CSE 421
Introduction to Algorithms

Lecture 5: Greedy Algorithms

Sorry about lack of audio in Wednesday's recording.

I'll answer — Any questions about it on Ed

Please listen for echo & confirm mike is on

Greedy Algorithms

Hard to define exactly but can give general properties

- Solution is built in small steps
- Decisions on how to build the solution are made to maximize some criterion without looking to the future
  - Want the 'best' current partial solution as if the current step were the last step

May be more than one greedy algorithm using different criteria to solve a given problem

- Not obvious which criteria will actually work

Greedy Algorithms

- Greedy algorithms
  - Easy to produce
  - Fast running times
  - Work only on certain classes of problems
    - Hard part is showing that they are correct

- Focus on methods for proving that greedy algorithms do work

Interval Scheduling

Interval Scheduling:

- Single resource
- Reservation requests of form:

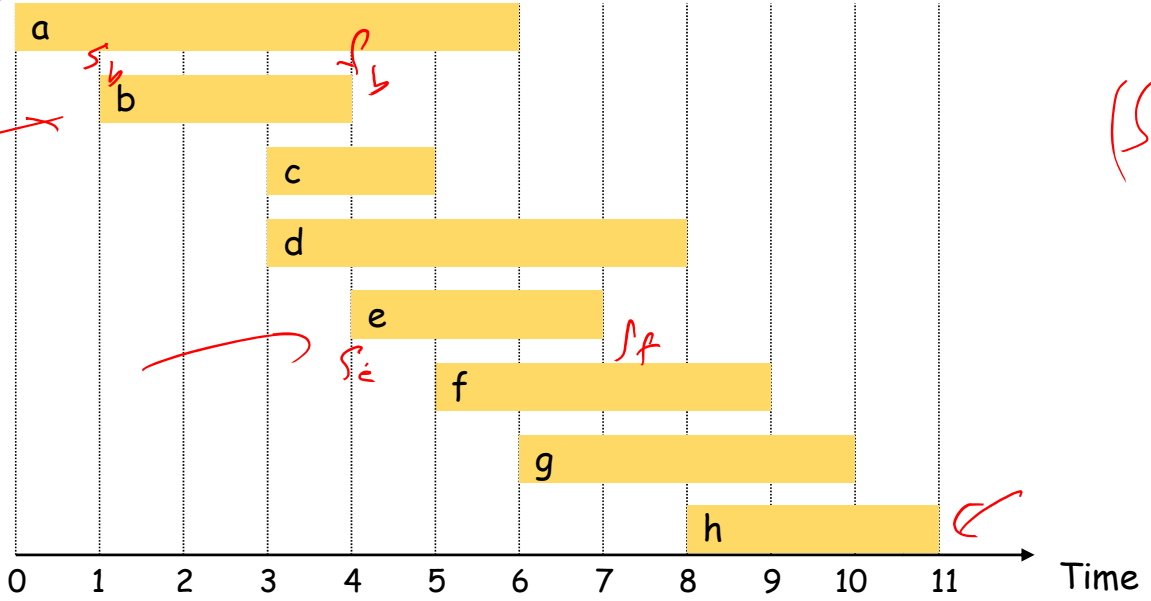
  "Can I reserve it from start time s to finish time f?"

$$s < f$$

Interval Scheduling

Interval scheduling:

- Job j starts at s_j and finishes at $f_j > s_j$.
- Two jobs i and j are compatible if they don't overlap: $f_i \leq s_j$ or $f_j \leq s_i$
- **Goal:** find maximum size subset of mutually compatible jobs.



Greedy Algorithms for Interval Scheduling

- What criterion should we try?

Greedy Algorithms for Interval Scheduling

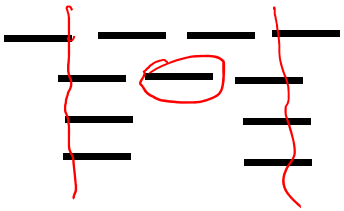
- What criterion should we try?
  - Earliest start time s_i
  - Shortest request time $f_i - s_i$
  - Fewest conflicts

Greedy Algorithms for Interval Scheduling

- What criterion should we try?
  - Earliest start time s_i
    - Doesn't work

  - Shortest request time $f_i - s_i$
    - Doesn't work

  - Fewest conflicts
    - Doesn't work



  - Earliest finish time f_i
    - Works!

Also latest start will work

Greedy (by finish time) Algorithm for Interval Scheduling

$R \leftarrow$ set of all requests

$A \leftarrow \emptyset$

while $R \neq \emptyset$ do

Choose request $i \in R$ with smallest finish time f_i

Add request i to A

Delete all requests in R not compatible with request i

return A

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

Interval Scheduling: Analysis

Claim: A is a compatible set of requests and
requests are added to A in order of finish time

- When we add a request to A we delete all incompatible ones from R

Name the finish times of requests in A as $a_1, a_2, \ldots, a_t$ in order.

Claim: Let $O \subseteq R$ be a set of compatible requests whose finish times in order are $o_1, o_2, \ldots, o_s$. Then for every integer $k \geq 1$ we have:

a) if O contains a k^{th} request then A does too, and

b) $a_k \leq o_k$ "A is ahead of O"

Note that a) alone implies that $t \geq s$ which means that A is optimal but we also need b) "stays ahead" to keep the induction going.

Inductive Proof of Claim

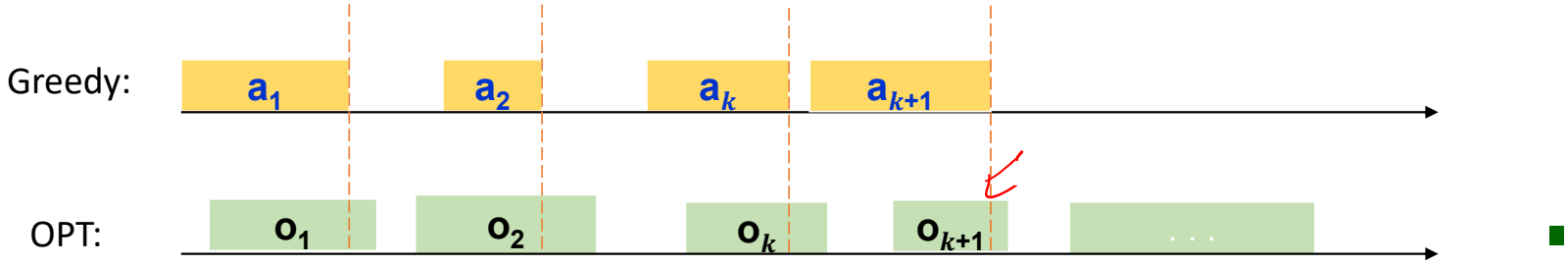
Base Case $k = 1$: A includes the request with smallest finish time, so if O is not empty then $a_1 \leq o_1$

Inductive Step: Suppose that $a_k \leq o_k$ and there is a k+1st request in O.

Then k+1st request in O is compatible with $a_1, a_2, \ldots, a_k$ since $a_k \leq o_k$ and $o_k \leq$ start time of k+1st request in O whose finish time is o_{k+1}

⇒ There is a k+1st request in A whose finish time is named a_{k+1}.

Also, since A would have considered both requests and chosen the one with the earlier finish time, $a_{k+1} \leq o_{k+1}$.



Interval Scheduling: Greedy Algorithm Implementation

```
Sort jobs by finish times so that 0 ≤ f_1 ≤ f_2 ≤ ... ≤ f_n.

A ← ϕ
last ← 0
for j = 1 to n {
    if (last ≤ s_j)
      A ← A ∪ {j}
      last ← f_j
}
return A
```

$O(n \log n)$

$O(n)$

$O(n \log n)$

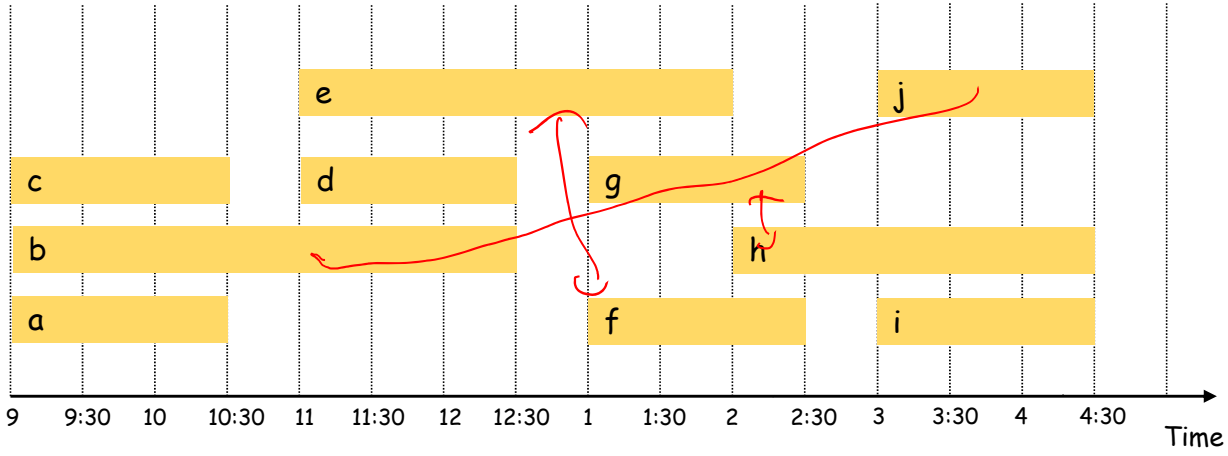
Scheduling All Intervals: Interval Partitioning

Interval Partitioning:

- Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of rooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses 4 rooms to schedule 10 lectures.



Can you do better?

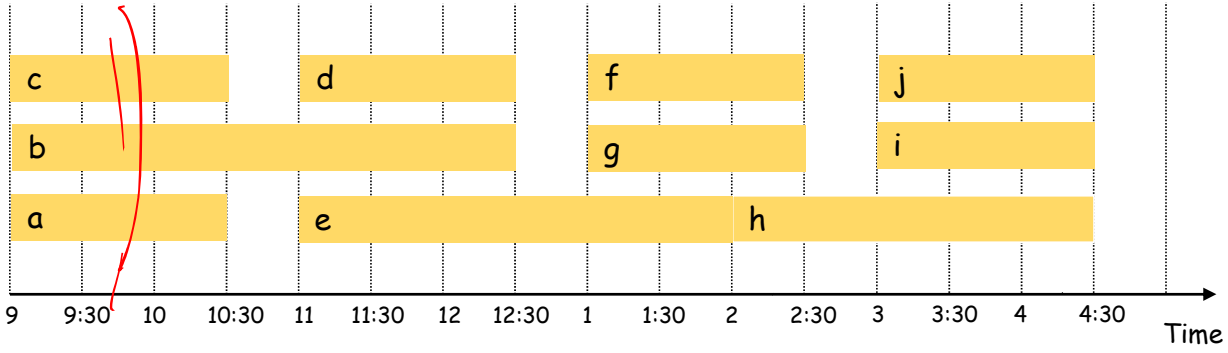
Scheduling All Intervals: Interval Partitioning

Interval Partitioning:

- Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of rooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses only 3 rooms.

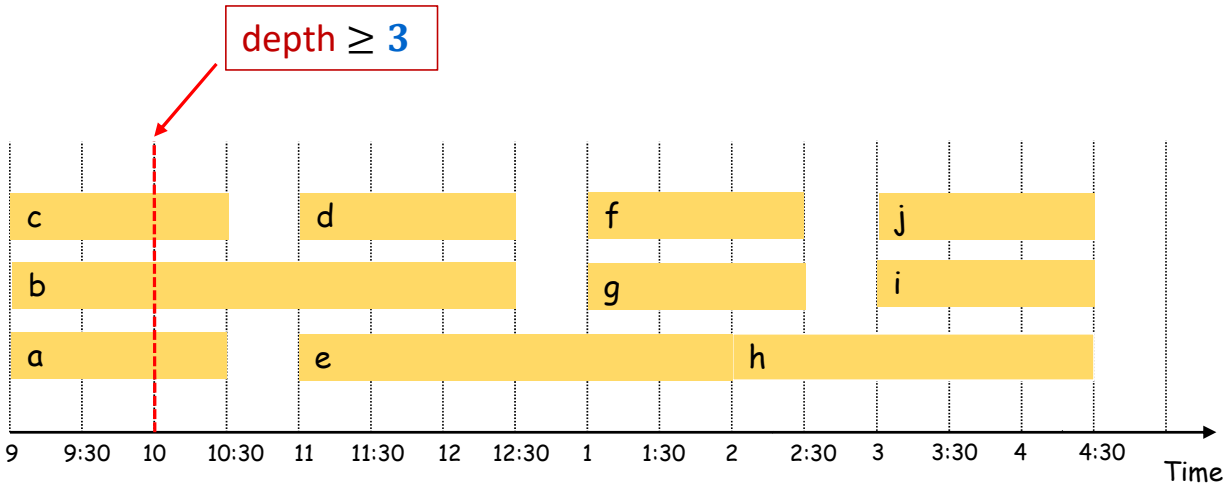


Scheduling All Intervals: Interval Partitioning

Defn: The depth of a set of open intervals is the maximum number that contain any given time.

Key observation: # of rooms needed $\geq$ depth.

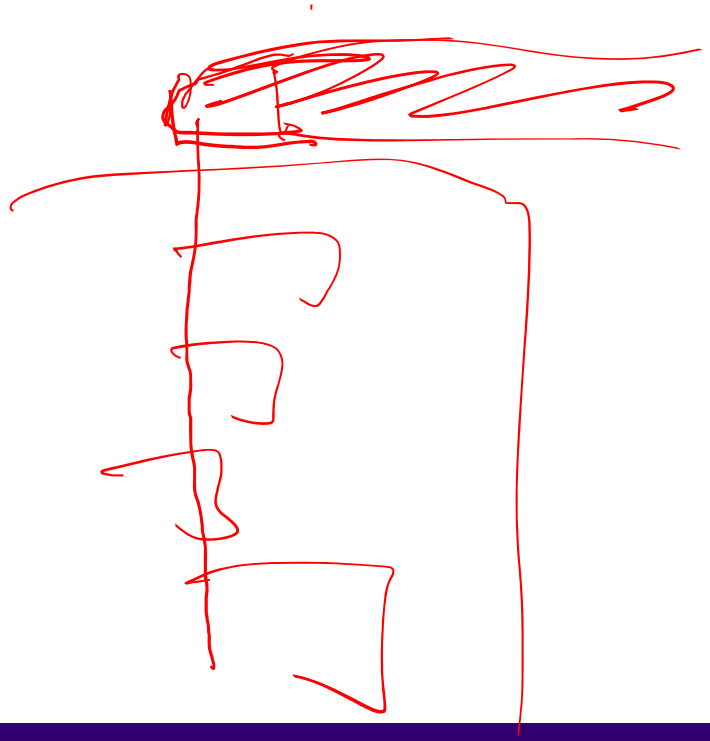
Example: This schedule uses only $\mathbf{3}$ rooms. Since depth $\geq \mathbf{3}$ this is optimal.



A simple greedy algorithm

Sort requests in increasing order of start times $(s_1, f_1), \ldots, (s_n, f_n)$

$last_1 \leftarrow 0$ // finish time of last request currently scheduled in room 1
for $i \leftarrow 1$ to n {
  $j \leftarrow 1$
  while (request i not scheduled) {
    if $s_i \geq last_j$ then
      schedule request i in room j
      $last_j \leftarrow f_i$
    $j \leftarrow j + 1$
    if $last_j$ undefined then $last_j \leftarrow 0$
  }
}



Interval Partitioning: Greedy Analysis

Observation: Greedy algorithm never schedules two incompatible lectures in the same room

- Only schedules request i in room j if $s_i \geq last_j$

Theorem: Greedy algorithm is optimal.

Proof:

Let d = number of rooms that the greedy algorithm allocates.

- Room d is allocated because we needed to schedule a request, say j, that is incompatible with some request in each of the other $d - 1$ rooms.
- Since we sorted by start time, these incompatibilities are caused by requests that start no later than s_j and finish after s_j.

So… we have d requests overlapping at time $s_j + \varepsilon$ for some tiny $\varepsilon > 0$.

Key observation $\Rightarrow$ all schedules use $\geq d$ rooms. ∎

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      $last_j \leftarrow f_i$

    $j \leftarrow j + 1$

    if $last_j$ undefined then $last_j \leftarrow 0$

  }

}

Might need to try all d rooms to schedule a request

Runtime analysis

$O(n \log n)$

$O(n\,d)$

d might be as big as n

Worst case $\Theta(n^2)$

A more efficient implementation: Priority queue

Sort requests in increasing order of start times $(s_1, f_1), \ldots, (s_n, f_n)$ $O(n \log n)$

$d \leftarrow 1$
schedule request 1 in room 1
$last_1 \leftarrow f_1$
insert 1 into priority queue Q with key = $last_1$
for $i \leftarrow 2$ to n {
  $j \leftarrow$ findmin(Q) $O(1)$
  if $s_i \geq last_j$ then {
    schedule request i in room j
    $last_j \leftarrow f_i$
    increasekey(j,Q) to $last_j$ } $O(\log d)$
  else {
    $d \leftarrow d + 1$
    schedule request i in room d
    $last_d \leftarrow f_i$
    insert d into priority queue Q with key = $last_d$} $O(\log d)$
}

$O(n \log d)$

$\Theta(n \log n)$ total

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

Structural: Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument: Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.