CSE 421

Linear Programs

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Linear Programming

Optimize a linear function subject to linear inequalities.

$$\max c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subjects to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

for $i = 1,2,\cdots,m$

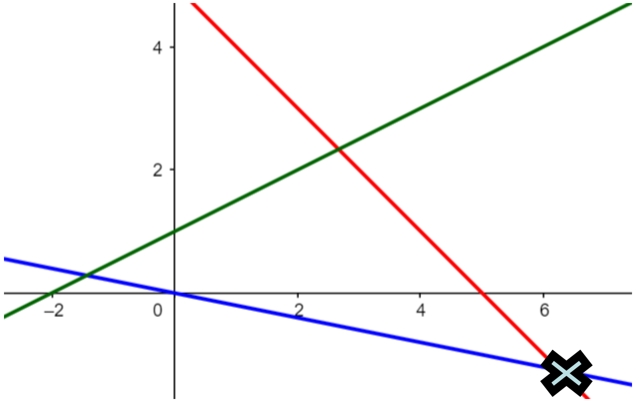
Example:

$$\min y \text{ subjects to}$$

$$x + y \le 5$$

$$x + 5y \ge 0$$

$$x - 2y \ge -2$$



Applications of Linear Programming

Generalizes: Ax=b, shortest path, max-flow, matching, minimum spanning tree, Dynamic Decision Problem, …

Why significant?

- We can solve linear programming in polynomial time.
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: CPLEX, Gorubi, ….
- CPLEX can solve LPs with millions of variables/constraints in seconds

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alories and (h)appiness per pound:

	veggies	**meat**	**fruits**	**dairy**
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat and 0.2lb of fruits we will be $0.5\ h_m + 0.2\ h_f$ happy

- You should eat at most 2000 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

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- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned} \max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\ s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \leq 20 \\ & x_v c_v + x_m c_m + x_f c_f + x_d c_d \leq 2000 \\ & x_v, x_m, x_f, x_d \geq 0 \end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

Diet Problem by LP

George Stigler (graduated from UW, a 1982 Nobel Laureate in economics) studied this.

See the precise linear program here.

Foie Linéaire à la Stigler

Annual Foods (in 1944 money):

Wheat Flour (Enriched): $10.8

Liver (Beef): $0.69

Cabbage: $4.09

Spinach: $1.83

Navy Beans, Dried: $22.3

(In today's dollar, 1.6 dollar per day)

How to Design an LP?

- Define the set of variables
- Put constraints on your variables,
  - should they be nonnegative?
- Write down the constraints
  - If a constraint is not linear try to approximate it with a linear constraint
- Write down the objective function
  - If it is not linear approximation with a linear function
- Decide if it is a minimize/maximization problem

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$
\begin{aligned}
\max \quad & \sum_{e \text{ out of } s} x_e \\
s.t. \quad & \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e & \forall v \neq s, t \\
& x_e \leq c(e) & \forall e \\
& x_e \geq 0 & \forall e
\end{aligned}
$$

Example 3: Min Cost Flow

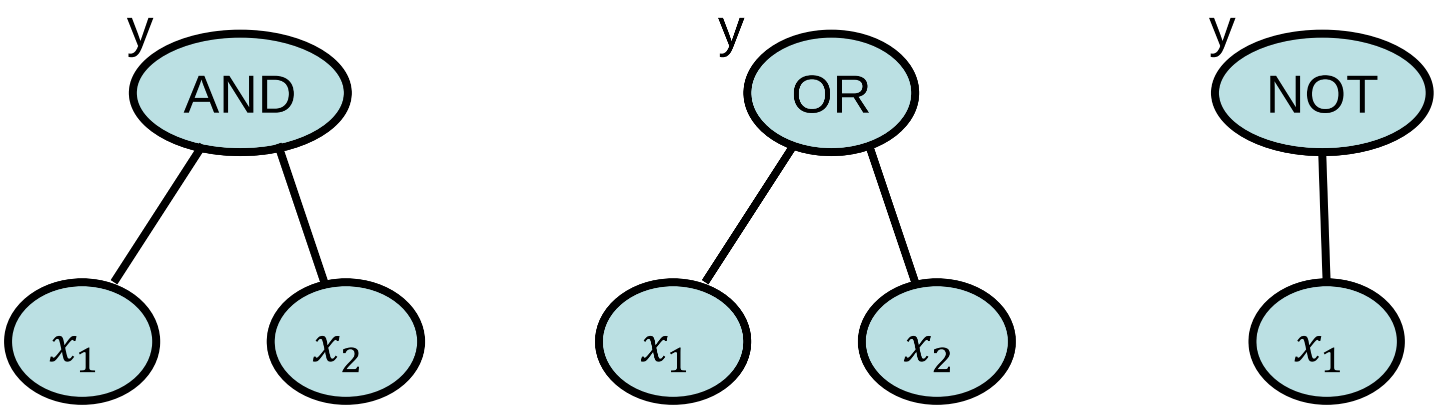
Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$ for each gallon of water that we send through e.

Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in E} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e \qquad \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) \qquad \forall e \\
& x_e \geq 0 \qquad \forall e
\end{aligned}
$$

Example 4: Circuit Evaluation

Given a circuit C with inputs x_i.
Goal: Output the result of the circuit.



$$y \leq x_1$$
$$y \leq x_2$$
$$y \geq x_1 + x_2 - 1$$
$$y \geq 0$$

$$y \geq x_1$$
$$y \geq x_2$$
$$y \leq x_1 + x_2$$
$$y \leq 1$$

$$y = 1 - x_1$$

Example 4: Circuit Evaluation

Define the set of variables

- One variable for each input
- One variable for each circuit to denote its output

Put constraints on your variables

- "Input = Input"
- All variables between 0 and 1

Write down the constraints

- For each gate, write down the inequalities like last slide

Write down the objective function

- 0

Decide if it is a minimize/maximization problem

- max/min

In a sense, every algorithm can be expressed as linear program!

Feasibility Problem

When there is no objective (namely, 0),
any solution satisfies all inequalities is an answer.

Note that feasibility version is not easier.

Reduction:
Suppose we want to solve $\min c^\top x$ subjects to $Ax \leq b$.
It is same as $\min 0$ subjects to $Ax \leq b$, $c^\top x \leq OPT$.

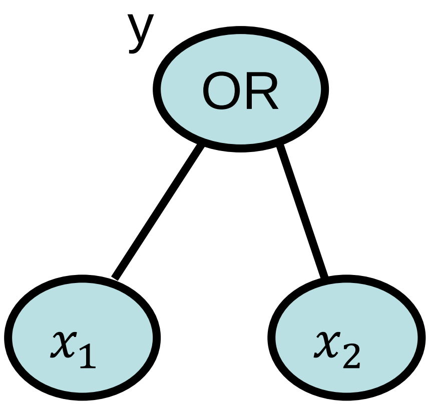
Why can't we solve 3SAT?

Instead of setting the input variables,
can we simply set the output variables to 1?

Problem:
If both x_1 and x_2 is $\{0,1\}$,
Then y is x_1 or x_2.

However, if we only specify y,
A feasible point can be fractional.

e.g. For $y = 1$.
One solution is $x_1 = x_2 = 1/2$.



$$y \geq x_1$$
$$y \geq x_2$$
$$y \leq x_1 + x_2$$
$$y \leq 1$$

This shows we can solve "fractional 3SAT" using LP.

Integer Programming

$$\max c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subjects to

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$$

for $i = 1,2,\cdots,m$

x_i are integer

We can write 3SAT as an integer programming (IP).

So, “IP” is NP-complete.

Universality of Linear Programs

NC = the set of decision problems decidable in $O(\log^{O(1)} n)$ time using polynomial many computers.

Theorem: If "LP" is in NC, then P is in NC.

In practice, we can often solve linear program in $O(\log^2 n)$ parallel time.

Proof:

Given a decision problem A in P.

We can write A as a circuit of polynomial size.

Then, we can write it as a linear program.

If we can solve linear program in $O(\log^{O(1)} n)$ parallel time,

Then, we can decide A in $O(\log^{O(1)} n)$ parallel time.

Difficulty of Linear Programs

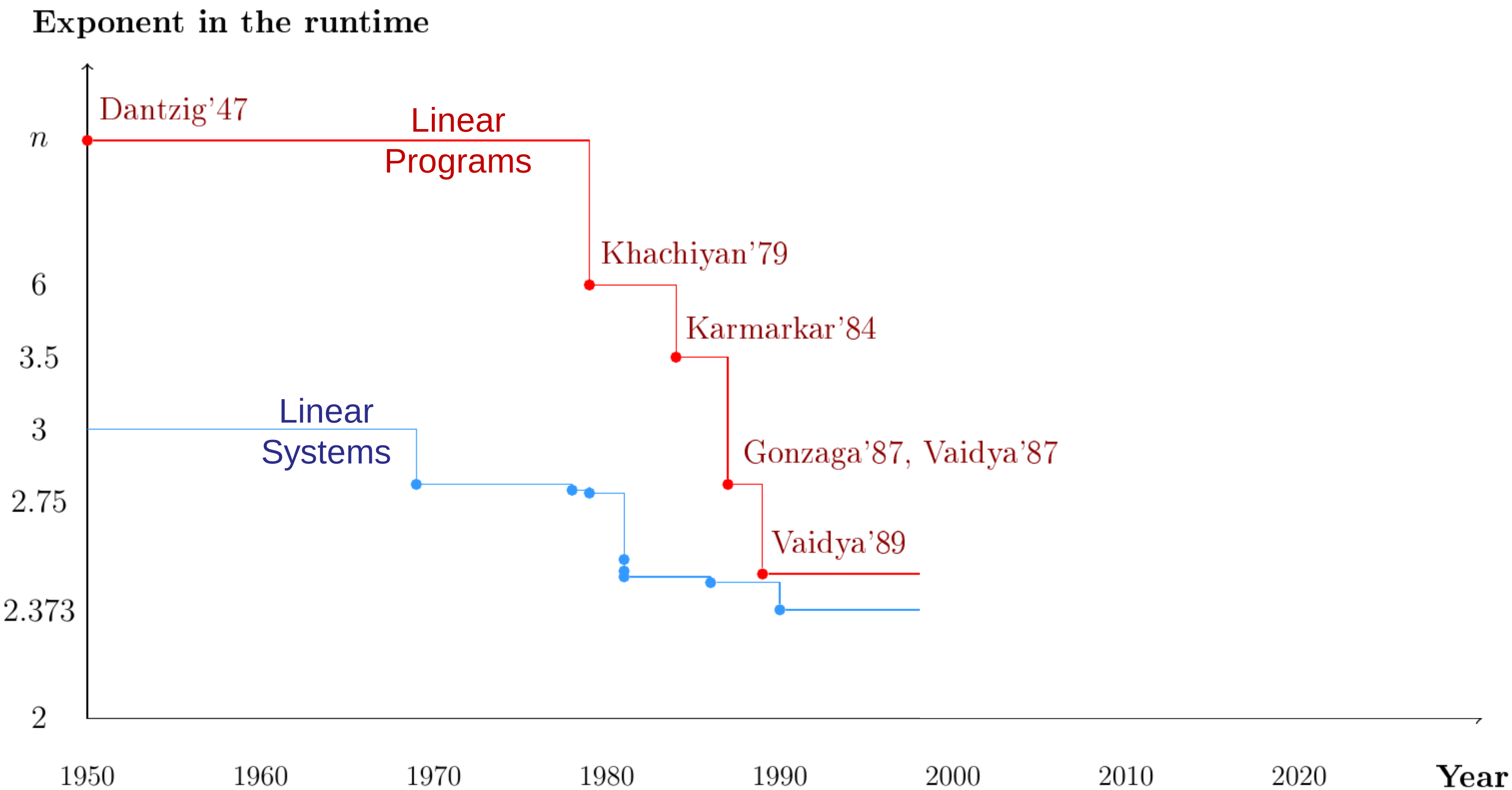
- Before 1979, some **believed** general linear programs can't be even solved efficiently.

- Now, we still cannot solve it **exactly** and efficiently.

(one of the 18 unsolved problems in mathematics by Smale)

- We can solve it approximately (aka, finding x with $Ax \leq b + \epsilon$) in

$$n^{O(1)} \log(1/\epsilon).$$


Exponent in the runtime
Dantzig'47
Linear
Programs
Khachiyan'79
Karmarkar'84
Gonzaga'87, Vaidya'87
Vaidya'89
Linear
Systems
n
6
3.5
3
2.75
2.373
2
1950
1960
1970
1980
1990
2000
2010
2020
Year

Going back to basic: verification

Given some x, how can we tell if it is an optimal solution?