CSE 421
Introduction to Algorithms

Lecture 7: Minimum Spanning Trees
Prim, Kruskal and more

Glasses left at Allen
Center breakout...

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

Structural: Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument: Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Minimum Spanning Trees (Forests)

MST

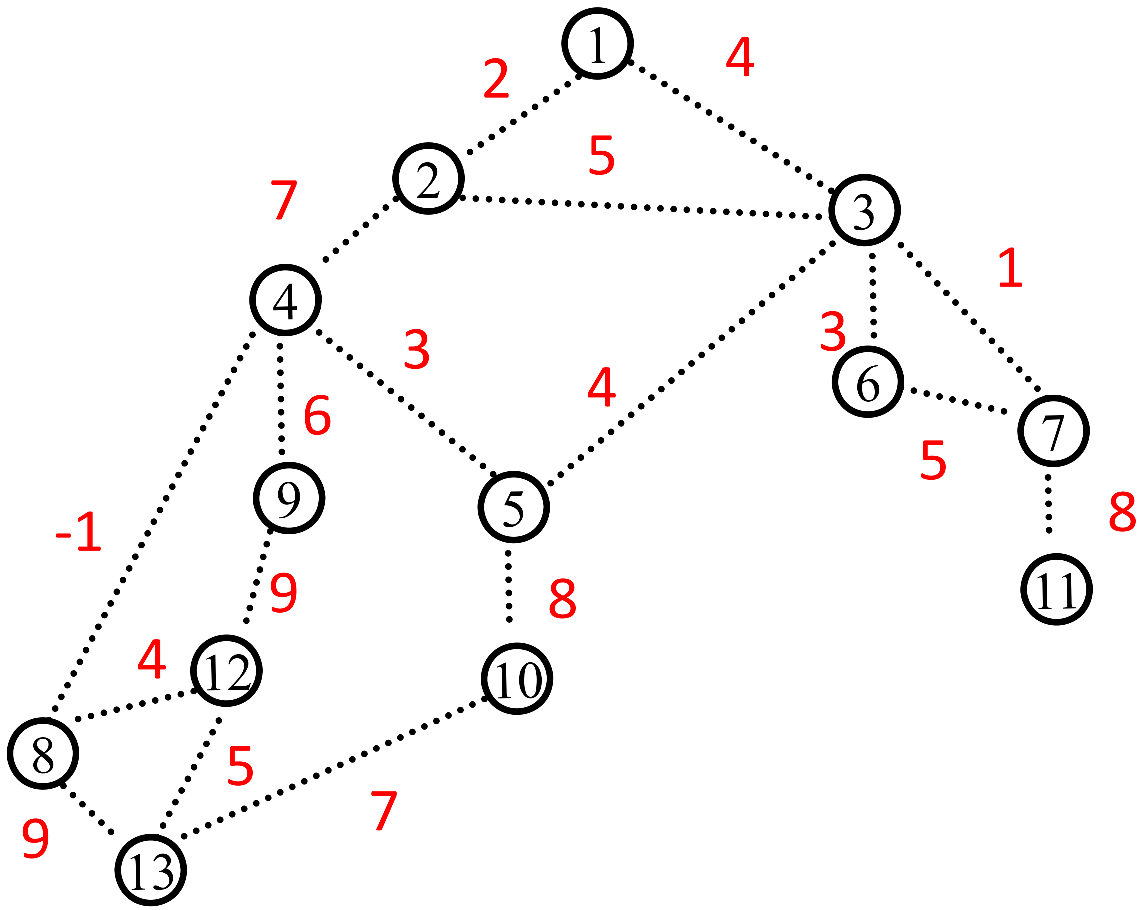
Given: an undirected graph $\boldsymbol{G = (V, E)}$ with each edge $\boldsymbol{e}$ having a weight $\boldsymbol{w(e)}$

Find: a subgraph $\boldsymbol{T}$ of $\boldsymbol{G}$ of minimum total weight s.t.
every pair of vertices connected in $\boldsymbol{G}$ are also connected in $\boldsymbol{T}$

If $\boldsymbol{G}$ is connected then $\boldsymbol{T}$ is a tree

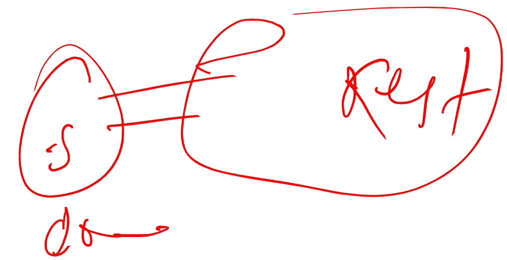
- Otherwise, $\boldsymbol{T}$ is still a forest

Weighted Undirected Graph



Greedy Algorithm

Prim’s Algorithm:



- start at a vertex s
- add the cheapest edge adjacent to s
- repeatedly add the cheapest edge that joins the vertices explored so far to the rest of the graph

Exactly like Dijsktra’s Algorithm but with a different objective

Dijsktra’s Algorithm

Dijkstra(G,w,s)

$S \leftarrow \{s\}$

$d[s] \leftarrow 0$

while $S \neq V$ {

among all edges $e = (u, v)$ s.t. $v \notin S$ and $u \in S$ select* one with the minimum value of $d[u] + w(e)$

$S \leftarrow S \cup \{v\}$

$d[v] \leftarrow d[u] + w(e)$

$pred[v] \leftarrow u$

}

*For each $v \notin S$ maintain $d'[v]$ = minimum value of $d[u] + w(e)$
over all vertices $u \in S$ s.t. $e = (u, v)$ is in G

Prim's Algorithm

Prim($\boldsymbol{G}$,$\boldsymbol{w}$,$\boldsymbol{s}$)

$\boldsymbol{S} \leftarrow \{\boldsymbol{s}\}$

while $\boldsymbol{S} \neq \boldsymbol{V}$ {

among all edges $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{v})$ s.t. $\boldsymbol{v} \notin \boldsymbol{S}$ and $\boldsymbol{u} \in \boldsymbol{S}$ select* one with the minimum value of $\boldsymbol{w}(\boldsymbol{e})$

$\boldsymbol{S} \leftarrow \boldsymbol{S} \cup \{\boldsymbol{v}\}$

$\boldsymbol{pred}[\boldsymbol{v}] \leftarrow \boldsymbol{u}$

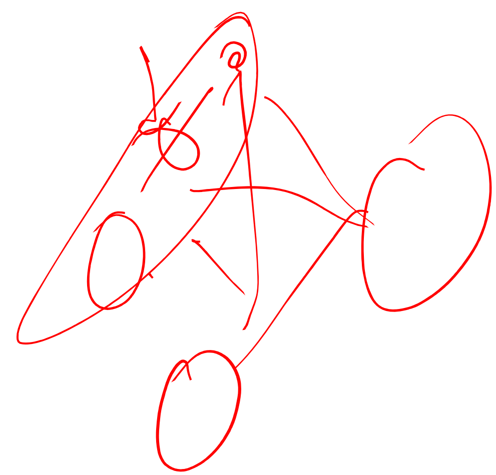
}

*For each $\boldsymbol{v} \notin \boldsymbol{S}$ maintain $\boldsymbol{small}[\boldsymbol{v}]$ = minimum value of $\boldsymbol{w}(\boldsymbol{e})$
over all vertices $\boldsymbol{u} \in \boldsymbol{S}$ s.t. $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{v})$ is in $\boldsymbol{G}$

Second Greedy Algorithm

Kruskal's Algorithm:

- Start with the vertices and no edges
- Repeatedly add the cheapest edge that joins two different components.
  - i.e. cheapest edge that doesn't create a cycle



Proving Greedy MST Algorithms Correct

Instead of specialized proofs for each one we'll have one unified argument ...

Cuts

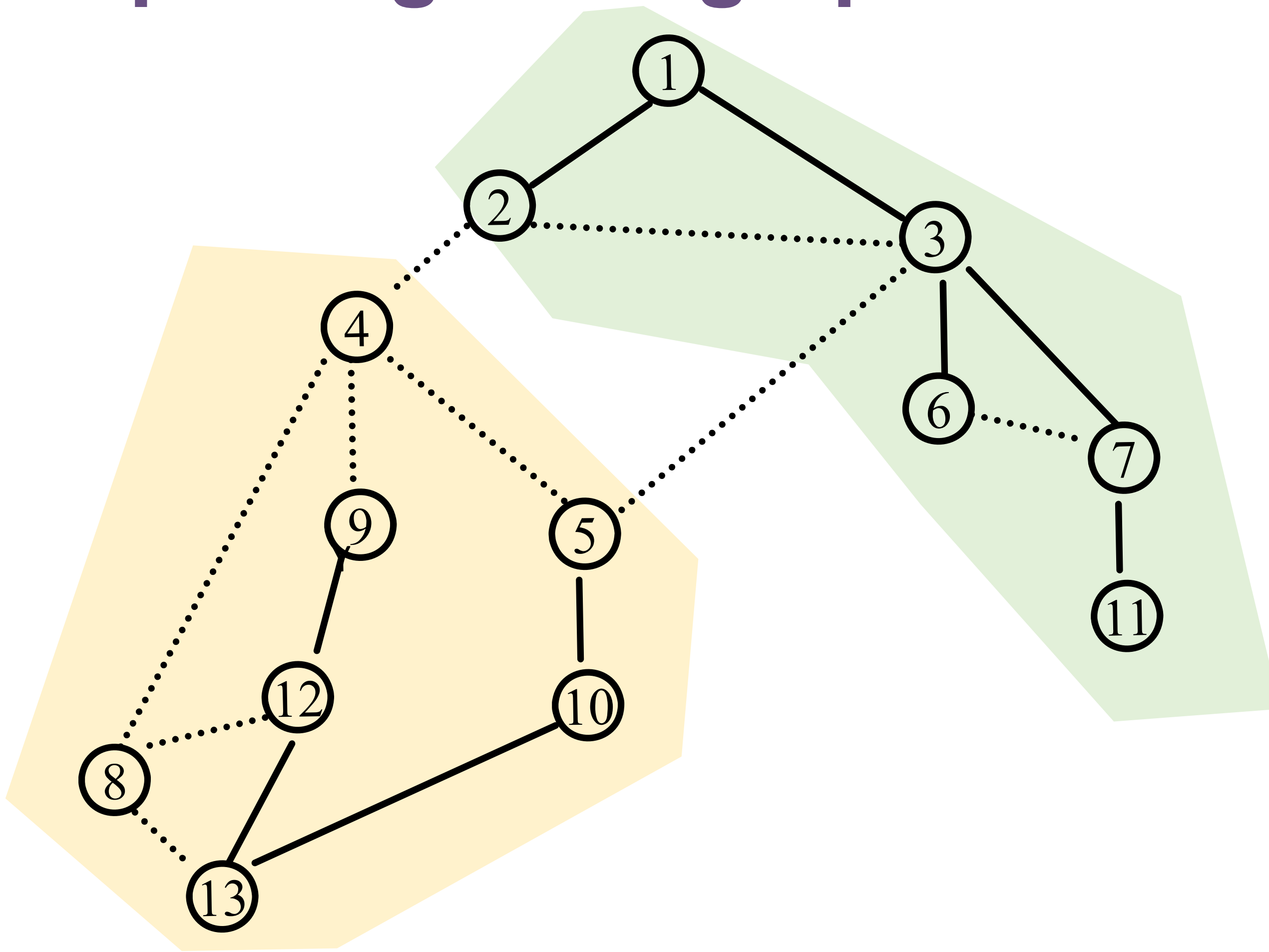
Defn: Given a graph $G = (V, E)$, a cut of G is a partition of V into two non-empty pieces, S and $V \setminus S$.

We write this cut as $(S, V \setminus S)$.

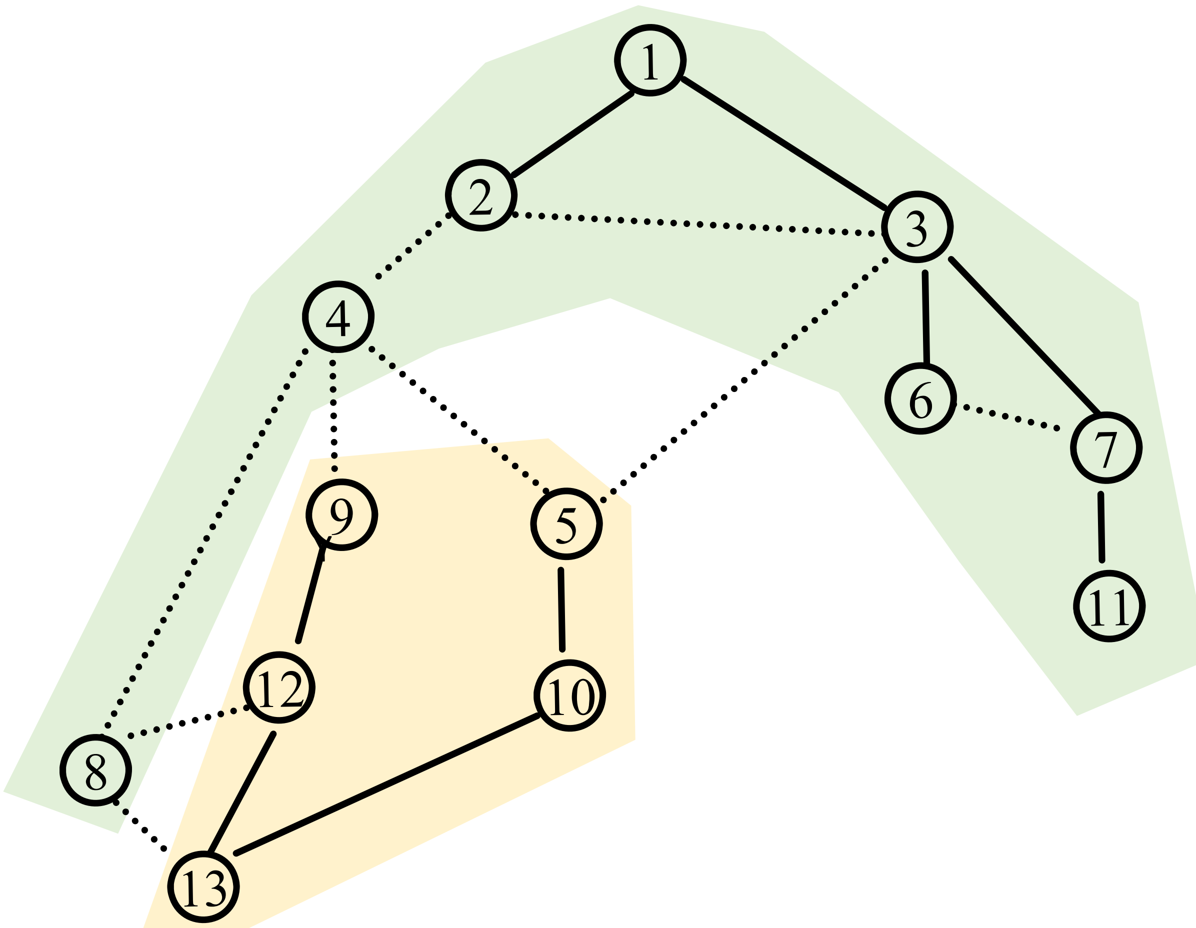
Defn: Edge e crosses cut $(S, V \setminus S)$ iff one endpoint of e is in S and the other is in $V \setminus S$

Defn: Given a graph $G = (V, E)$, and a subgraph G' of G we say that a cut $(S, V \setminus S)$ respects G' iff no edge of G' crosses $(S, V \setminus S)$

A cut respecting a subgraph



Another cut respecting the subgraph



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Generic Greedy MST Algorithms and Safe Edges

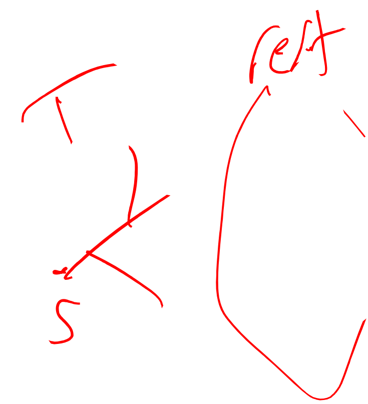
Greedy algorithms for MST build up the tree/forest edge-by-edge as follows:

$\boldsymbol{T} \leftarrow \emptyset$

while ($\boldsymbol{T}$ isn't spanning)

  choose* some "best" edge $\boldsymbol{e}$ (that won't create a cycle)

  $\boldsymbol{T} \leftarrow \boldsymbol{T} \cup \{\boldsymbol{e}\}$



Defn: An edge $\boldsymbol{e}$ of $\boldsymbol{G}$ is called **safe** for $\boldsymbol{T}$

iff there is *some* cut $(\boldsymbol{S}, \boldsymbol{V} \setminus \boldsymbol{S})$ that respects $\boldsymbol{T}$

s.t $\boldsymbol{e}$ is a *cheapest* edge crossing $(\boldsymbol{S}, \boldsymbol{V} \setminus \boldsymbol{S})$

Theorem: Any greedy algorithm that always chooses* an edge $\boldsymbol{e}$ that is safe for $\boldsymbol{T}$ correctly computes an MST

Greedy algorithms: Choose safe edges that don't create cycles

Prim's Algorithm:

- Always chooses cheapest edge from current tree to rest of the graph
- This is cheapest edge across a cut that has all the vertices of current tree on one side.

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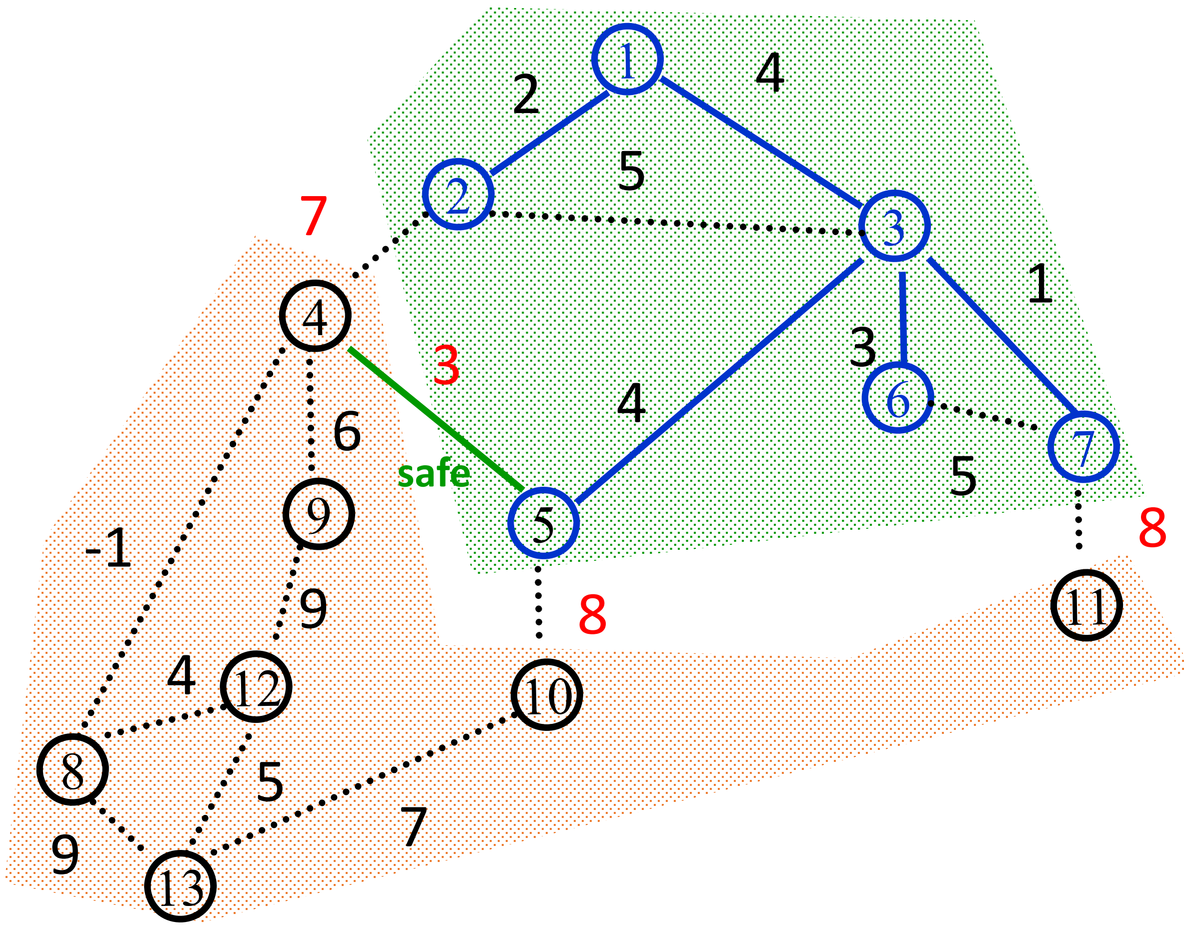
Prim's Algorithm



Greedy algorithms: Choose safe edges that don't create cycles

Kruskal's Algorithm:

- Always choose cheapest edge connecting two pieces of the graph that aren't yet connected
- This is the cheapest edge across any cut that has those two pieces on different sides and doesn't split any other current pieces (respects the cut).

Side Bar: Induction wrt Algorithms
- only useful if you have loops

induct over
- size of problem
- # of steps / times through loops
- values of vars
- # of outputs produced so far

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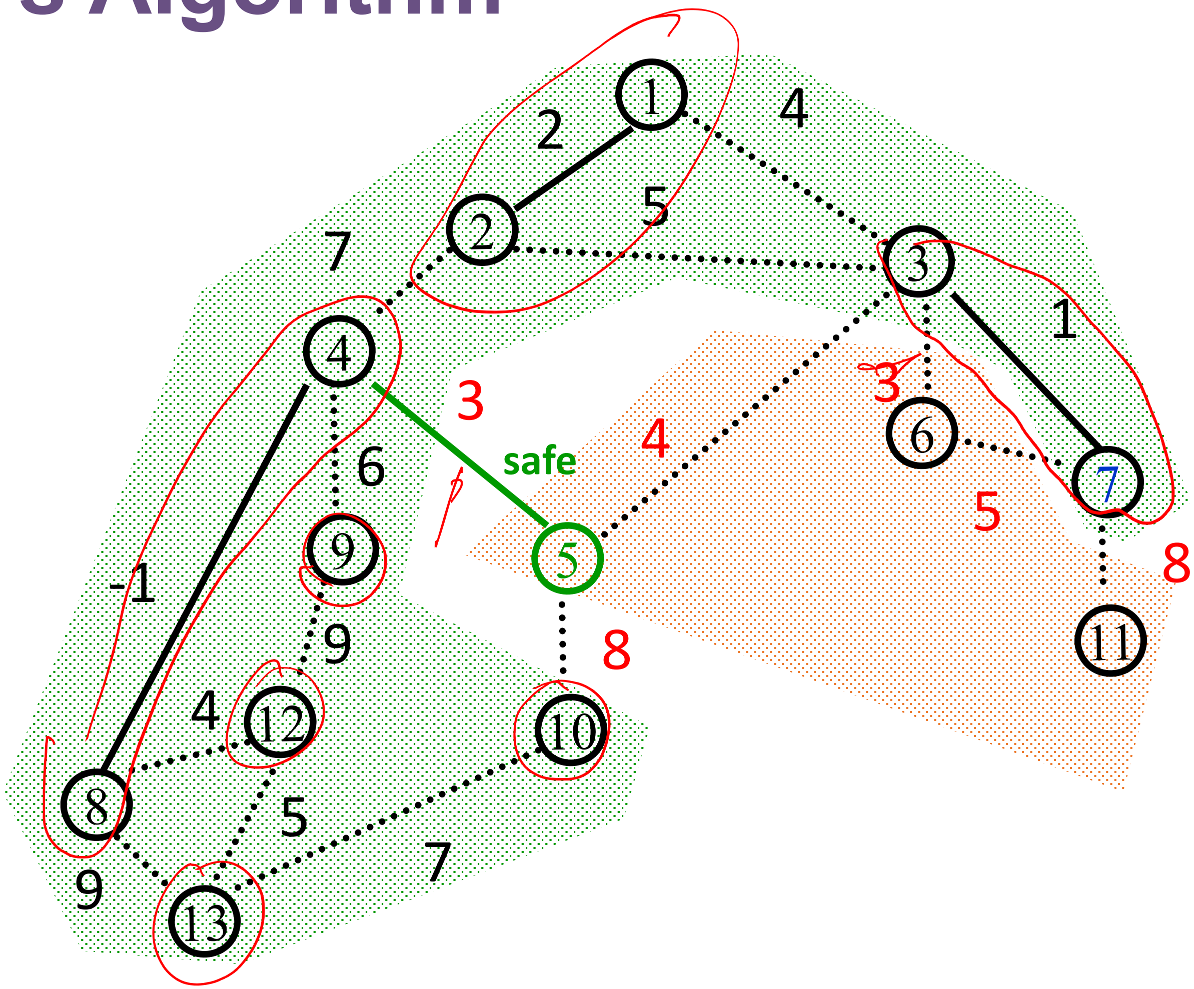
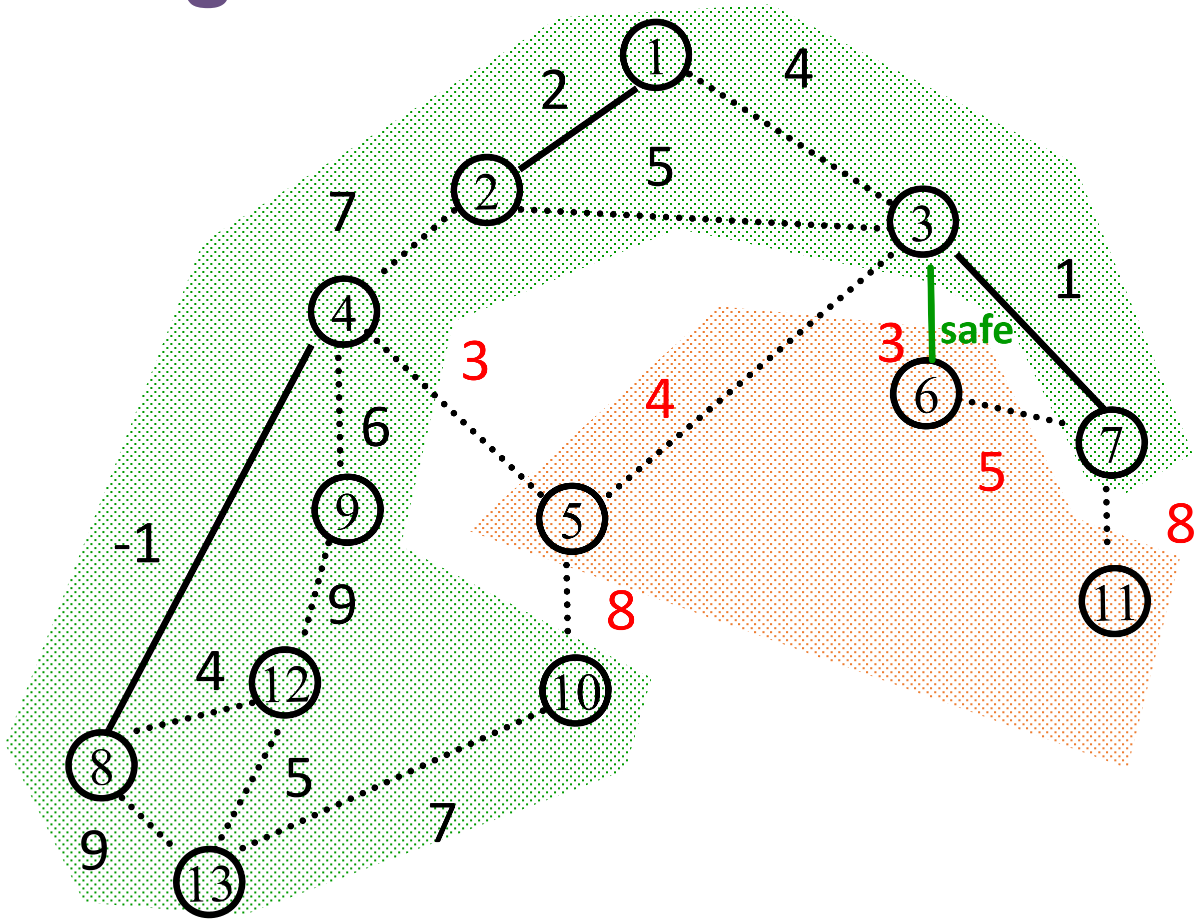
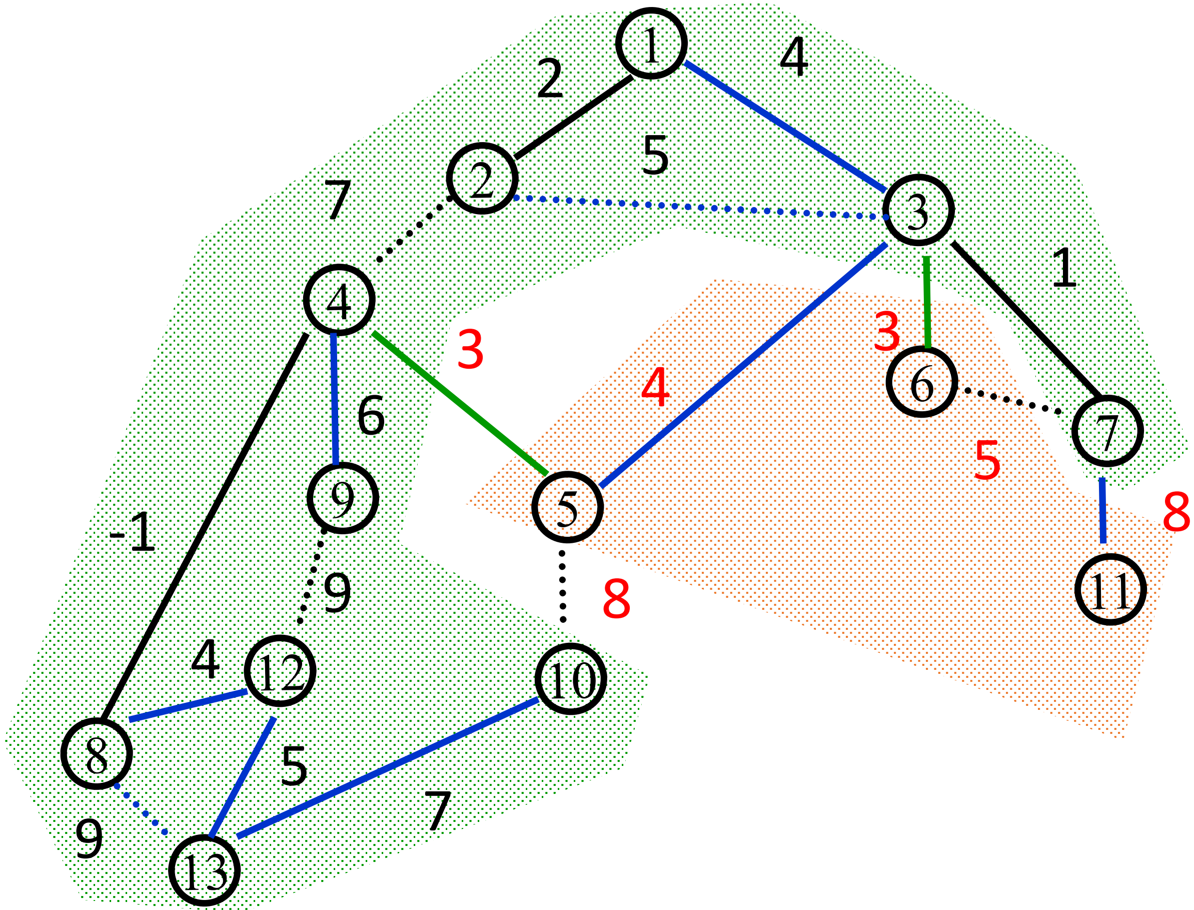
Kruskal’s Algorithm



Kruskal's Algorithm



Kruskal's Algorithm



Generic Greedy MST Algorithms and Safe Edges

Defn: An edge e of G is called **safe** for T
iff there is *some* cut $(S, V \setminus S)$ that respects T
s.t e is a *cheapest* edge crossing $(S, V \setminus S)$

Theorem: Any greedy algorithm that always chooses* an edge e that is safe for T
correctly computes an MST

Proof: We prove via induction and an exchange argument that at every step,
the subgraph T is contained in some MST of G.

Base Case: $T = \emptyset$. This is trivially true since $\emptyset$ is contained in every set.

IH: Suppose that T is contained in some MST of G.

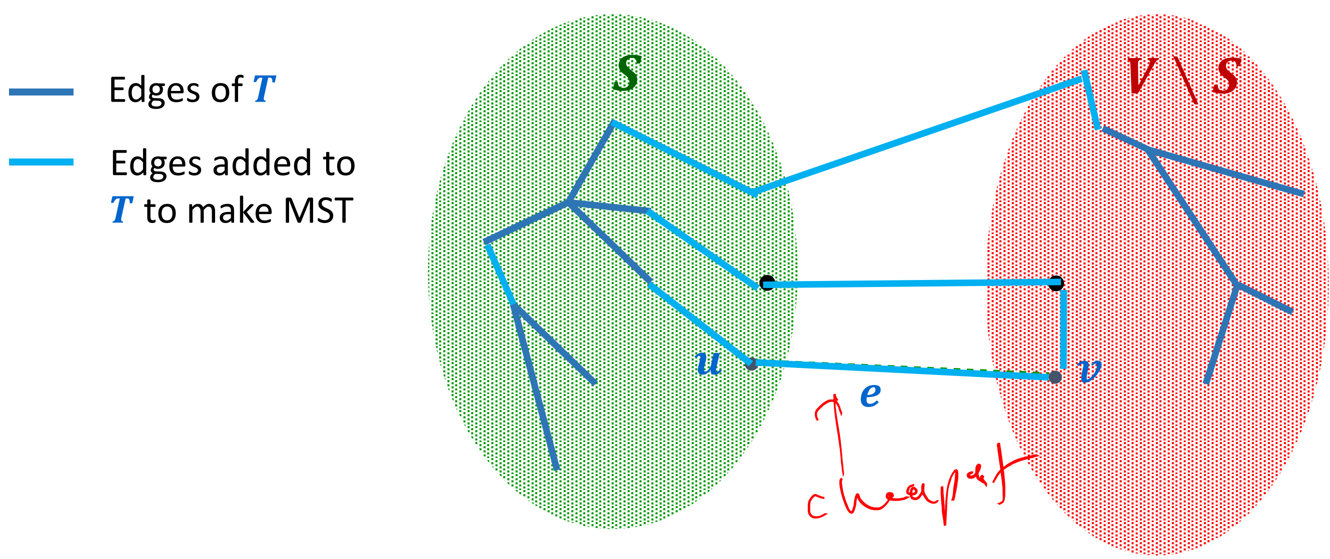
IS: We need to show that if e is safe for T then $T \cup \{e\}$ is contained
in an MST of G.

Proof of Lemma: An Exchange Argument

<u>IS</u>: e is a safe edge for T so e must be a cheapest edge crossing some cut $(S, V \setminus S)$ respecting T

By IH, T is contained in an MST. If this MST contains $e = (u, v)$ we're done.

Otherwise, this MST must contain a path from u to v.



All the same points are connected by the new tree

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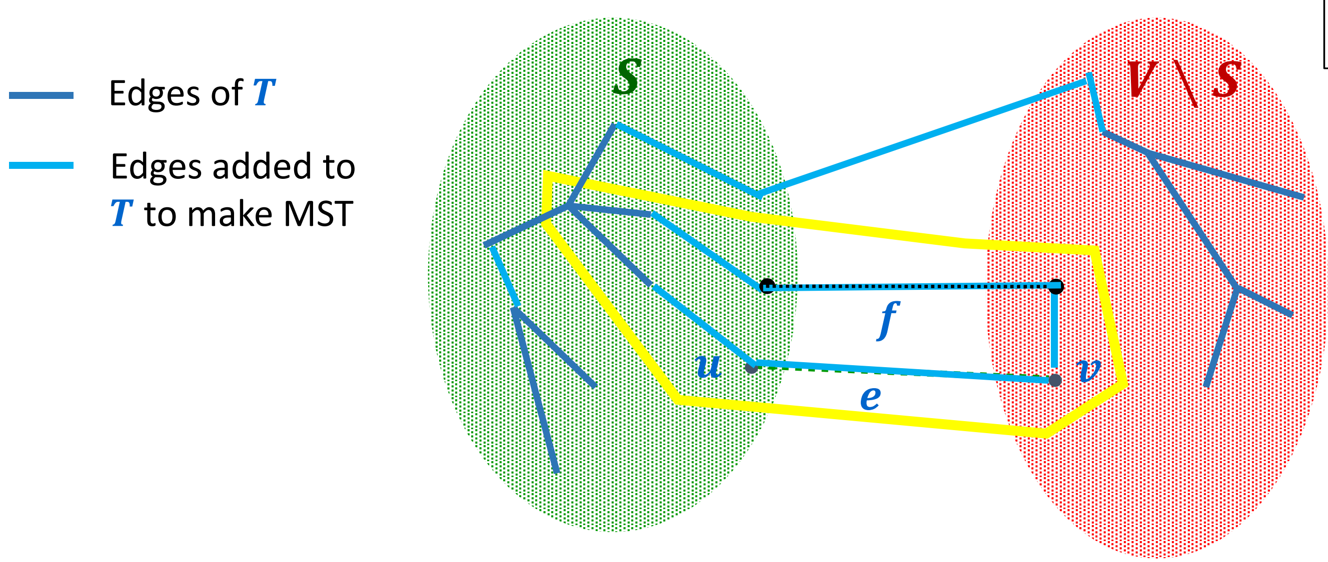
Proof of Lemma: An Exchange Argument

IS: e is a safe edge for T so e must be a cheapest edge crossing some cut $(S, V \setminus S)$ respecting T

By IH, T is contained in an MST. If this MST contains $e = (u, v)$ we're done.

Otherwise, this MST must contain a path from u to v.

This must contain some edge f crossing the cut.



Since e was cheapest $w(e) \le w(f)$

Exchange e for f to get a new spanning subgraph that is at least as cheap and contains $T \cup \{e\}$. ∎

All the same points are connected by the new tree

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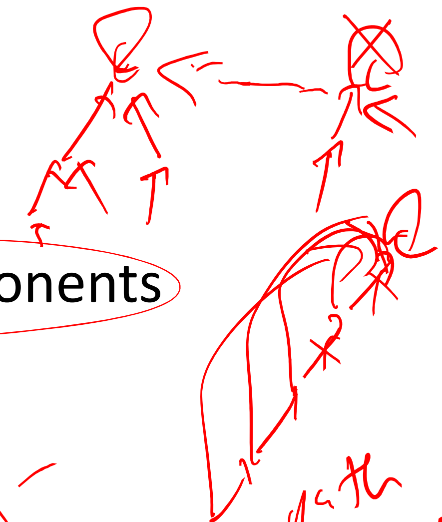
Kruskal's Algorithm: Implementation & Analysis

- First sort the edges by weight $O(m \log m)$
- Go through edges from smallest to largest
  - if endpoints of edge e are currently in different components
    - then add to the graph
    - else skip



path compression

Union-Find data structure handles test for different components
- Total cost of union find: $O(m \cdot \alpha(n))$ where $\alpha(n) \ll \log m$

2^{2^n} 2^n

Overall $O(m \log m)$ which is $O(m \log n)$

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Union-Find disjoint sets data structure

Maintaining components

- start with n different components
  - one per vertex
- find components of the two endpoints of e
  - $2m$ finds
- union two components when edge connecting them is added
  - $n - 1$ unions

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Prim's Algorithm with Priority Queues

- For each vertex u not in tree maintain current cheapest edge from tree to u
  - Store u in priority queue with key = weight of this edge
- Operations:
  - $n - 1$ insertions (each vertex added once)
  - $n - 1$ delete-mins (each vertex deleted once)
    - pick the vertex of smallest key, remove it from the p.q. and add its edge to the graph
  - $< m$ decrease-keys (each edge updates one vertex)

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Prim's Algorithm with Priority Queues

Priority queue implementations: same complexity as Dijkstra

- Array
  - insert $O(\mathbf{1})$, delete-min $O(\boldsymbol{n})$, decrease-key $O(\mathbf{1})$
  - total $O(\boldsymbol{n} + \boldsymbol{n}^{\mathbf{2}} + \boldsymbol{m}) = O(\boldsymbol{n}^2)$
- Heap
  - insert, delete-min, decrease-key all $O(\log \boldsymbol{n})$
  - total $O(\boldsymbol{m} \log \boldsymbol{n})$

Worse if $\boldsymbol{m} = \boldsymbol{\Theta}(\boldsymbol{n}^2)$

- $\boldsymbol{d}$-Heap ($\boldsymbol{d} = \boldsymbol{m}/\boldsymbol{n}$)
  - $\boldsymbol{m}$ insert, decrease-key $O(\log_{\boldsymbol{m}/\boldsymbol{n}} \boldsymbol{n})$
  - $\boldsymbol{n}-\mathbf{1}$ delete-min $O((\boldsymbol{m}/\boldsymbol{n})\log_{\boldsymbol{m}/\boldsymbol{n}} \boldsymbol{n})$
  - total $O(\boldsymbol{m} \log_{\boldsymbol{m}/\boldsymbol{n}} \boldsymbol{n})$

Better for all values of $\boldsymbol{m}$

Boruvka's Algorithm (1927)

A bit like Kruskal's Algorithm

- Start with n components consisting of a single vertex each
- At each step:
  - Each component chooses to add its cheapest outgoing edge
  - Two components may choose to add the same edge
  - Need to add a tiebreaker on edge weights (no equal weights) to avoid cycles

Useful for parallel algorithms since components may be processed (almost) independently

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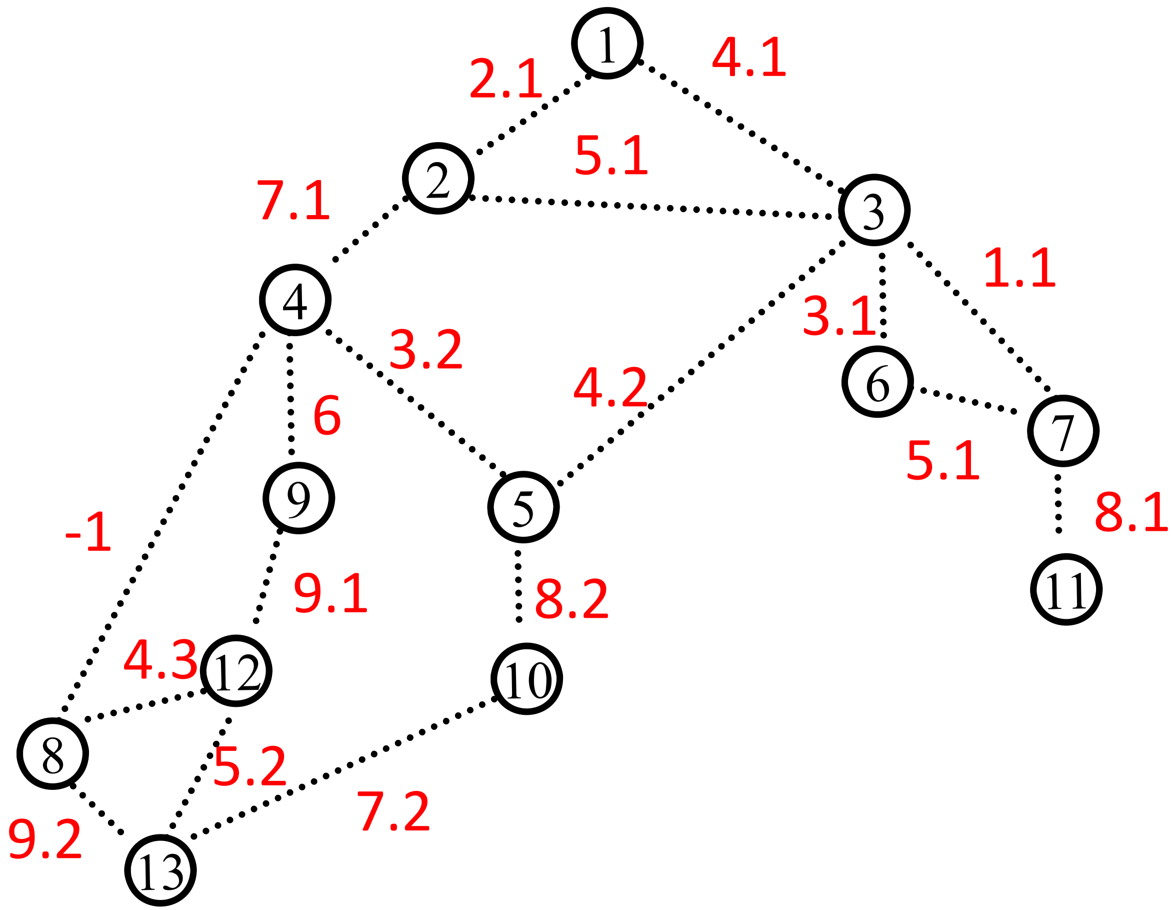
Boruvka



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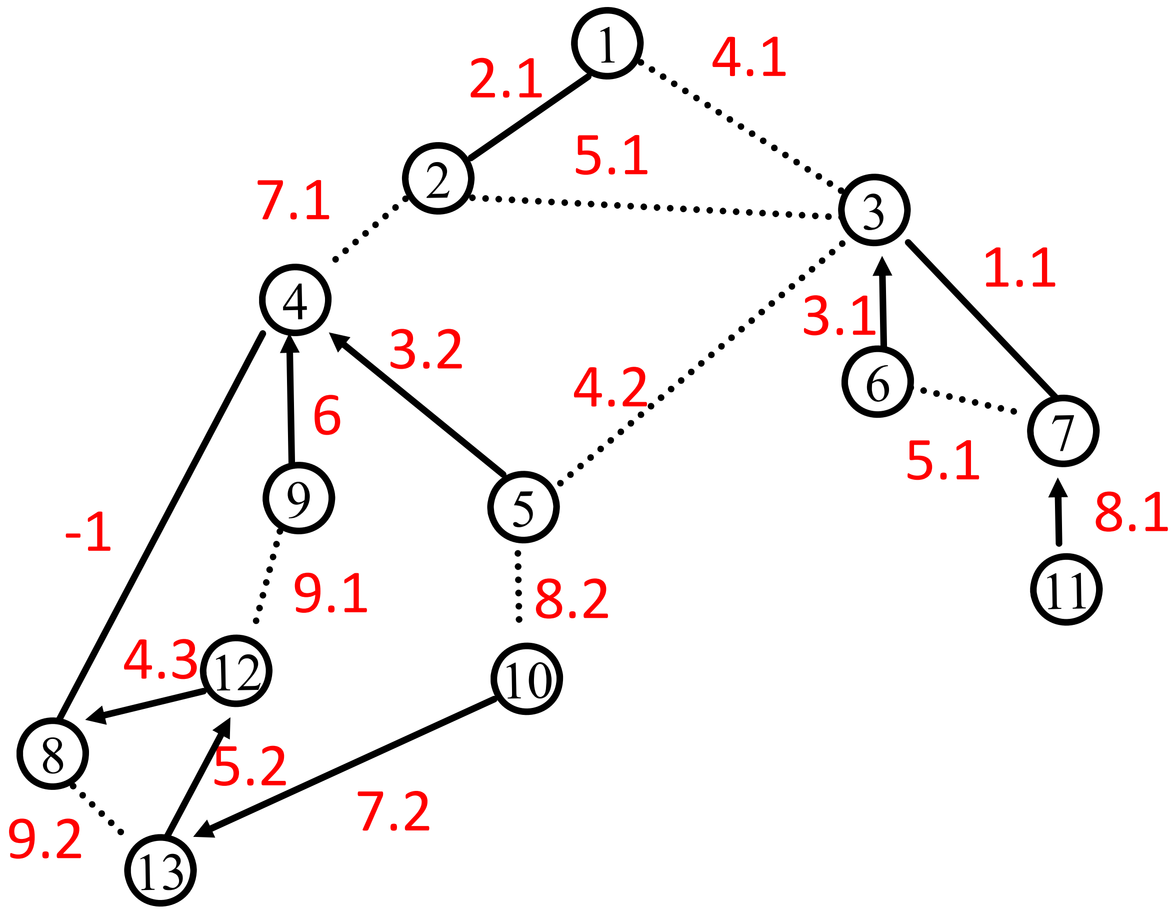
Boruvka



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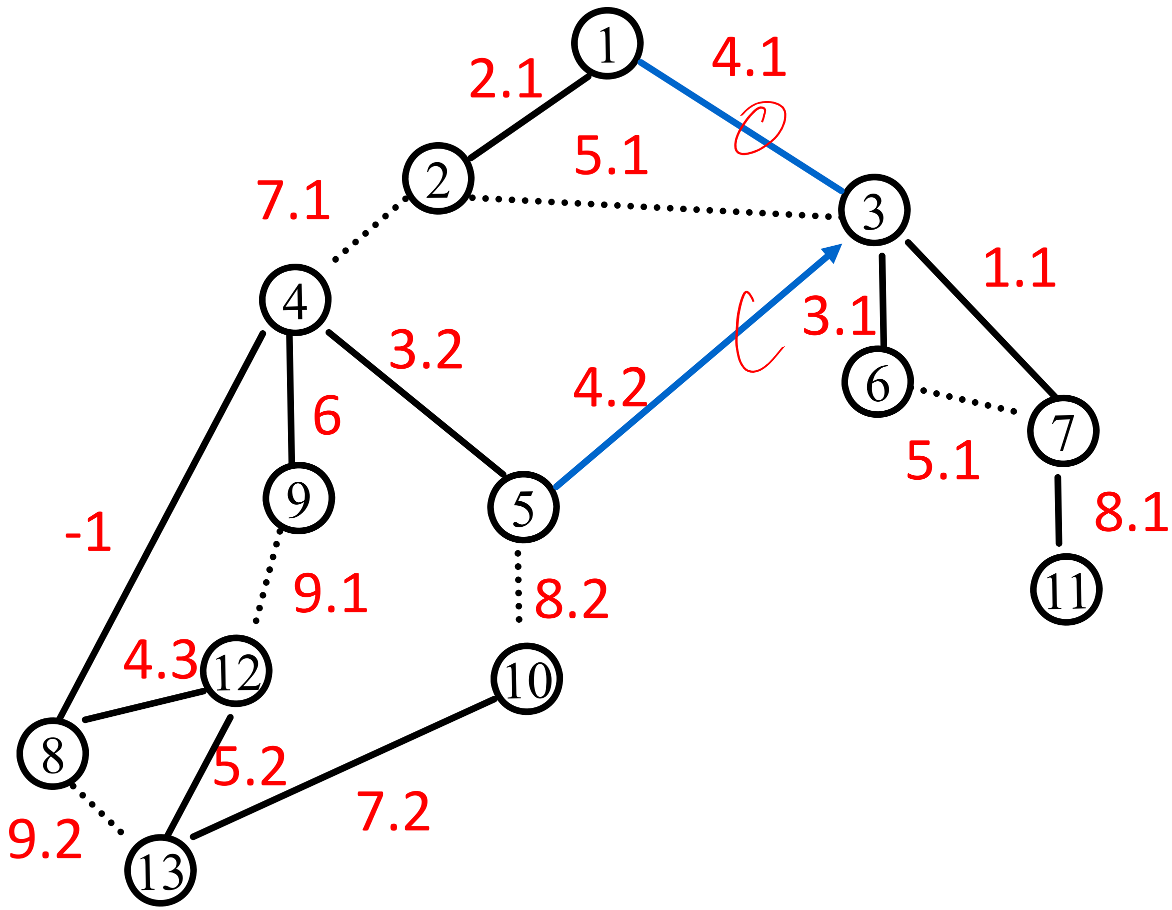
Boruvka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Many other minimum spanning tree algorithms, most of them greedy

Cheriton & Tarjan

- Use a queue of components
  - Component at head chooses cheapest outgoing edge
  - New merged component goes to tail of the queue.
- $O(\boldsymbol{m} \operatorname{loglog} \boldsymbol{n})$ time

Chazelle

- $O(\boldsymbol{m} \cdot \boldsymbol{\alpha}(\boldsymbol{m}) \cdot \log(\boldsymbol{\alpha}(\boldsymbol{m})))$ time
  - Incredibly hairy algorithm

Karger, Klein & Tarjan

- $O(\boldsymbol{m} + \boldsymbol{n})$ time randomized algorithm that works most of the time

Applications of Minimum Spanning Tree Algorithms

MST is a fundamental problem with diverse applications

- Network design
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms
  - travelling salesperson problem, Steiner tree
- Indirect applications
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - reducing data storage in sequencing amino acids
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid network cycles
- Clustering

Applications of Minimum Spanning Tree Algorithms

Minimum cost network design:

- Build a network to connect all locations $\{v_1, \dots, v_n\}$
- Cost of connecting v_i to v_j is $w(v_i, v_j) > 0$.
- Choose a collection of links to create that will be as cheap as possible
- Any minimum cost solution is an MST
    - If there is a solution containing a cycle then we can remove any edge and get a cheaper solution

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Applications of Minimum Spanning Tree Algorithms

Maximum Spacing Clustering:

Given:

- Collection $\boldsymbol{U}$ of $\boldsymbol{n}$ points $\{\boldsymbol{p_1}, \ldots, \boldsymbol{p_n}\}$
- Distance measure $\boldsymbol{d(p_i, p_j)}$ satisfying
  - Zero base: $\boldsymbol{d(p_i, \cancel{p_j}\, i) = 0}$
  - Nonnegativity: $\boldsymbol{d(p_i, p_j) > 0}$ for $\boldsymbol{i \neq j}$
  - Symmetry: $\boldsymbol{d(p_i, p_j) = d(p_j, p_i)}$
- Positive integer $\boldsymbol{k \leq n}$

Find: a $\boldsymbol{k}$-clustering, i.e. partition of $\boldsymbol{U}$ into $\boldsymbol{k}$ clusters $\boldsymbol{C_1, \ldots, C_k}$, s.t.
the spacing between the clusters is as large possible where
spacing = min$\{\boldsymbol{d(p_i, p_j)}$: $\boldsymbol{p_i}$ and $\boldsymbol{p_j}$ are in different clusters$\}$

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Greedy Algorithm for Maximum Spacing Clustering

- Start with n clusters each consisting of a single point
- Repeat until only k clusters remain
  - find the closest pair of points in different clusters under distance d
  - merge their clusters

Gets the same components as Kruskal's Algorithm does if we stop early!

- The sequence of closest pairs is exactly the MST

- Alternatively...
  - we could run any MST algorithm once and for any k we could get the maximum spacing k-clustering by deleting the $k-1$ most expensive edges in the MST

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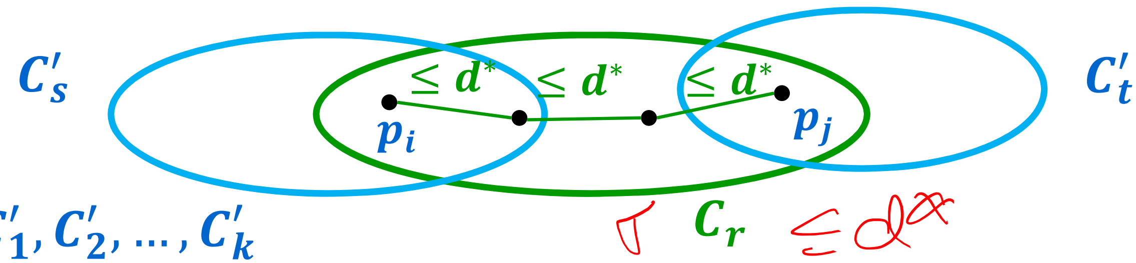
Proof that this works

- Removing the $k-1$ most expensive edges from an MST yields k components $C_1, \dots, C_k$ and the spacing for them is precisely the cost d^* of the $k-1^{\text{st}}$ most expensive edge in the tree



- Consider any other k-clustering $C'_1, C'_2, \dots, C'_k$
  - There is some pair of points p_i, p_j s.t. p_i, p_j are in some cluster C_r but p_i, p_j are in different clusters C'_s and C'_t
  - Since both are in C_r, points p_i and p_j are joined by a path with each hop of distance at most d^*
  - This path must have some *adjacent* pair in different clusters of $C'_1, C'_2, \dots, C'_k$ so the spacing of $C'_1, C'_2, \dots, C'_k$ must be at most d^* ∎

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