CSE 421
Introduction to Algorithms

Lecture 4: BFS, DFS Properties/Applications, Topological Sort

Undirected Graph Search Application: Connected Components

Want to answer questions of the form:

Given: vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{G}$

Is there a path from $\boldsymbol{u}$ to $\boldsymbol{v}$?

Idea: create array $\boldsymbol{A}$ s.t

$\boldsymbol{A}[\boldsymbol{u}]$ = smallest numbered vertex connected to $\boldsymbol{u}$

Answer is yes iff $\boldsymbol{A}[\boldsymbol{u}] = \boldsymbol{A}[\boldsymbol{v}]$

Q: Why is this better than an array **Path**$[\boldsymbol{u}, \boldsymbol{v}]$?

Undirected Graph Search Application: Connected Components

```
Initial state: all $v$ unvisited
for $s \leftarrow 1$ to $n$ do
    if state($s$) $\neq$ fully-explored then
        BFS($s$): setting $A[u] \leftarrow s$ for each $u$ found
            (and marking $u$ visited/fully-explored)
endfor
```

Total cost: $O(n+m)$

- Each vertex is touched once in outer procedure and edges examined in different BFS runs are disjoint
- Works also with Depth First Search ...

DFS(u) – Recursive Procedure

Global Initialization: mark all vertices "unvisited"

```
DFS($u$)
    mark $u$ "visited" and add $u$ to $R$
    for each edge $(u, v)$
        if ($v$ is "unvisited")
            DFS($v$)
    end for
    mark $u$ "fully-explored"
```

Properties of DFS(s)

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x
- Edges into undiscovered vertices define depth-first spanning tree of G

Unlike the BFS tree:

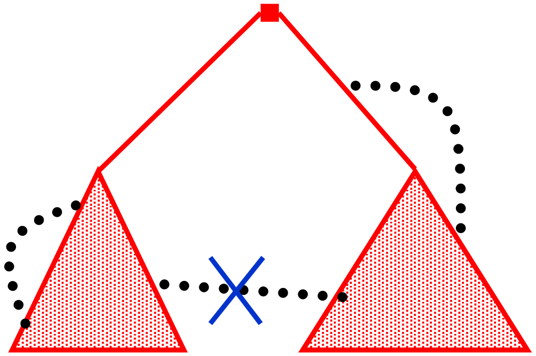
- the DFS spanning tree *isn't* minimum depth
- its levels *don't* reflect min distance from the root
- non-tree edges *never* join vertices on the same or adjacent levels

BUT…

Non-tree edges in DFS tree of undirected graphs

Claim: All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- In other words ... No “cross edges”.



No cross edges in DFS on undirected graphs

Claim: During DFS(x) every vertex marked "visited" is a descendant of x in the DFS tree T

Claim: For every x, y in the DFS tree T, if (x, y) is an edge *not* in T then one of x or y is an ancestor of the other in T

Proof:

  - One of DFS(x) or DFS(y) is called first, suppose WLOG that DFS(x) was called before DFS(y)
  - During DFS(x), the edge (x, y) is examined
- Since (x, y) is a *not* an edge of T, y was already visited when edge (x, y) was examined during DFS(x)
- Therefore y was visited during the call to DFS(x) so y is a descendant of x. ■

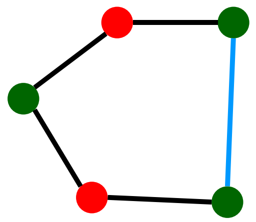
Applications of Graph Traversal: Bipartiteness Testing

Definition: An undirected graph G is bipartite iff we can color its vertices **red** and **green** so each edge has different color endpoints

Input: Undirected graph G
Goal: If G is bipartite, output a coloring;
otherwise, output "NOT Bipartite".

Fact: Graph G contains an odd-length cycle ⇒ it is not bipartite

Just coloring the cycle part of G is impossible



On a cycle the two colors must alternate, so

- **green** every 2nd vertex
- **red** every 2nd vertex

Can't have either if length is not divisible by 2.

Applications of Graph Traversal: Bipartiteness Testing

WLOG ("without loss of generality"): Can assume that G is connected

- Otherwise run on each component

Simple idea: start coloring nodes starting at a given node s

- Color s **red**
- Color all neighbors of s **green**
- Color all their neighbors **red**, etc.
- If you ever hit a node that was already colored
    - the **same** color as you want to color it, ignore it
    - the **opposite** color, output *"*NOT Bipartite*"* and halt

BFS gives Bipartiteness

Run BFS assigning all vertices from layer $\boldsymbol{L_i}$ the color $\boldsymbol{i} \bmod \mathbf{2}$

- i.e., **red** if they are in an even layer, **green** if in an odd layer

- if no edge joining two vertices of the same color
  - then it is a good coloring
- otherwise
  - there is a bad edge; output “Not Bipartite”

Why is that “Not Bipartite” output correct?

Why does BFS work for Bipartiteness?

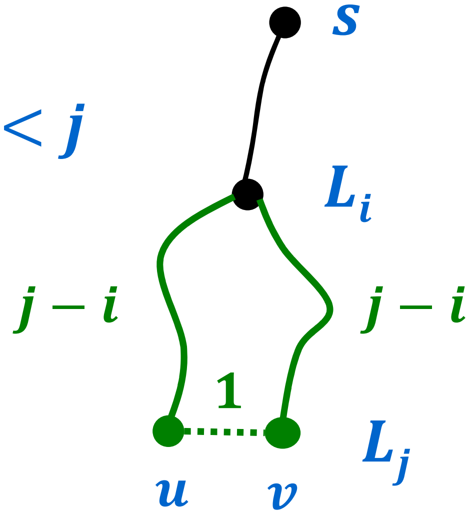
Recall: All edges join vertices on the same or adjacent BFS layers

⇒ Any bad edge must join two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in the same layer

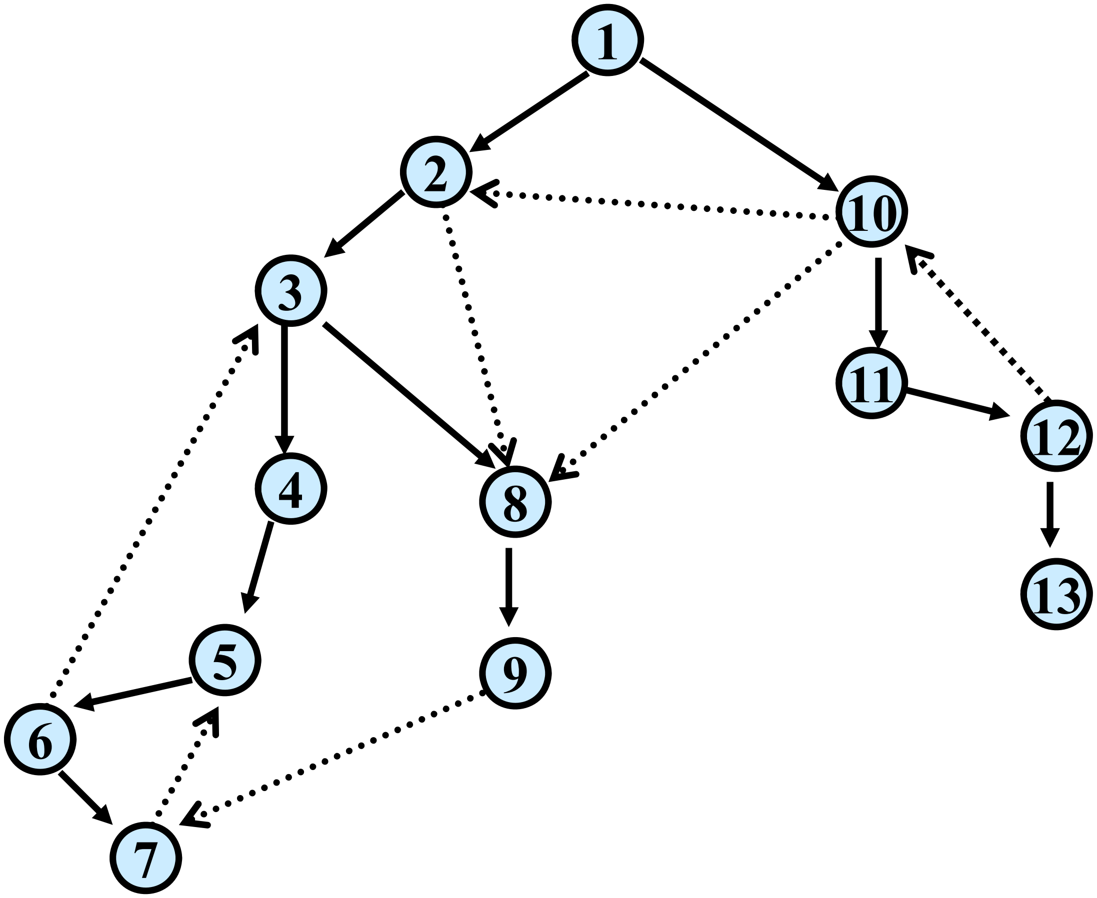
Say the layer with $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{L_j}$

$\boldsymbol{u}$ and $\boldsymbol{v}$ have common ancestor at some level $\boldsymbol{L_i}$ for $\boldsymbol{i < j}$

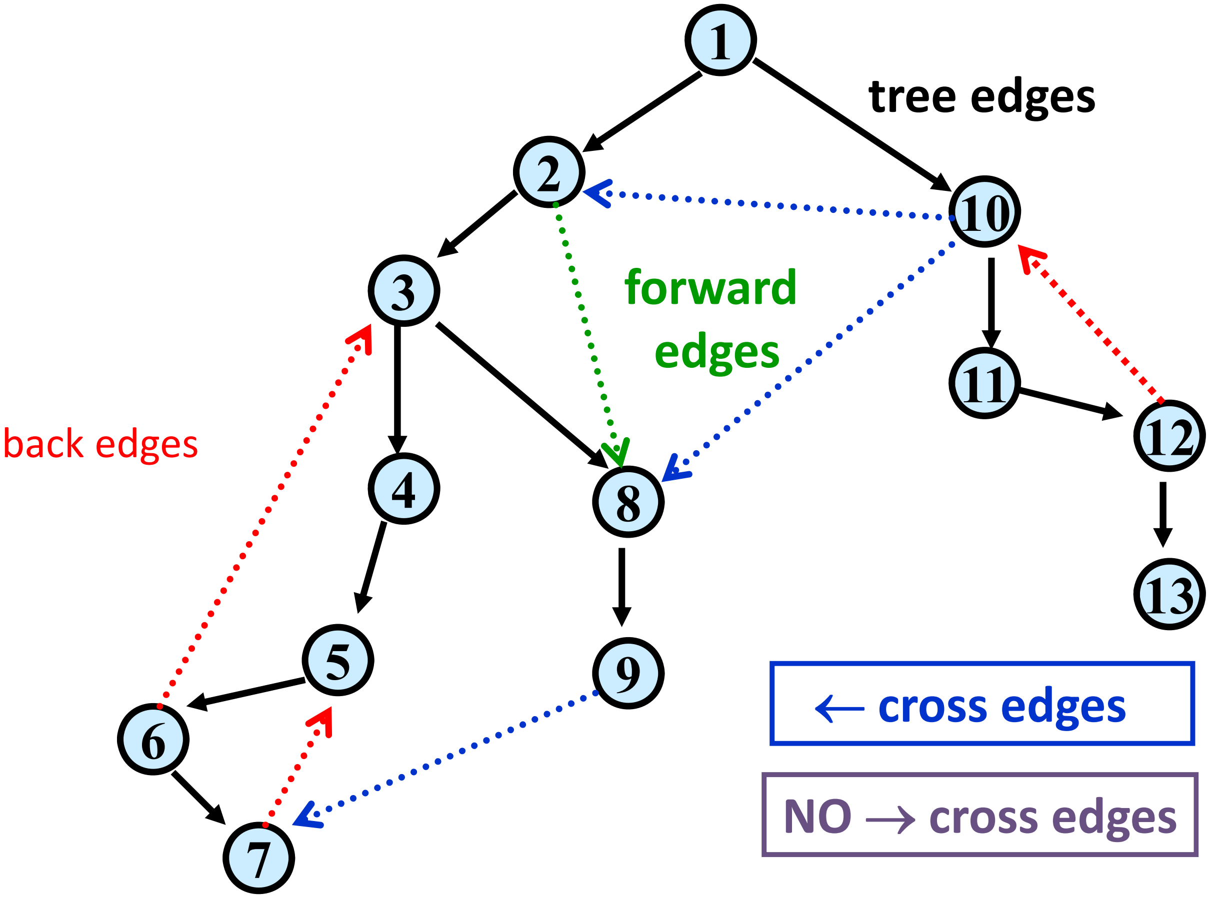
Odd cycle of length $\mathbf{2(j-i)+1}$
⇒ Not Bipartite



DFS(v) for a directed graph



DFS(v)



← cross edges

NO → cross edges

Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

- Every cycle contains a back edge in the DFS tree

Strongly Connected Components of Directed Graphs

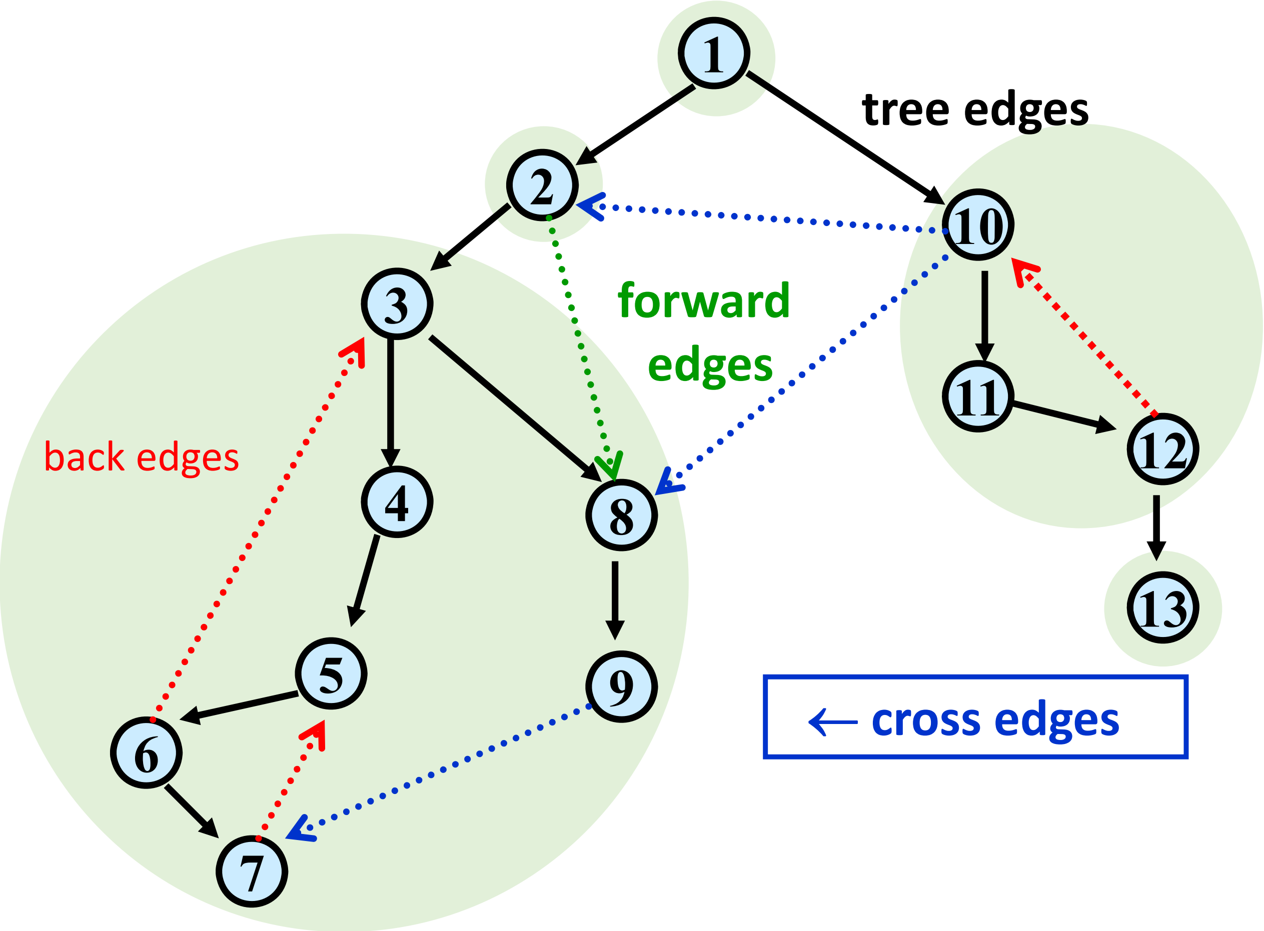
Defn: Vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are strongly connected iff they are on a directed cycle (there are paths from $\boldsymbol{u}$ to $\boldsymbol{v}$ and from $\boldsymbol{v}$ to $\boldsymbol{u}$).

Defn: Can partition vertices of any directed graph into strongly connected components:

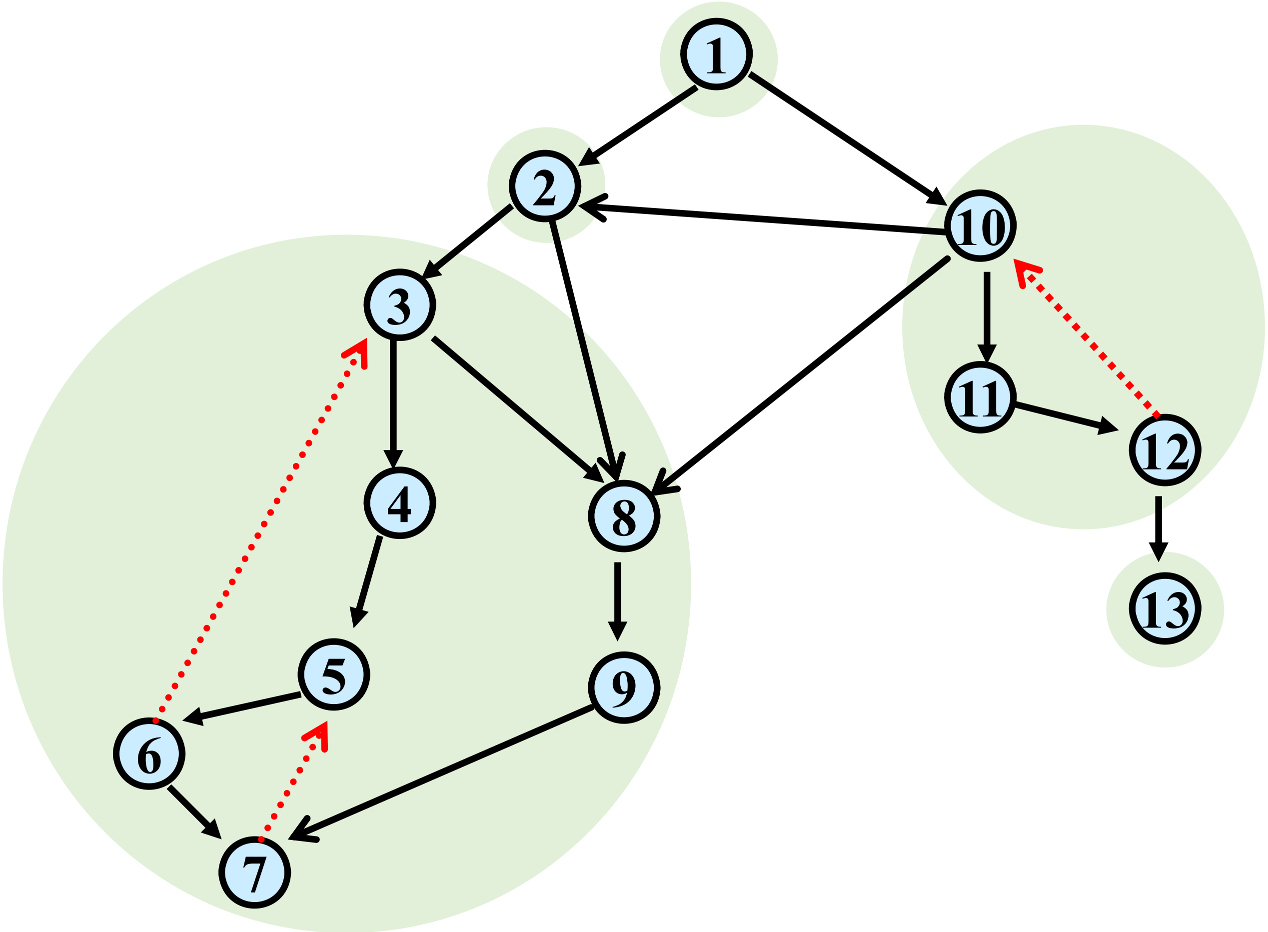
1. all pairs of vertices in the same component are strongly connected
2. can't merge components and keep property 1

- Strongly connected components can be stored efficiently just like connected components
- Can be found by extending DFS algorithm in $O(\boldsymbol{n}+\boldsymbol{m})$ time using extra bookkeeping
  - We won't cover the details

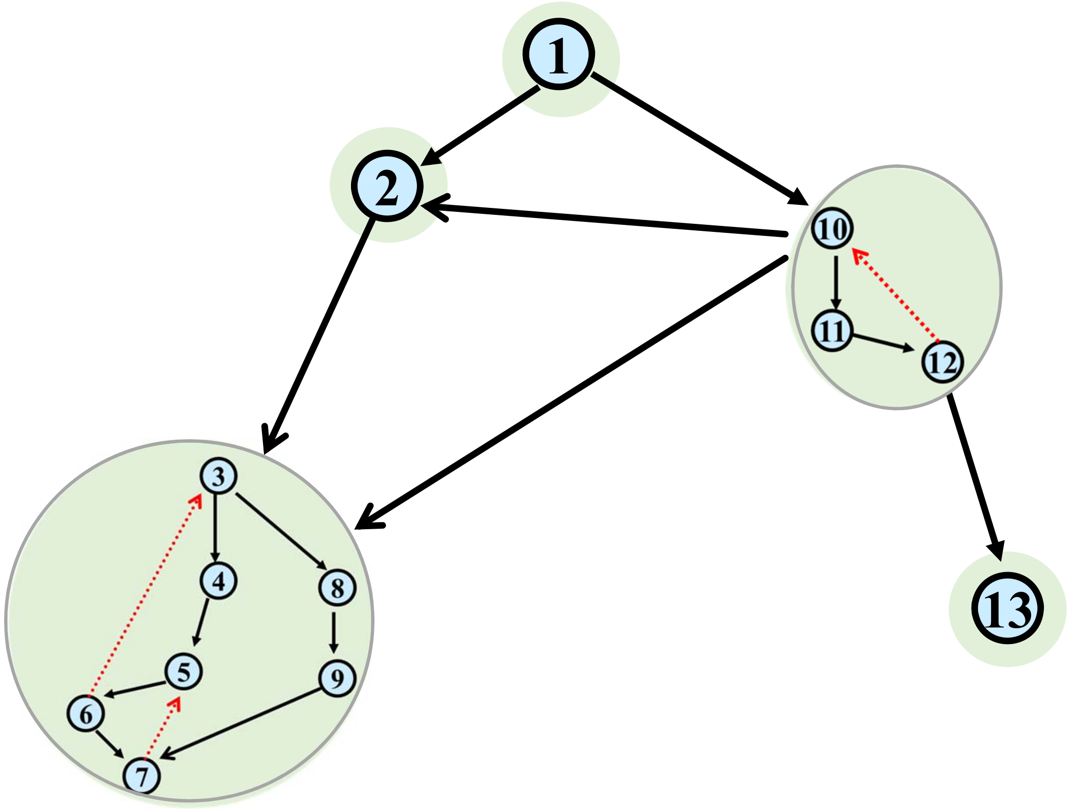
Strongly Connected Components



Strongly Connected Components



Strongly Connected Components



Directed Acyclic Graphs

A directed graph $\boldsymbol{G = (V, E)}$ is acyclic iff it has no directed cycles

Terminology: A directed acyclic graph is also called a DAG

After shrinking the strongly connected components of a directed graph to single vertices, the result is a DAG

Topological Sort

Given: a directed acyclic graph (DAG) $\boldsymbol{G = (V, E)}$

Output: numbering of the vertices of $\boldsymbol{G}$ with distinct numbers from $\mathbf{1}$ to $\boldsymbol{n}$ so that edges only go from lower numbered to higher numbered vertices

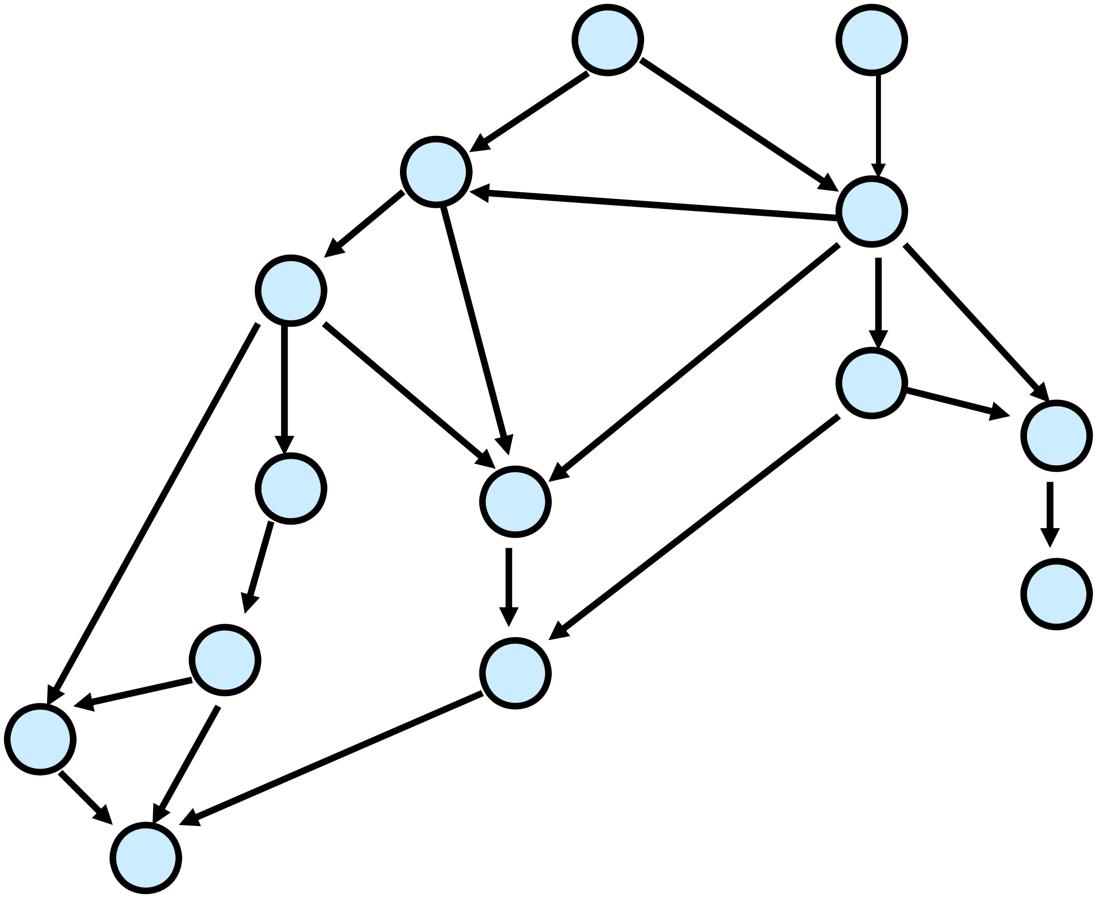
Applications:

- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them

Nice algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.

Directed Acyclic Graph



In-degree 0 vertices

Claim: Every DAG has a vertex of in-degree 0

Proof: By contradiction

Suppose every vertex has some incoming edge

Consider following procedure:

```
while (true) do
    $v \leftarrow$ some predecessor of $v$
```

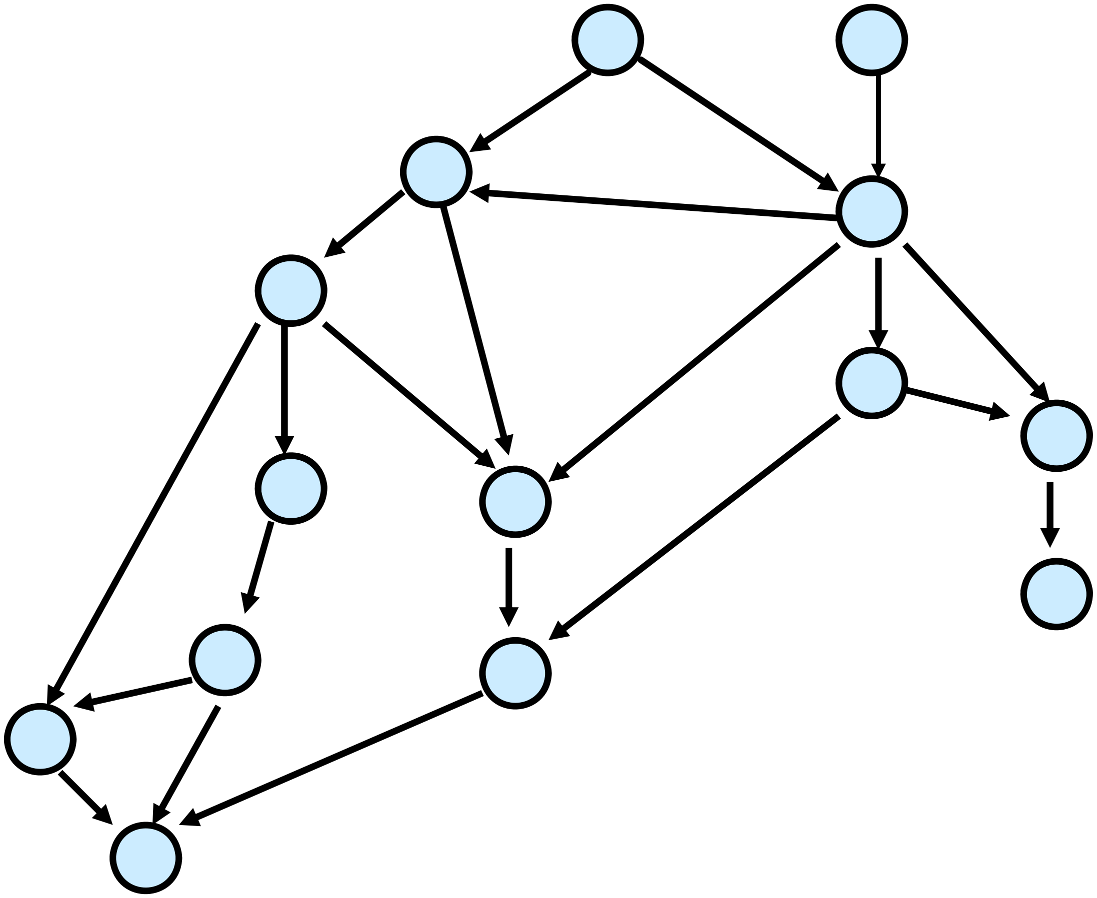
- After $n+1$ steps where $n=|V|$ there will be a repeated vertex
  - This yields a cycle, contradicting that it is a DAG. ■

Topological Sort

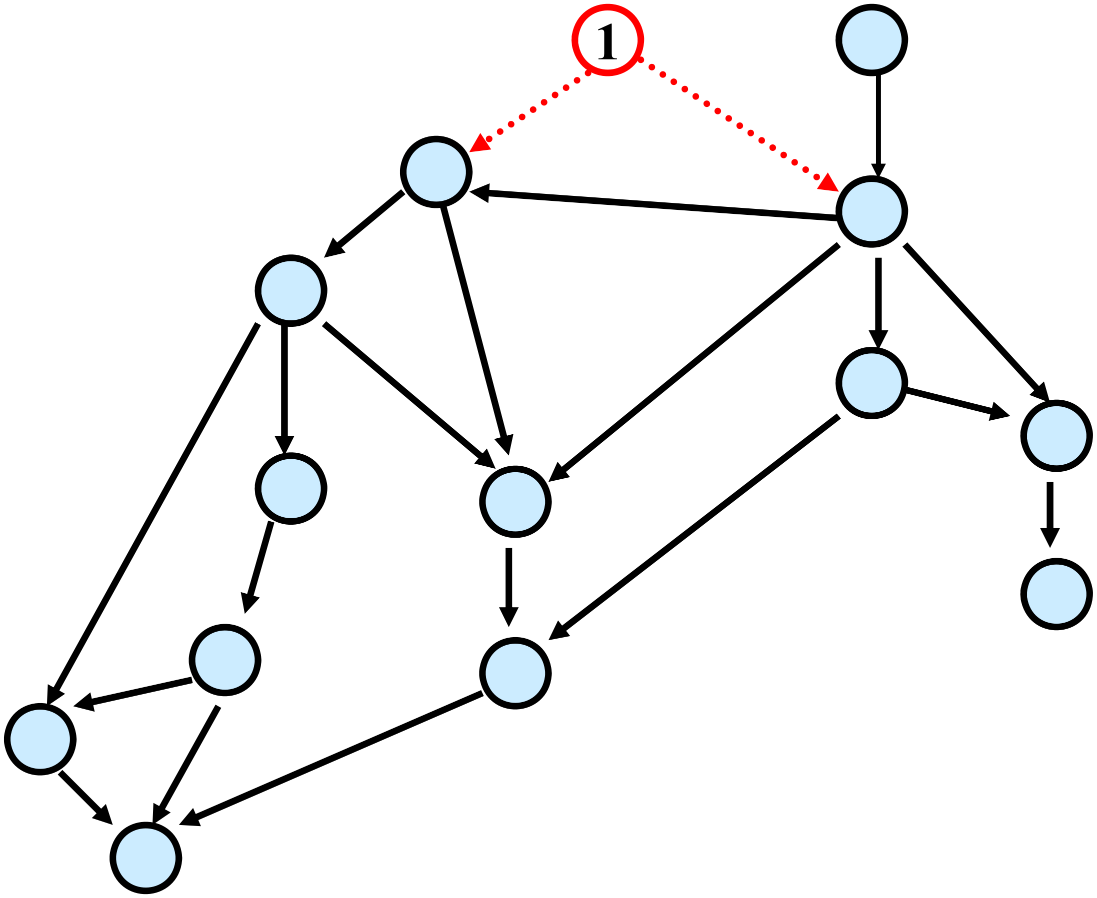
- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph
  - Then give a vertex of in-degree 0 number 2
  - Etc.

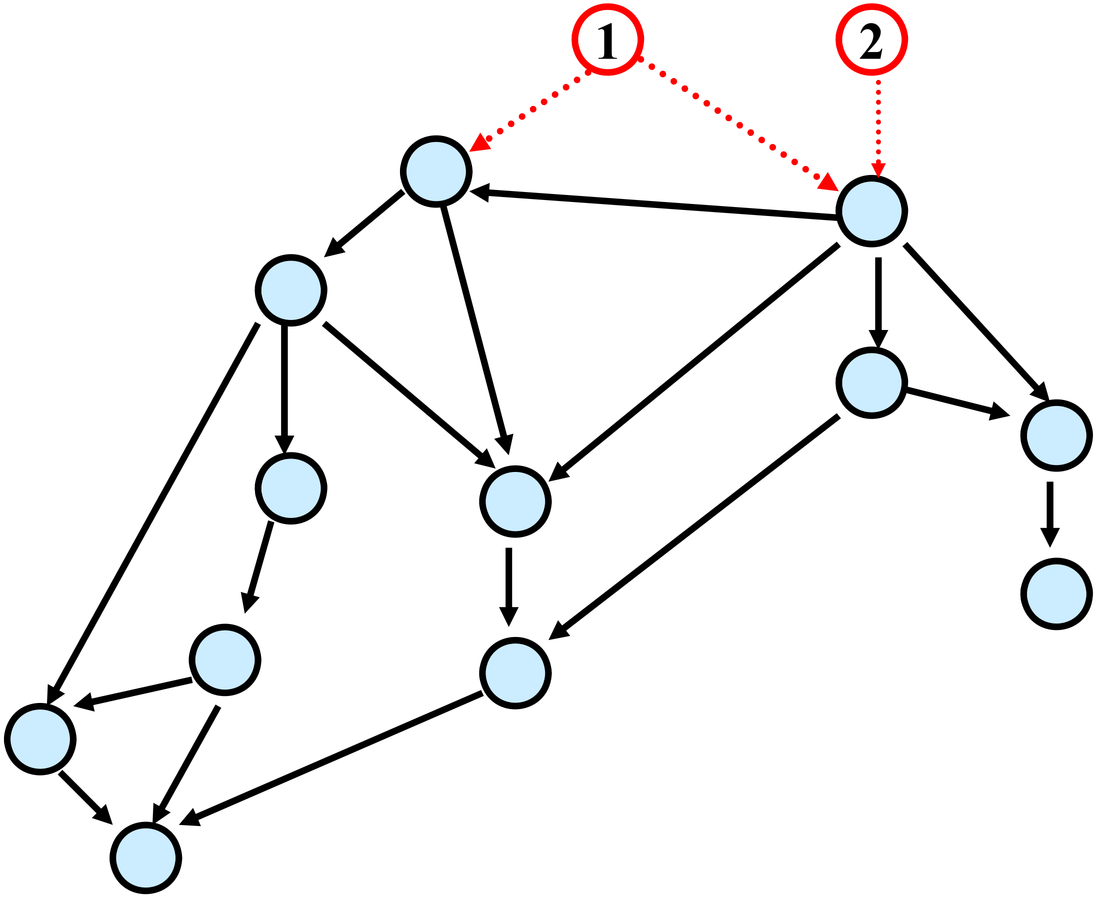
Topological Sort



Topological Sort



Topological Sort



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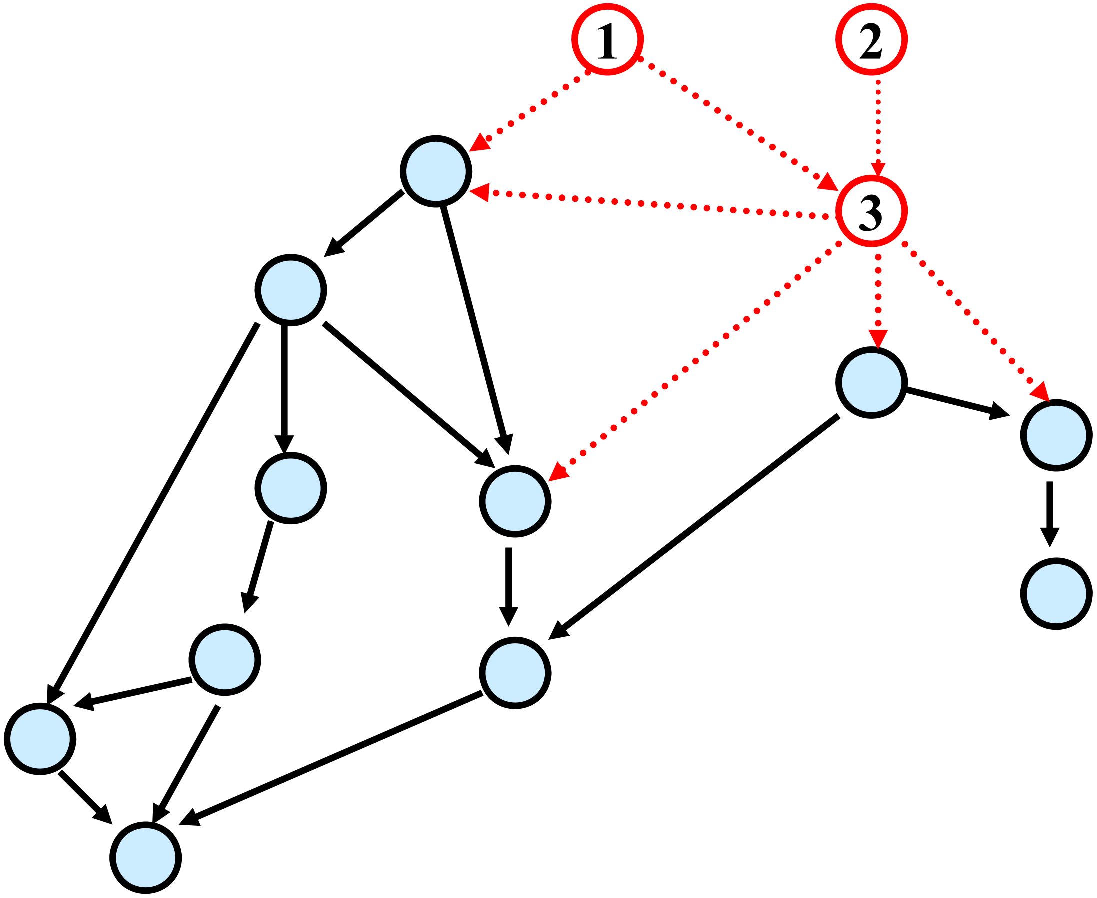

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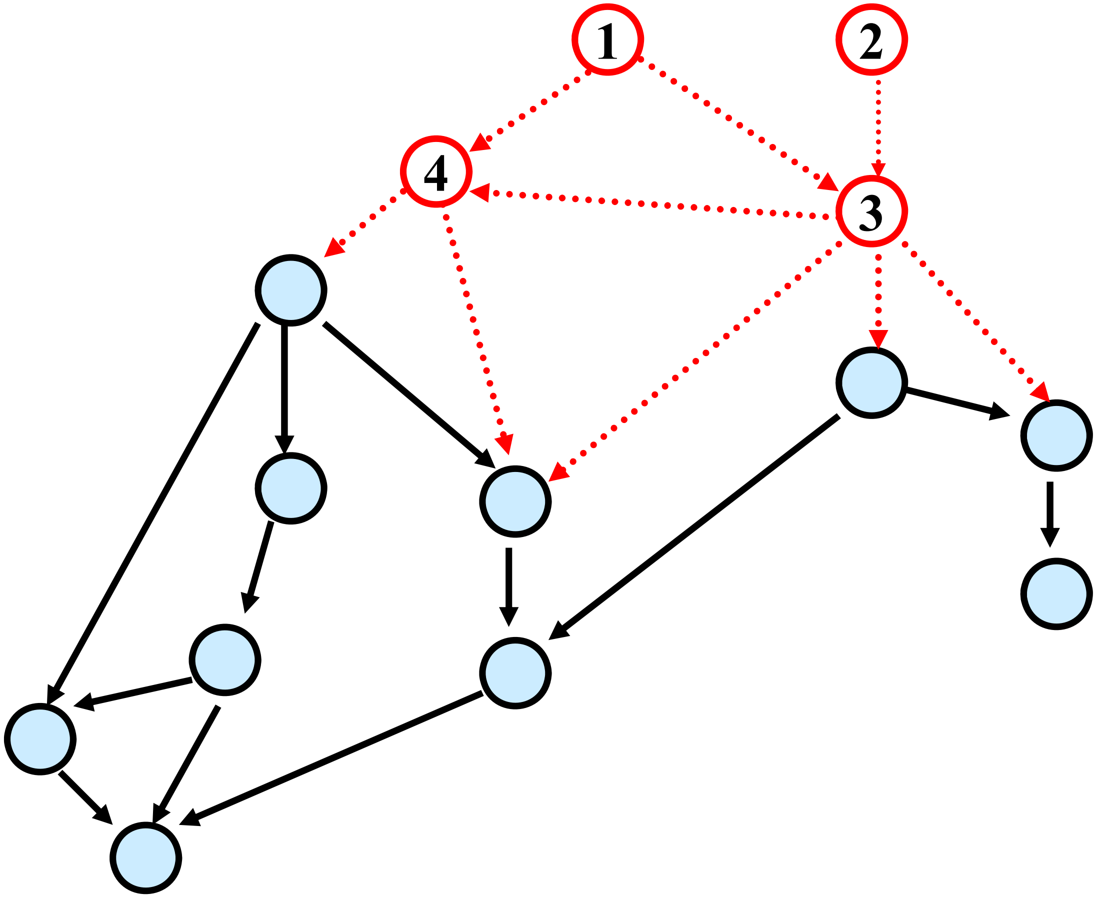

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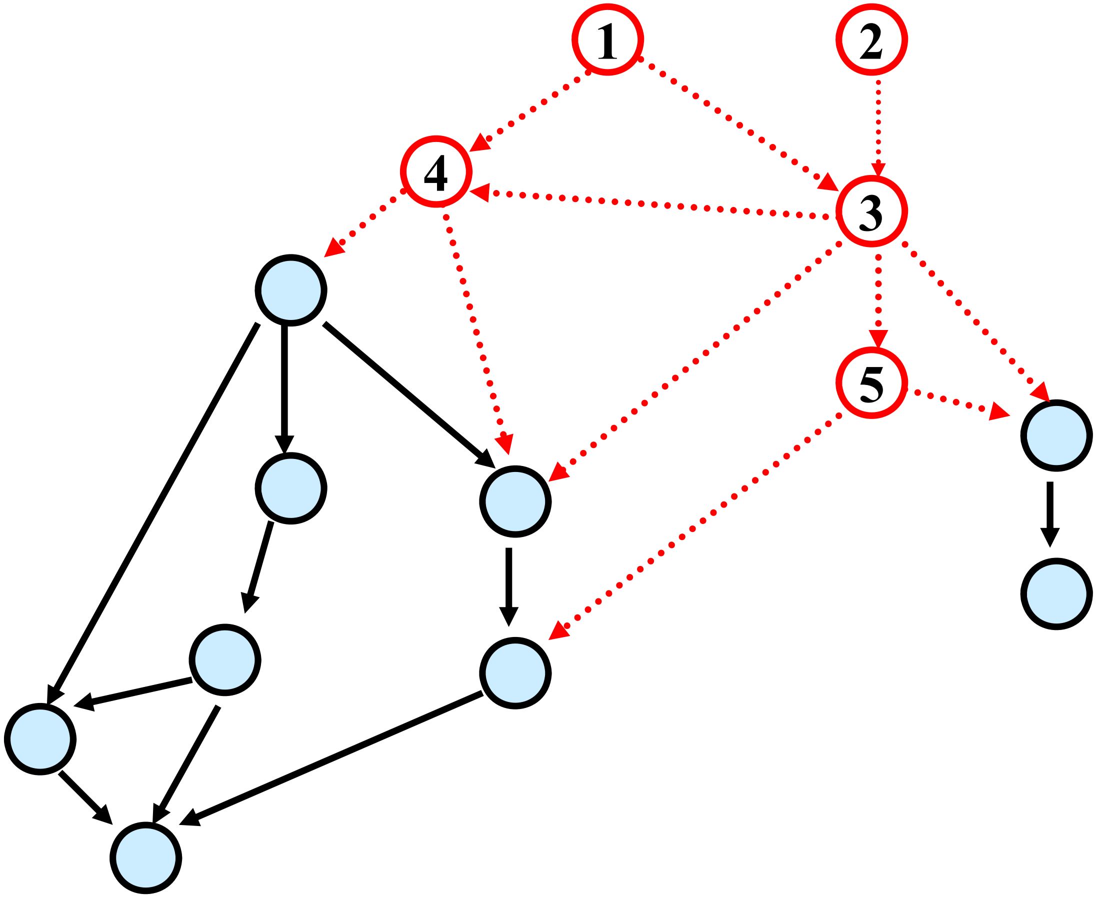

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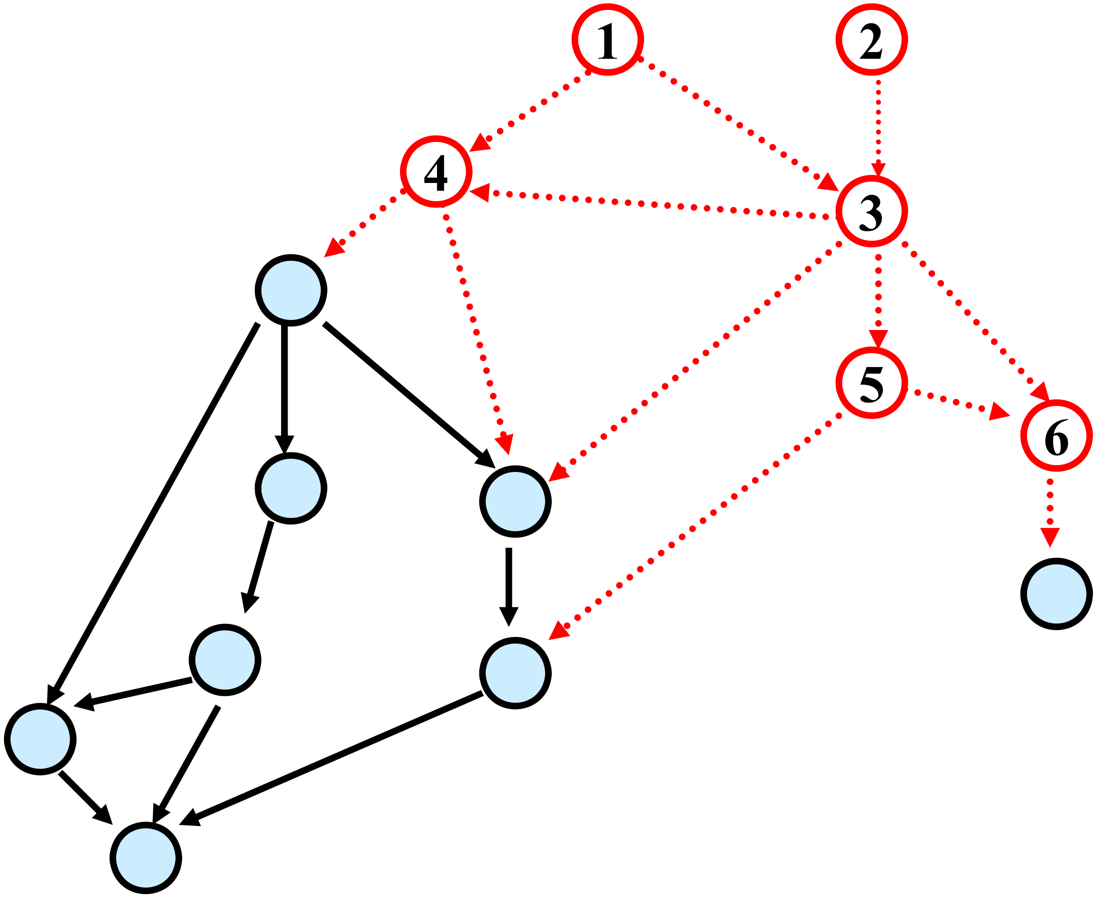

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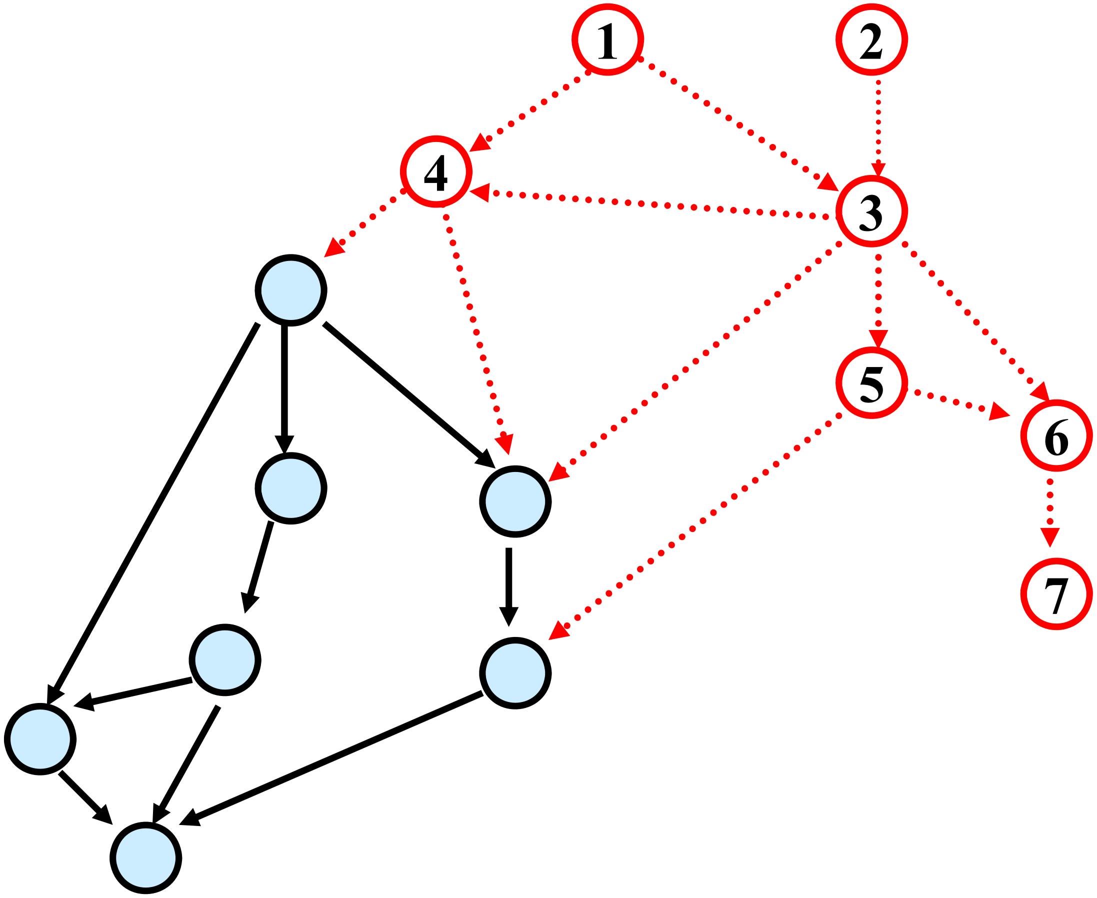

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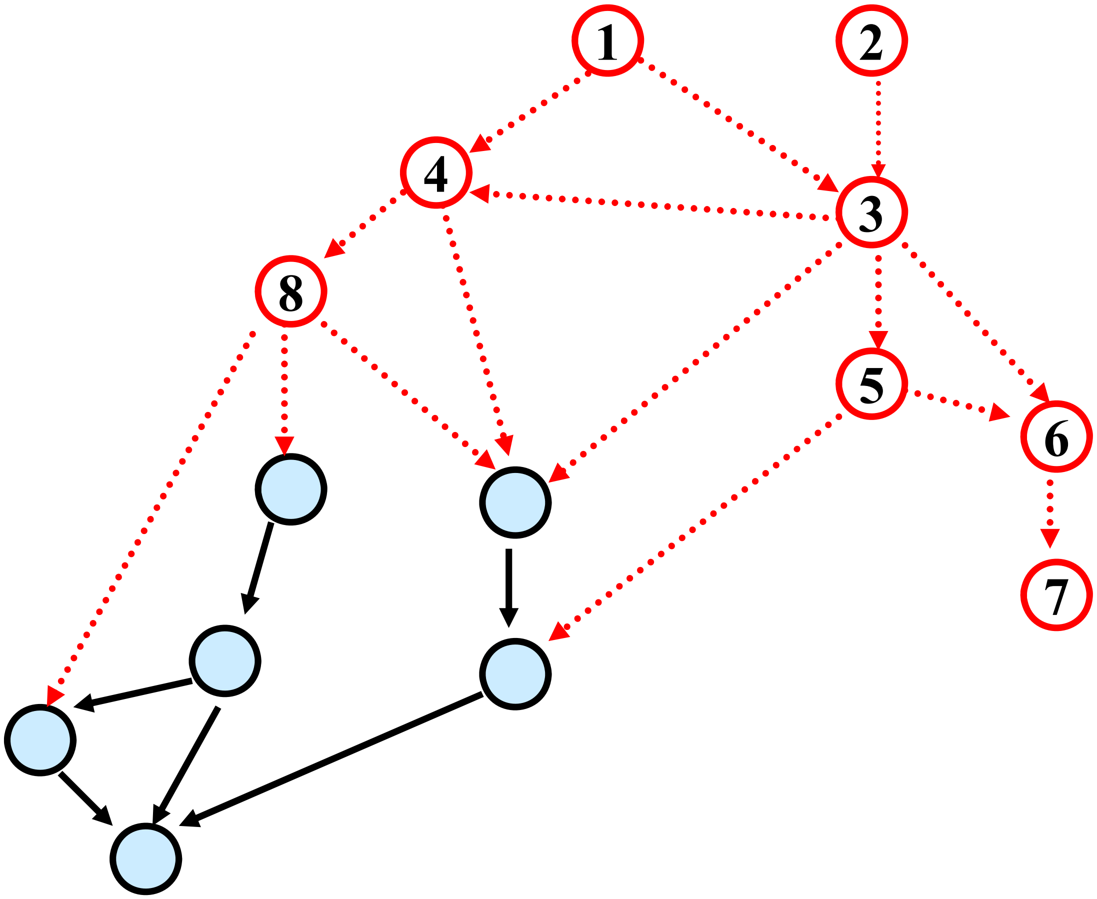

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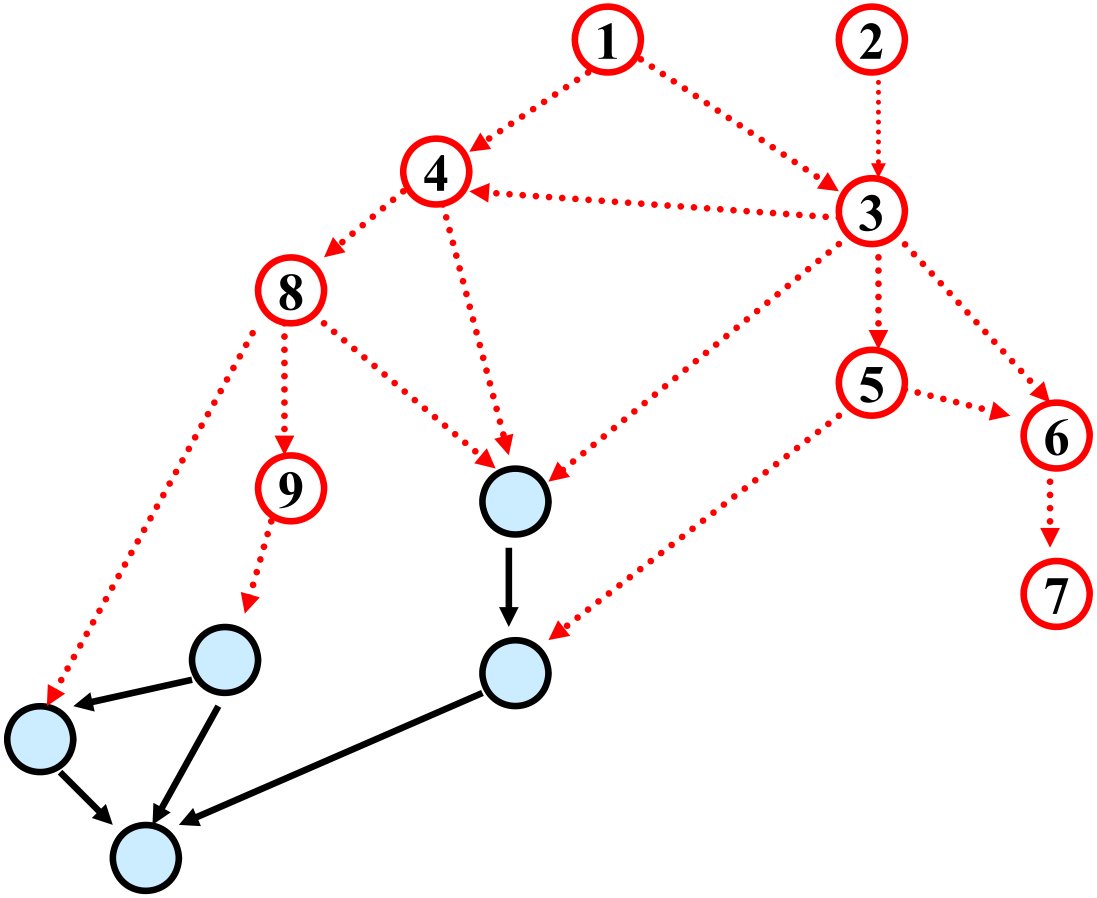

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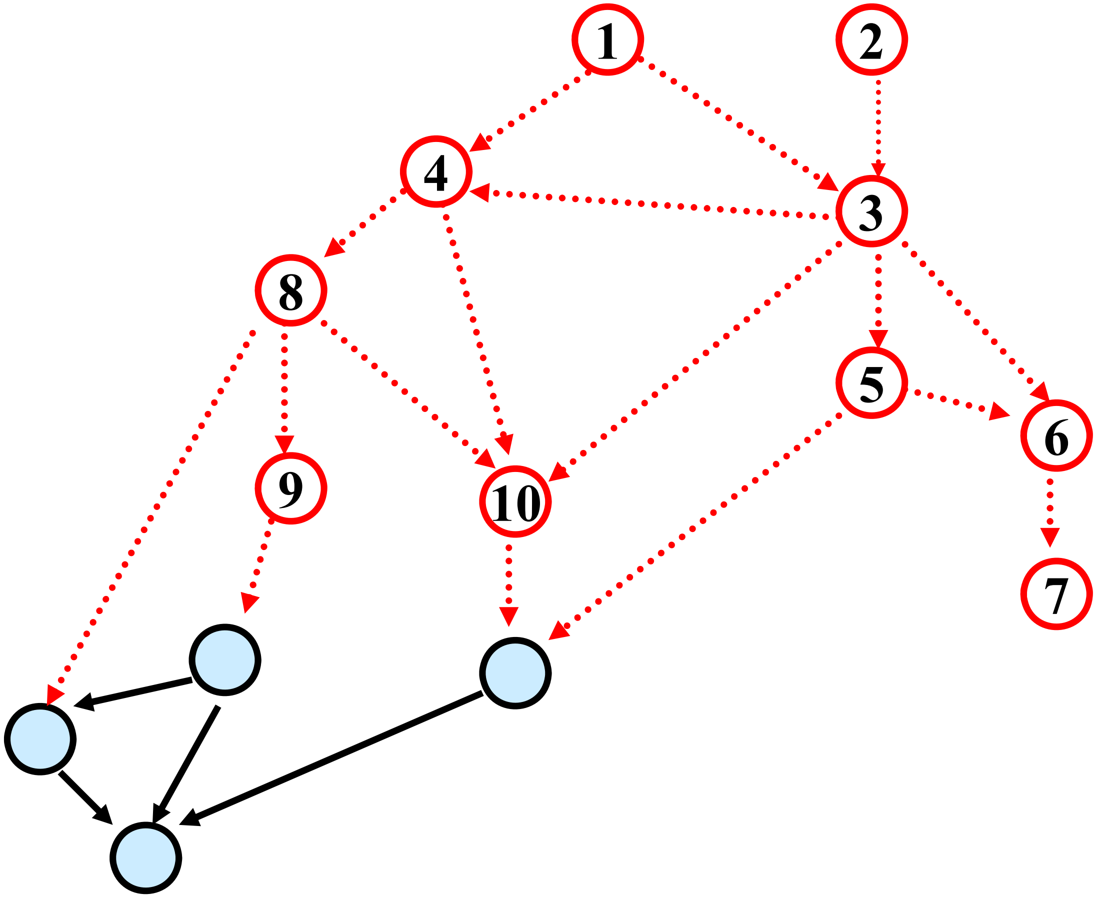

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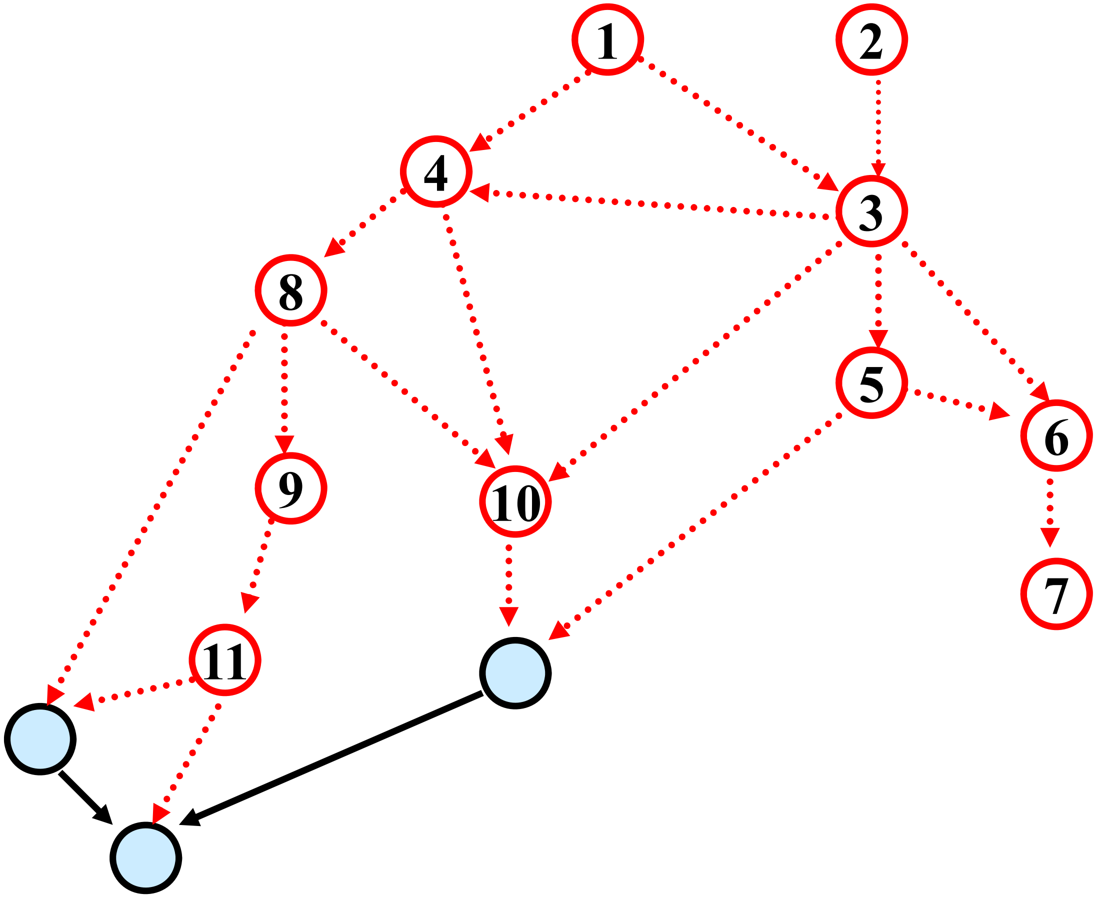

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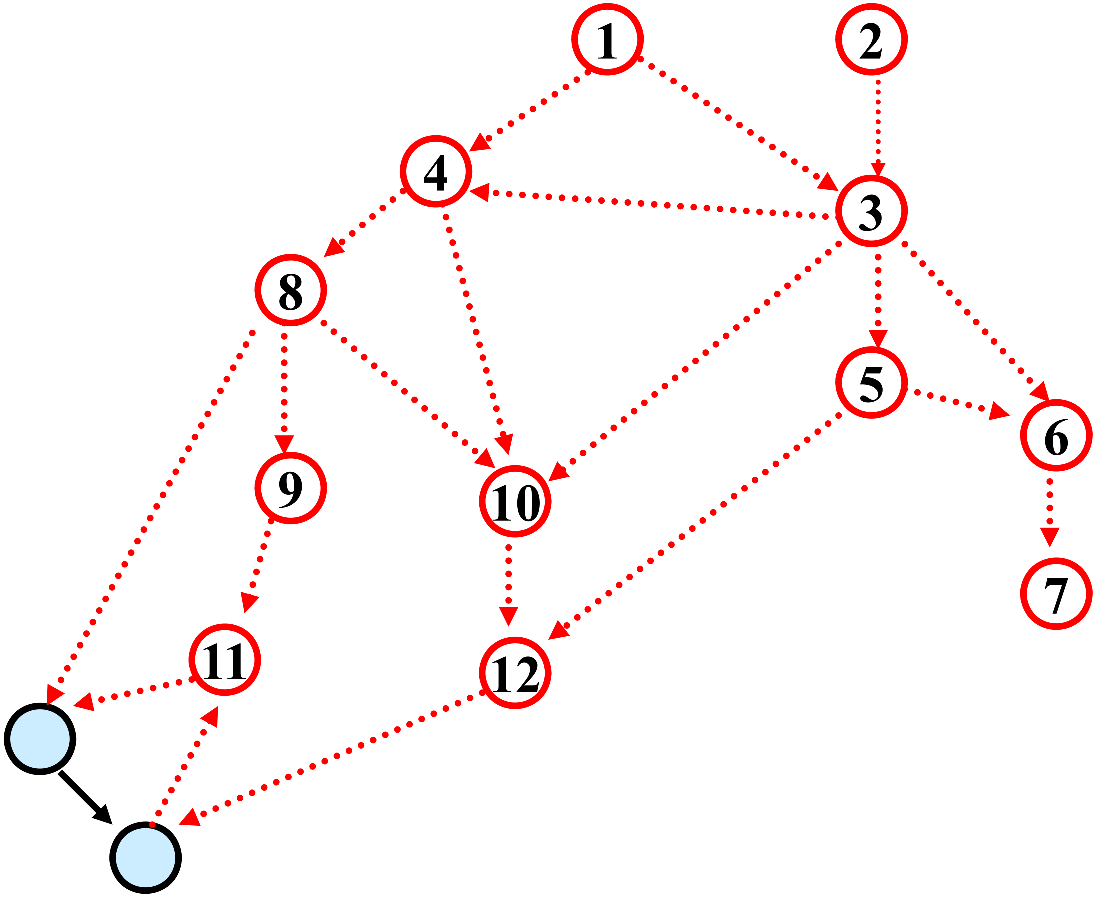

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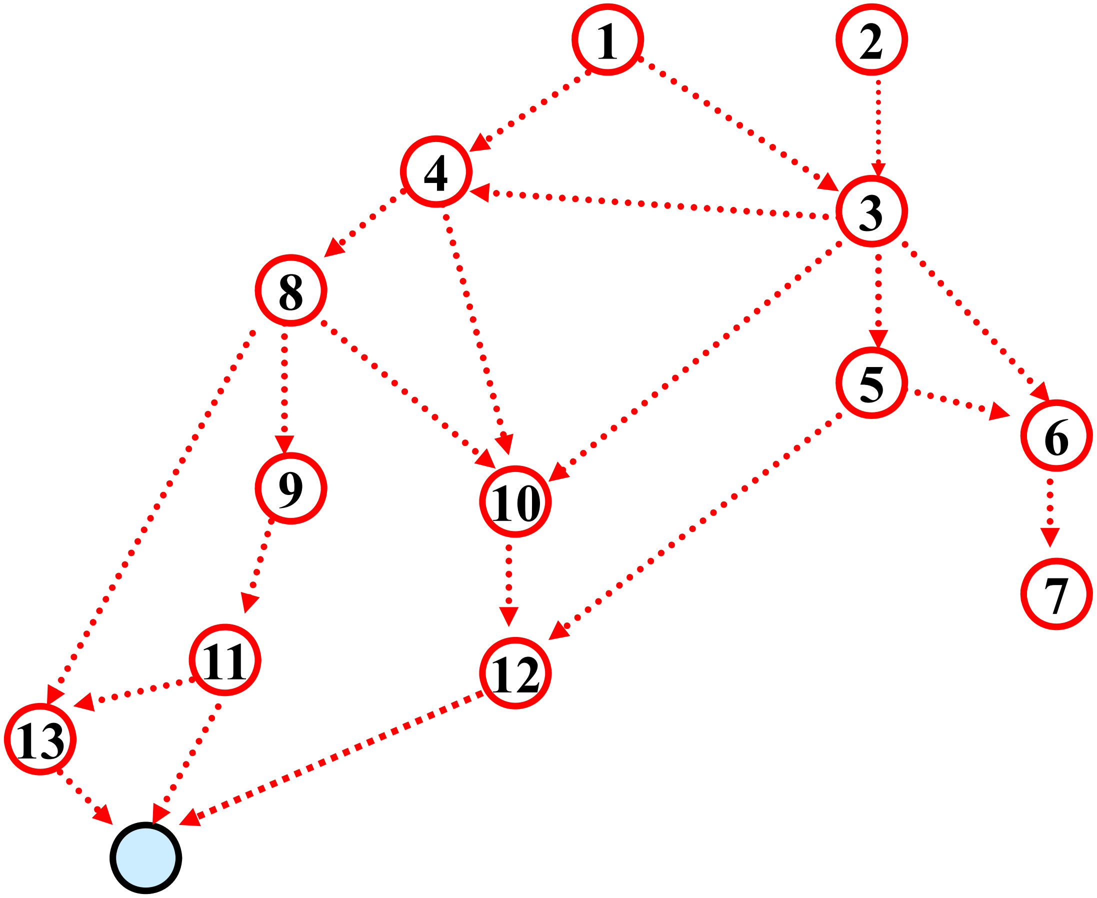

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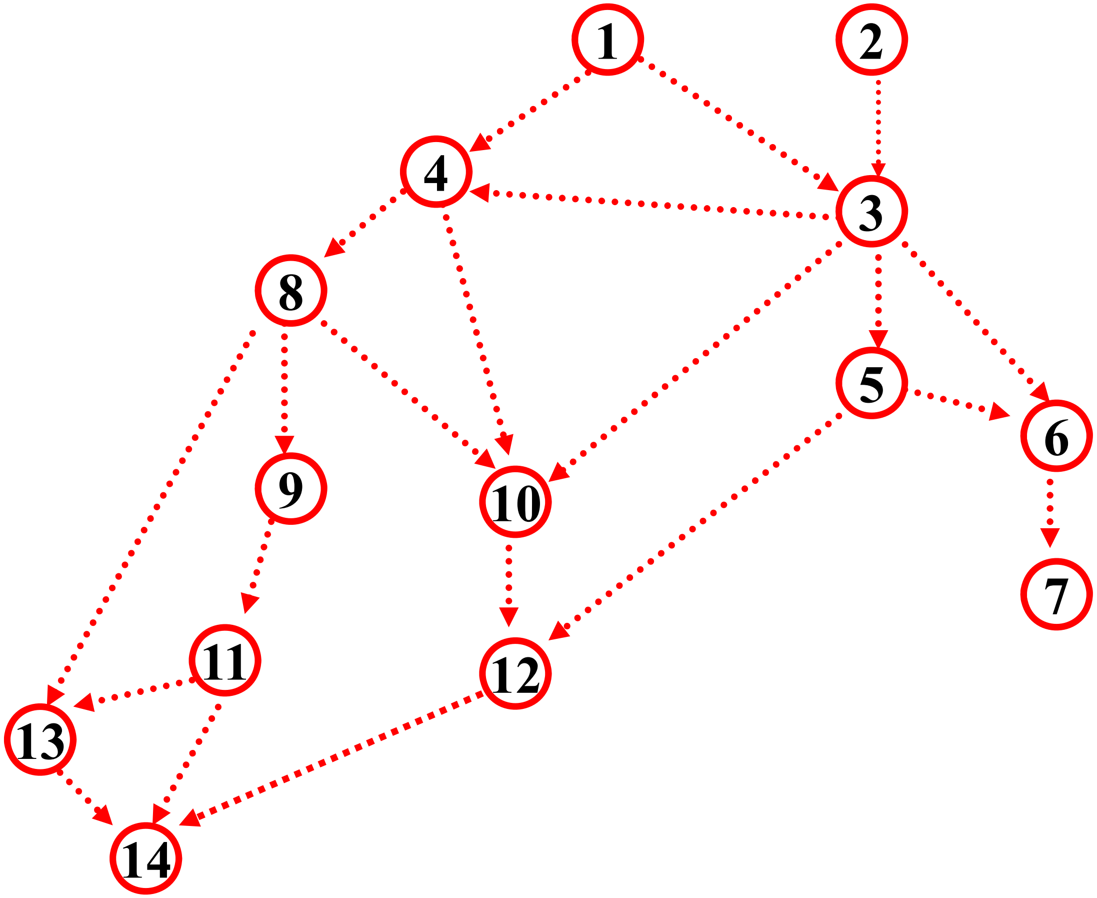

Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex
  $O(m+n)$
- Maintain a list of vertices of in-degree $\mathbf{0}$
- Remove any vertex in list and number it
- When a vertex is removed, decrease in-degree of each neighbor by $\mathbf{1}$ and add them to the list if their degree drops to $\mathbf{0}$

Total cost: $O(m+n)$