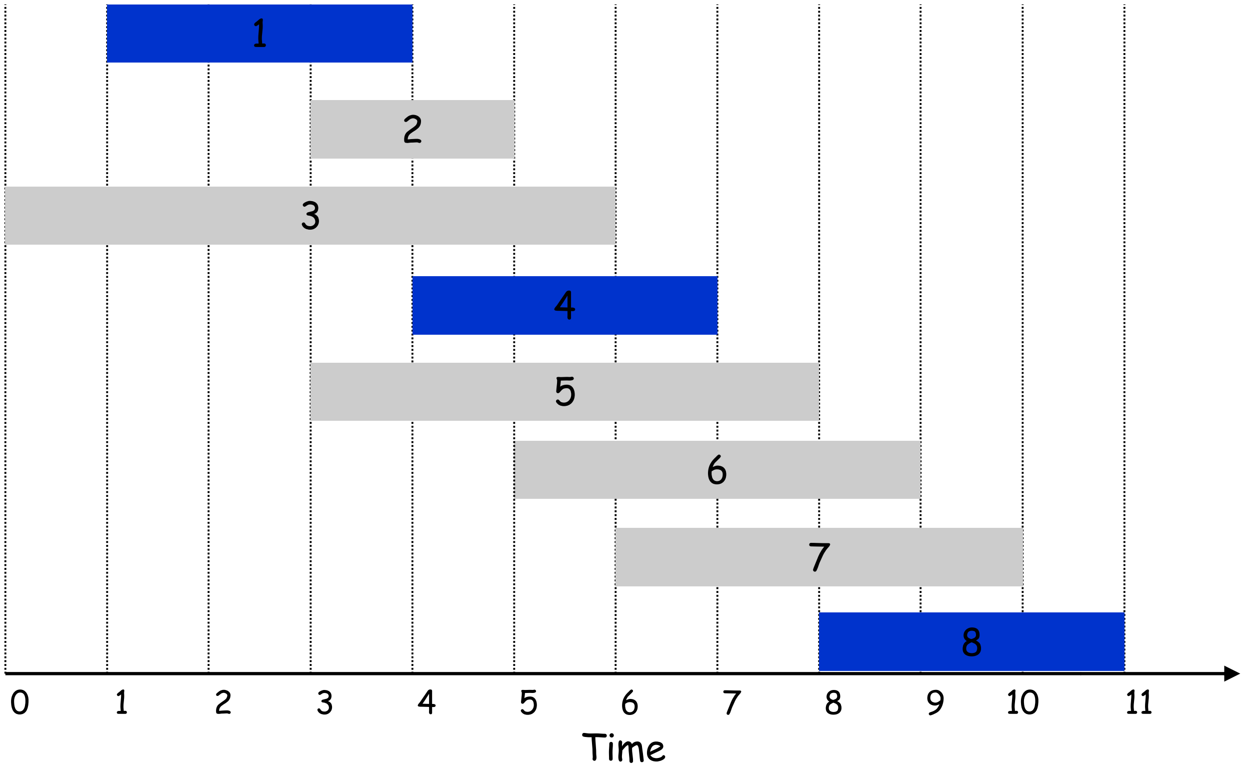

j	w_j	p(j)	OPT(j)
0			0
1	3	0	3
2	4	0	4
3	1	0	4
4	3	1	6
5	4	0	6
6	3	2	7
7	2	3	7
8	4	5	10

Dynamic Programming

- Give a solution of a problem using smaller (overlapping) sub-problems where

  the parameters of all sub-problems are determined in-advance

- Useful when the same subproblems show up again and again in the solution.

How to recover the solution?

We can simply maintain the solution.

```
Input: n, s(1),...,s(n)  and f(1),...,f(n) and w_1,...,w_n.

Sort jobs by finish times so that f(1) ≤ f(2) ≤ ⋯ f(n).

Compute p(1),p(2),...,p(n)

OPT(j){
   M[0] = 0
   S[0] = {}
   for j = 1 to n
      if w_j + M[p(j)] > M[j-1]
         M[j] = w_j + M[p(j)].
         S[j] = {j} ∪ S[p(j)]
      else
         M[j] = M[j-1]
         S[j] = S[j-1]
}

Output M[n] and S[n]
```

$O(1)$ time (for S[j] = {j} ∪ S[p(j)])

$O(1)$ time (for S[j] = S[j-1])

What is the runtime of this new algorithm?

Each S[j] points to some vertices of a tree.
← We add leaf j with its parent $S[p(j)]$.

Quiz

Jeremy Lin has created a time machine. Now, he knows exactly the price of $GME for the next n days, which are $p_1, p_2, \cdots, p_n$.

Somehow, Jeremy doesn't want to be labeled as greedy.

So, can you use dynamic programming to help Jeremy instead?

W Let w_k be the Jeremy Lin's net worth on the k-th day. Then, we have

$$w_k = w_{k-1} \times p_k / p_{k-1}$$

$$w_k = \max(w_{k-1} \times p_k / p_{k-1}, w_{k-1})$$

$$w_k = \max(w_{k-1} + p_k - p_{k-1}, w_{k-1})$$

$$w_k = w_{k-1} + p_k - p_{k-1}$$

$$w_k = w_{k-1} \times \max(p_k / p_{k-1}, 0)$$

Total Results: 0

Quiz

Life is not easy.
Robinhood doesn't want someone to hold $GME to the moon 🚀🚀

Now, Jeremy can only hold $GME for at most 2 consecutive days.

So, what is the formula for w_k?

$$w_k = \max\left(w_{k-1}, w_{k-2}\frac{p_k}{p_{k-1}}, w_{k-3}\frac{p_k}{p_{k-2}}\right).$$

Knapsack Problem

Knapsack Problem

Given n objects and a "knapsack."

Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.

Knapsack has capacity of W kilograms.

Goal: fill knapsack so as to maximize total value.

Ex: OPT is { 3, 4 } with value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: First Attempt

Let $OPT(i) =$ Max value of subsets of items $1, \ldots, i$ of weight $\leq W$.

Case 1: $OPT(i)$ does not select item i

- In this case $OPT(i) = OPT(i-1)$

Case 2: $OPT(i)$ selects item i

- In this case, item i does not immediately imply we have to reject other items
- The problem does not reduce to $OPT(i-1)$ because we now want to pack as much value into box of weight $\leq W - w_i$

Conclusion: We need more subproblems, we need to strengthen IH.

Stronger DP (Strengthening Hypothesis)

What is the ordering of item we should pick?

Let $OPT(i, w)$ = Max value of subsets of items $1, \dots, i$ of weight $\leq w$

Case 1: $OPT(i, w)$ selects item i

- In this case, $OPT(i, w) = v_i + OPT(i - 1, w - w_i)$

Case 2: $OPT(i, w)$ does not select item i

- In this case, $OPT(i, w) = OPT(i - 1, w)$.

Take best of the two

Therefore,

$$OPT(i, w) = \begin{cases} 0 & \text{If } i = 0 \\ OPT(i - 1, w) & \text{If } w_i > w \\ \max(OPT(i - 1, w), v_i + OPT(i - 1, w - w_i)) & \text{o.w.,} \end{cases}$$

DP for Knapsack

```
Comp-OPT(i,w)
   if M[i,w] == empty
     if (i==0)
       M[i,w]=0
     else if (w_i > w)
       M[i,w]= Comp-OPT(i-1,w)
     else
       M[i,w]= max {Comp-OPT(i-1,w), v_i + Comp-OPT(i-1,w-w_i)}
   return M[i, w]
```

recursive

```
for w = 0 to W
   M[0, w] = 0
for i = 1 to n
   for w = 1 to W
      if (w_i > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}

return M[n, W]
```

Non-recursive

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0											
{ 1, 2 }	0											
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0											
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7								
{ 1, 2, 3 }	0	1										
{ 1, 2, 3, 4 }	0	1										
{ 1, 2, 3, 4, 5 }	0	1										

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
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```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19					
{ 1, 2, 3, 4 }	0	1										
{ 1, 2, 3, 4, 5 }	0	1										

W = 11

```
if (wi > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}
```

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

DP for Knapsack

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29		
{ 1, 2, 3, 4, 5 }	0	1										

W = 11

```
if (wi > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}
```

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

DP for Knapsack

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  M[i, w] = M[i-1, w]
else
  M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}
```

Quiz

Life is not easy.
Robinhood doesn't want someone to hold $GME to the moon 🚀🚀

Now, Jeremy can only hold $GME for at most 2 consecutive days.
and can only trade $GME for at most t times.

So, what is the best trading?
Let $w_{k,t}$ be the network at k-th day using t trades.

$$w_{k,t} = \max\left(w_{k-1,t}, w_{k-2,t-1}\frac{p_k}{p_{k-1}}, w_{k-3,t-1}\frac{p_k}{p_{k-2}}\right).$$

Knapsack Problem: Running Time

Running time: $\Theta(n \cdot W)$

- Not polynomial in input size!
- "Pseudo-polynomial."“
- Decision version of Knapsack is NP-complete.

Knapsack approximation algorithm:

There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum in time Poly(n).

UW Expert

DP Ideas so far

- You may have to define an ordering to decrease #subproblems

- You may have to strengthen DP, equivalently the induction, i.e., you have may have to carry more information to find the Optimum.

- This means that sometimes we may have to use two dimensional or three dimensional induction