CSE 421
Introduction to Algorithms

Lecture 25: More NP-completeness

NP-hardness & NP-completeness

Notion of hardness we **can** prove that is useful unless $\mathbf{P} = \mathbf{NP}$:

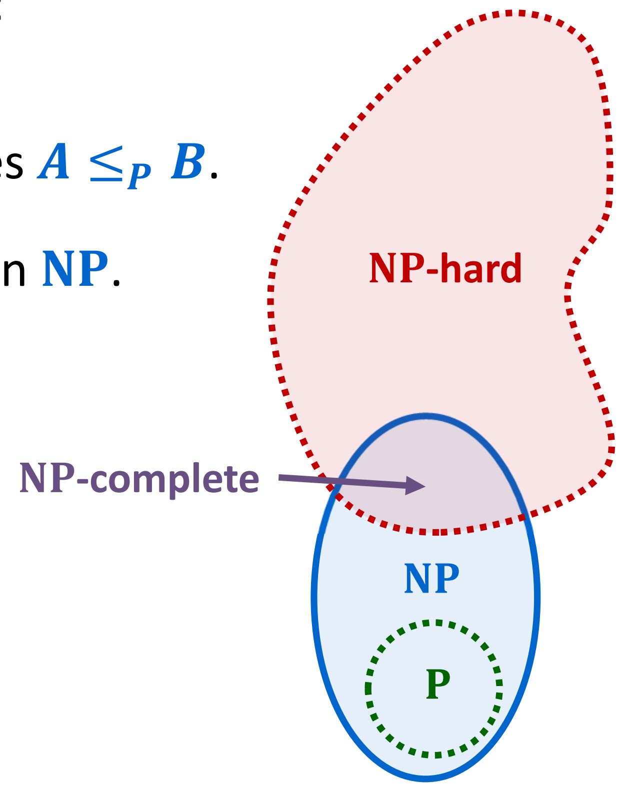
Defn: Problem $\boldsymbol{B}$ is $\mathbf{NP}$-hard iff **every** problem $\boldsymbol{A} \in \mathbf{NP}$ satisfies $\boldsymbol{A} \leq_{\boldsymbol{P}} \boldsymbol{B}$.

This means that $\boldsymbol{B}$ is at least as hard as every problem in $\mathbf{NP}$.

Defn: Problem $\boldsymbol{B}$ is $\mathbf{NP}$-complete iff

- $\boldsymbol{B} \in \mathbf{NP}$ and
- $\boldsymbol{B}$ is $\mathbf{NP}$-hard.

This means that $\boldsymbol{B}$ is a hardest problem in $\mathbf{NP}$.



Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- 6,000 citations per year (title, abstract, keywords).
  - more than "compiler", "operating system", "database"
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager solves 2D case in tour de force.
- 19xx: Feynman and other top minds seek 3D solution.
- 2000: Istrail proves 3D problem NP-complete.

Cook-Levin Theorem and implications

Theorem [Cook 1971, Levin 1973]: **3SAT** is **NP**-complete

Proof: See CSE 431.

Corollary: If **3SAT** $\leq_P$ **B** then **B** is **NP**-hard.

By the same kind of reasoning we have

Theorem: If **A** $\leq_P$ **B** for some **NP**-hard **A** then **B** is **NP**-hard.

NP-complete problems so far

So far:

3SAT → Independent-Set → Clique

$$\begin{array}{ccc} \textbf{3SAT} \to & \textbf{Independent-Set} \to & \textbf{Clique} \\ & \downarrow & \\ & \textbf{Vertex-Cover} \to \textbf{01-Programming} \to \textbf{Integer-Programming} & \end{array}$$

Steps to Proving Problem $\boldsymbol{B}$ is $\mathbf{NP}$-complete

- Show $\boldsymbol{B}$ is in $\mathbf{NP}$
    - State what the hint/certificate is.
    - Argue that it is polynomial-time to check.
- Show $\boldsymbol{B}$ is $\mathbf{NP}$-hard:
    - State: "Reduction is from $\mathbf{NP}$-hard Problem $\boldsymbol{A}$"
    - Show what the reduction function $\boldsymbol{f}$ is.
    - Argue that $\boldsymbol{f}$ is polynomial time.
    - Argue correctness in two directions:
        - $\boldsymbol{x}$ a **YES** for $\boldsymbol{A}$ implies $\boldsymbol{f(x)}$ is a **YES** for $\boldsymbol{B}$
            - Do this by showing how to convert a certificate for $\boldsymbol{x}$ being **YES** for $\boldsymbol{A}$ to a certificate for $\boldsymbol{f(x)}$ being a **YES** for $\boldsymbol{B}$.
        - $\boldsymbol{f(x)}$ a **YES** for $\boldsymbol{B}$ implies $\boldsymbol{x}$ is a **YES** for $\boldsymbol{A}$
            - … by converting certificates for $\boldsymbol{f(x)}$ to certificates for $\boldsymbol{x}$

Reduction from a Special Case to a General Case

Set-Cover:

Given a set $\boldsymbol{U}$ (universe) of $\boldsymbol{m}$ elements, a collection $\boldsymbol{S_1, \ldots, S_n}$ of subsets of $\boldsymbol{U}$, and an integer $\boldsymbol{k}$

Is there a sub-collection (the cover) of $\leq \boldsymbol{k}$ sets whose union is equal to $\boldsymbol{U}$?

Theorem: Set-Cover is $\mathbf{NP}$-complete

Proof:

1. **Set-Cover** is in $\mathbf{NP}$:
   a) Certificate is a set $\boldsymbol{T} \subseteq \{\mathbf{1}, \ldots, \boldsymbol{n}\}$ defining a supposed cover.
   b) Verifier outputs **YES** if $|\boldsymbol{T}| \leq \boldsymbol{k}$ and $\bigcup_{i \in T} \boldsymbol{S_i} = \boldsymbol{U}$; otherwise, answer **NO**.
   This computation is clearly polynomial-time

Set-Cover is NP-complete

Proof (continued):

2. **Set-Cover** is **NP**-hard

   Claim: Vertex-Cover $\leq_P$ Set-Cover

   a) Reduction function f takes and input a graph $G = (V, E)$ and integer k and produces a universe U, sets $S_1, \ldots, S_n \subseteq U$ and integer k' as follows:
      - $U = E$ (good idea since the objects being covered in **Vertex-Cover** are edges.)
      - Write $V = \{v_1, \ldots, v_n\}$.
        For each $i = 1, \ldots, n$ define S_i to be the set of edges in E that v_i touches.
      - $k' = k$.

   b) Clearly function f is polynomial time to compute.

   c) Correctness (⇒): Suppose that graph G has a vertex cover W of size $\leq k$.
      Define the set $T = \{i \mid v_i \in W\}$. Then $|T| = |W| \leq k$.
      Also since W is a vertex cover, $\bigcup_{i \in T} S_i = \{\, e \in E \mid \text{some } v_i \in W \text{ touches } e \,\} = E = U$.
      Therefore U has set cover T from $S_1, \ldots, S_n$ of size $\leq k$.

Set-Cover is NP-complete

Proof (continued):

2. **Set-Cover** is **NP**-hard

   Claim: Vertex-Cover $\leq_P$ Set-Cover

   a) Reduction function f takes and input a graph $G = (V, E)$ and integer k and produces a universe U, sets $S_1, \dots, S_n \subseteq U$ and integer k' as follows:
      - $U = E$ (good idea since the objects being covered in Vertex-Cover are edges.)
      - Write $V = \{v_1, \dots, v_n\}$.
        For each $i = 1, \dots, n$ define S_i to be the set of edges in E that v_i touches.
      - $k' = k$.

   b) c) …

   d) Correctness ($\Leftarrow$): Suppose that U has a set cover T from $S_1, \dots, S_n$ of size $\leq k$.
   Define the set $W = \{v_i \mid i \in T\}$. Then $|W| = |T| \leq k$.
   Also since T is a vertex cover, $U = E = \bigcup_{i \in T} S_i = \bigcup_{i \in T} \{ e \in E \mid v_i \text{ touches } e \}$. But this is the same as $E = \bigcup_{v \in W} \{ e \in E \mid v \text{ touches } e \}$, so graph G has vertex cover W of size $\leq k$. ■

Recall: Graph Colorability

Defn: A undirected graph $\boldsymbol{G = (V, E)}$ is $\boldsymbol{k}$-colorable iff
we can assign one of $\boldsymbol{k}$ colors to each vertex of $\boldsymbol{V}$ s.t.
for every edge $(\boldsymbol{u}, \boldsymbol{v})$ has different colored endpoints, $\chi(\boldsymbol{u}) \neq \chi(\boldsymbol{v})$.
"edges are not monochromatic"

Theorem: 3Color is $\mathbf{NP}$-complete

Proof:

1. **3Color** is in $\mathbf{NP}$:
   - We already showed this; the certificate was the coloring.
2. **3Color** is $\mathbf{NP}$-hard:

   Claim: 3SAT$\leq_P$3Color

   We need to find a function $\boldsymbol{f}$ that maps a 3CNF formula $\boldsymbol{F}$ to a graph $\boldsymbol{G}$ s.t.
   $\boldsymbol{F}$ is satisfiable $\Leftrightarrow$ $\boldsymbol{G}$ is 3-colorable.

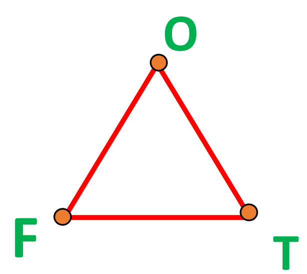
3SAT $\leq_P$ 3Color

Start with a base triangle with vertices **T**, **F**, and **O**.

We can assume that **T**, **F**, and **O** are the three colors used.

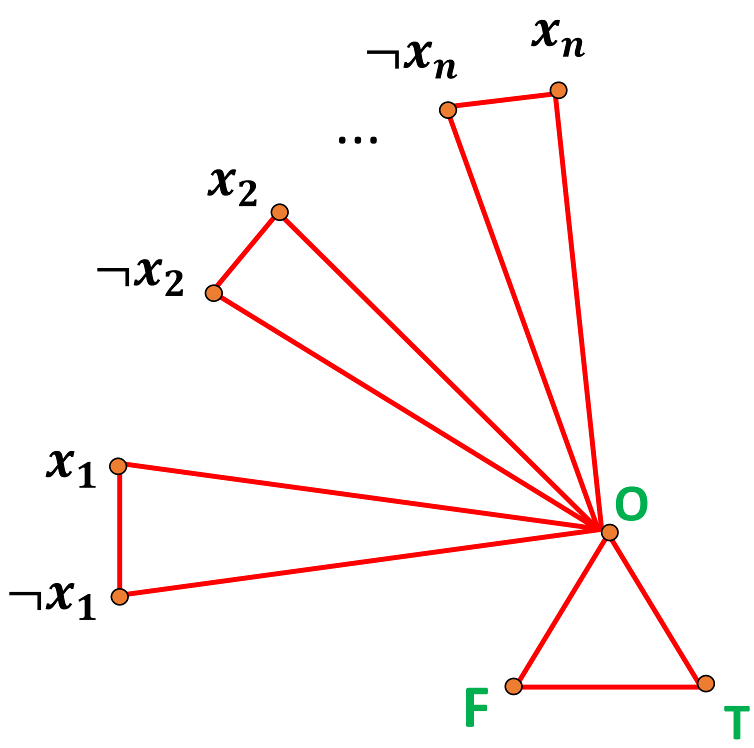
- Intuition: **T** and **F** will stand for *true* and *false*; **O** will stand for *other*.

To represent the properties of the 3CNF formula $\boldsymbol{F}$ we will need both a Boolean variable part and a clause part.



Base Triangle

3SAT $\leq_P$ 3Color

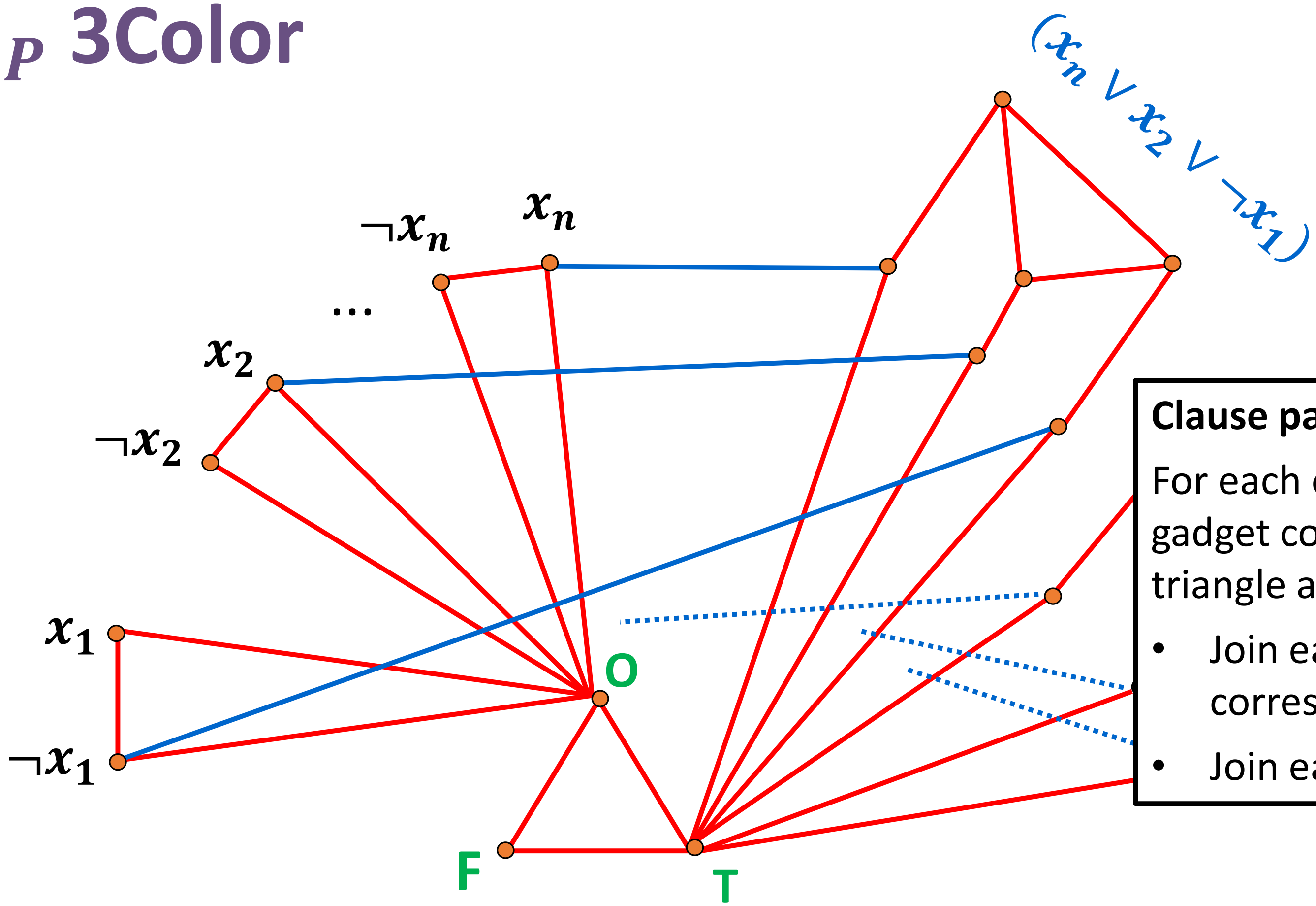


Base Triangle

Boolean variable part:

- For each Boolean variable add a triangle with two nodes labelled by literals as shown.
- Since both nodes are joined to node **O** and to each other, they must have opposite colors **T** and **F** in any 3-coloring.
- So, any 3-coloring corresponds to a unique truth assignment.

3SAT $\leq_P$ 3Color



Clause Part

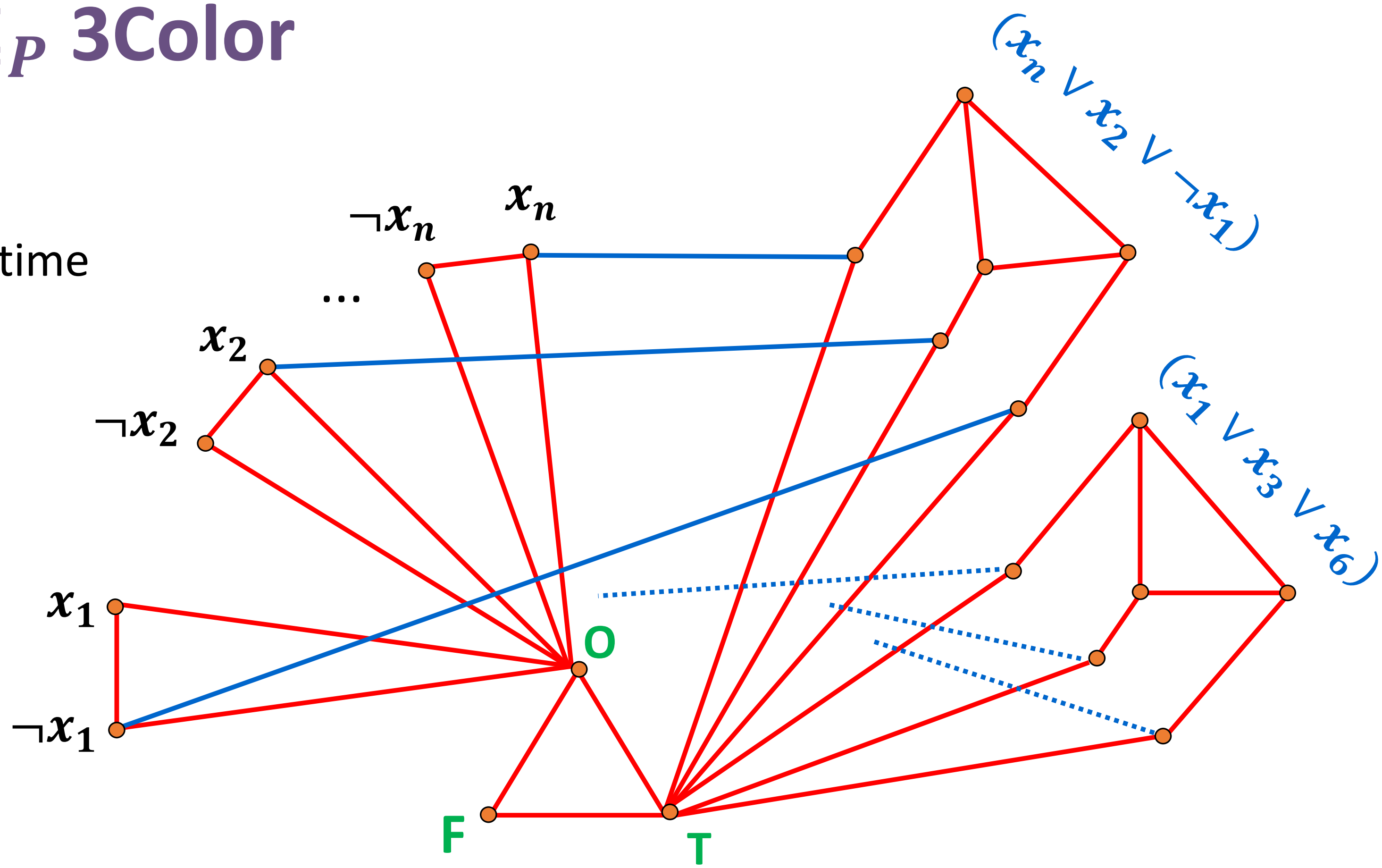
Clause part:

For each clause of F add a gadget consisting of a triangle and 3 "outer" nodes.

- Join each outer node to a corresponding literal node
- Join each outer node to **T**

$3SAT \leq_P 3Color$

Clearly only polynomial-time to produce.



Clause Part

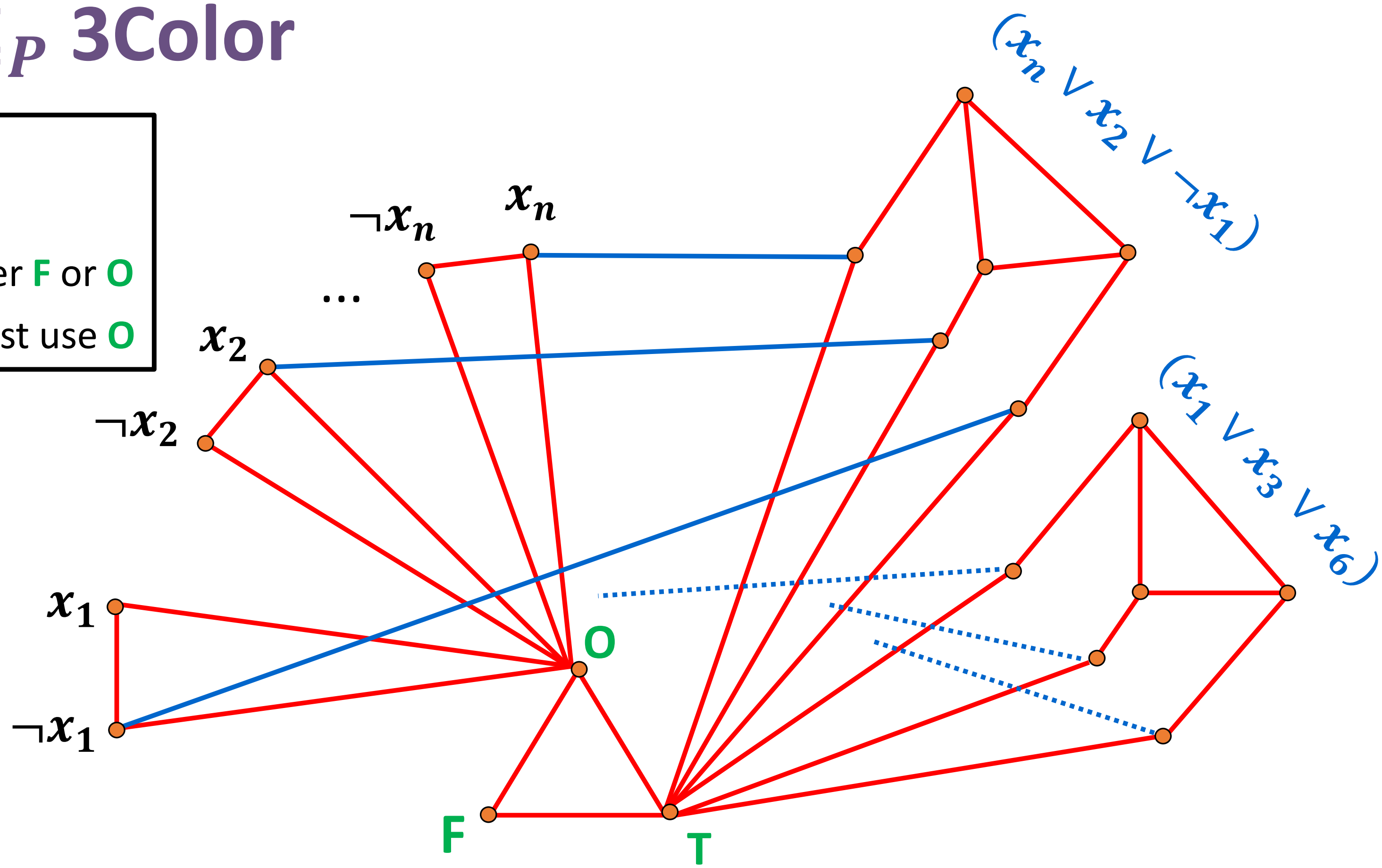
3SAT $\leq_P$ 3Color

Key property:

In any 3-coloring:

outer nodes either **F** or **O**

inner triangle must use **O**



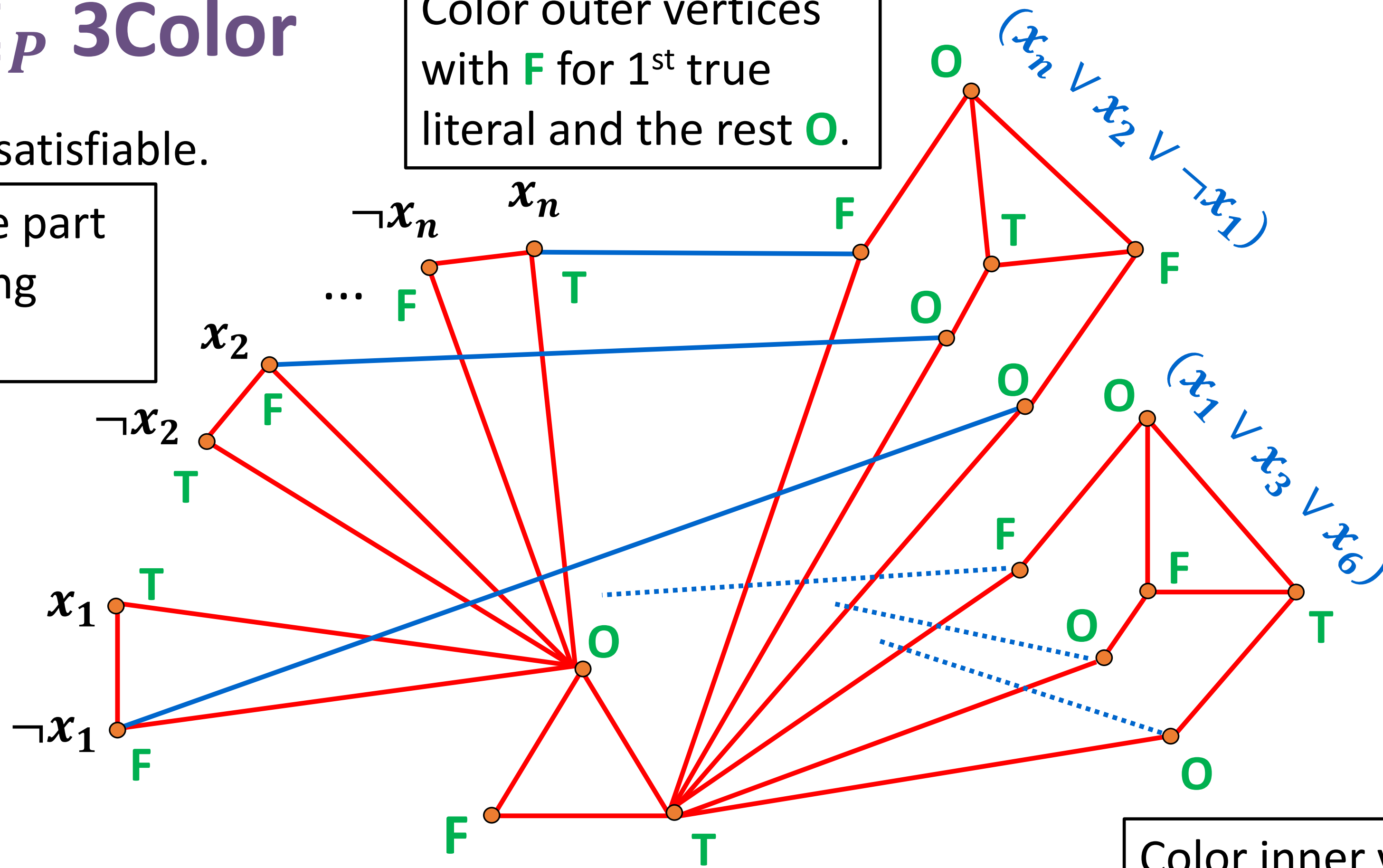
Clause Part

3SAT $\leq_P$ 3Color

Suppose F is satisfiable.

Color variable part using satisfying assignment.

Color outer vertices with F for 1st true literal and the rest O.



Color inner vertices with O opposite F.

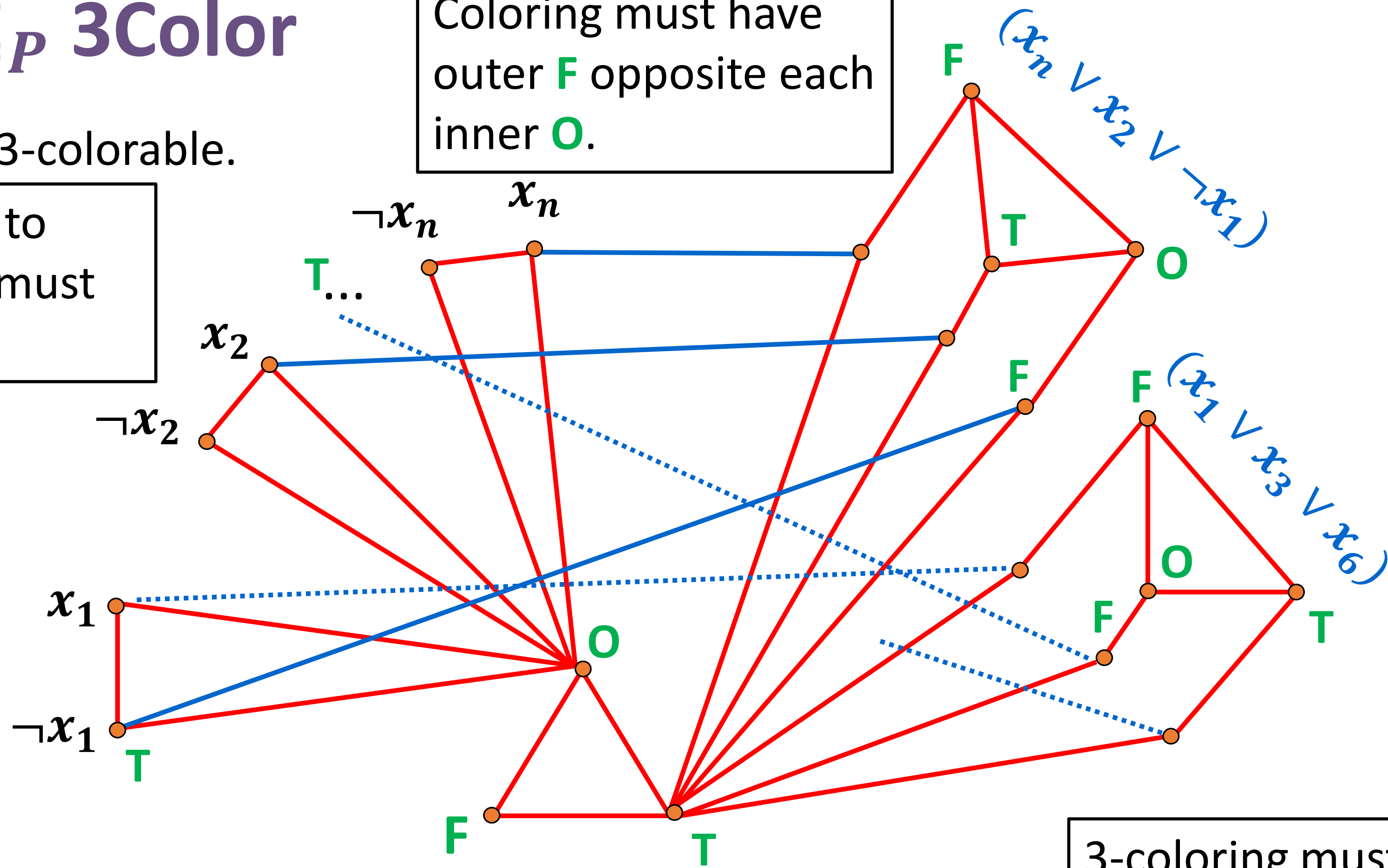
Therefore G is 3-colorable

3SAT $\leq_P$ 3Color

Suppose G is 3-colorable.

Literal joined to each outer F must be colored T.

Coloring must have outer F opposite each inner O.



3-coloring must use O on each inner triangle

Each clause has a literal that is T satisfying F ■

More NP-completeness

Subset-Sum: (Decision version of **Knapsack**)

Given: n integers $w_1, \ldots, w_n$ and integer W

Is there a subset of the n input integers that adds up to exactly W?

$O(nW)$ solution from dynamic programming but if W and each w_i can be n bits long then this is exponential time.

Theorem: Subset-Sum is $\mathbf{NP}$-complete

Proof:

1. **Subset-Sum** is in $\mathbf{NP}$:
   a) Certificate is n bits representing a subset S of $\{1, \ldots, n\}$.
   b) Check that $\sum_{i \in S} w_i = W$.
2. **Subset-Sum** is $\mathbf{NP}$-hard

Claim: 3SAT$\leq_P$Subset-Sum

3SAT$\leq_P$Subset-Sum

Given a 3-CNF formula F with m clauses and n variables

- We will create an input for **Subset-Sum** with $2m + 2n$ numbers that are $m + n$ digits long.
- We will ensure that no matter how we sum them there won't be any carries so each digit in the target W will force a separate constraint.
- Instead of calling them $w_1, \ldots, w_{2n+2m}$ we will use mnemonic names:
  - Two numbers for each variable x_i
    - t_i and f_i (corresponding to x_i being true or x_i being false)
  - Two extra numbers for each clause C_j
    - a_j and b_j (two identical filler numbers to handle number of false literals in clause C_j)
- We define them by giving their decimal representation...

3SAT$\leq_P$Subset-Sum

$C_1 = (x_1 \lor \neg x_2 \lor x_3)$
$C_2 = (\neg x_1 \lor x_2 \lor x_5)$
$C_3 = (\neg x_3 \lor x_4 \lor x_7)$
$C_4 = (\neg x_1 \lor \neg x_3 \lor x_9)$
…
$C_m = (x_1 \lor \neg x_8 \lor x_{22})$

Boolean variable part:
First n digit positions ensure that exactly one of t_i or f_i is included in any subset summing to W.

				i						j		
	1	2	3	4	…	n	1	2	3	4	…	m
$t_1 =$	1	0	0	0	…	0	1	0	0	0	…	1
$f_1 =$	1	0	0	0	…	0	0	1	0	1	…	0
$t_2 =$	0	1	0	0	…	0	0	1	0	0	…	0
$f_2 =$	0	1	0	0	…	0	1	0	0	0	…	0
$t_3 =$	0	0	1	0	…	0	1	0	0	0	…	0
$f_3 =$	0	0	1	0	…	0	0	0	1	1	…	0
…	…	…	…	…	…	…	…	…	…	…	…	…
$a_1 =$	0	0	0	0	…	0	1	0	0	0	…	0
$b_1 =$	0	0	0	0	…	0	1	0	0	0	…	0
$a_2 =$	0	0	0	0	…	0	0	1	0	0	…	0
$b_2 =$	0	0	0	0	…	0	0	1	0	0	…	0
…	…	…	…	…	…	…	…	…	…	…	…	…
$W =$	1	1	1	1	…	1	3	3	3	3	…	3

Clause part:
Three $\mathbf{1}$'s in each digit position j corresponding to the literals that would make clause C_j true.

Two extra $\mathbf{1}$'s one can choose in each clause position to add up to $\mathbf{3}$ and match W in case there are fewer than $\mathbf{3}$ satisfied literals per clause with satisfied assignment.

3SAT$\leq_P$Subset-Sum

If F satisfiable choose one of t_i or f_i depending on the satisfying assignment. Their sum will have exactly one 1 in each of the first n digits and at least one 1 in every clause digit position. Also include none, one, or both of each a_j, b_j pair to add to W.

				i						j		
	1	2	3	4	…	n	1	2	3	4	…	m
$t_1 =$	1	0	0	0	…	0	1	0	0	0	…	1
$f_1 =$	1	0	0	0	…	0	0	1	0	1	…	0
$t_2 =$	0	1	0	0	…	0	0	1	0	0	…	0
$f_2 =$	0	1	0	0	…	0	1	0	0	0	…	0
$t_3 =$	0	0	1	0	…	0	1	0	0	0	…	0
$f_3 =$	0	0	1	0	…	0	0	0	1	1	…	0
…	…	…	…	…	…	…	…	…	…	…	…	…
$a_1 =$	0	0	0	0	…	0	1	0	0	0	…	0
$b_1 =$	0	0	0	0	…	0	1	0	0	0	…	0
$a_2 =$	0	0	0	0	…	0	0	1	0	0	…	0
$b_2 =$	0	0	0	0	…	0	0	1	0	0	…	0
…	…	…	…	…	…	…	…	…	…	…	…	…
$W =$	1	1	1	1	…	1	3	3	3	3	…	3

If some subset sums to W must have exactly one of t_i or f_i for each i.
Set variable x_i to true if t_i used and false if f_i used.
Must have three 1's in each clause digit column j since things sum to W.
At most two of these can come from a_j, b_j to one of these 1's must come from the choices of the truth assignment which means that every clause C_j is satisfied so F is satisfiable. ∎

Some other NP-complete examples you should know

Hamiltonian-Cycle: Given a directed graph $G = (V, E)$. Is there a cycle in G that visits each vertex in V exactly once?

Hamiltonian-Path: Given a directed graph $G = (V, E)$. Is there a path p in G of length $n - 1$ that visits each vertex in V exactly once?

Same problems are also $\mathbf{NP}$-complete for undirected graphs

Note: If we asked about visiting each *edge* exactly once instead of each vertex, the corresponding problems are called **Eulerian-Cycle**, **Eulerian-Path** and are polynomial-time solvable.

Travelling-Salesperson Problem (TSP)

Travelling-Salesperson Problem (TSP):

Given: a set of n cities $v_1, \dots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities

What is the length of the shortest tour that visits all n cities?

DecisionTSP:

Given: a set of n cities $v_1, \dots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities *and* a distance D

Is there a tour of total length at most D that visits all n cities?

Hamiltonian-Cycle $\leq_P$ DecisionTSP

Define the reduction given $G = (V, E)$:

- Vertices $V = \{v_1, \dots, v_n\}$ become cities
- Define $d(v_i, v_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 2 & \text{if not} \end{cases}$
- Distance $D = |V|$.

Claim: There is a Hamiltonian cycle in $G \Leftrightarrow$ there is a tour of length $|V|$

NP-complete problems we've covered

3SAT → Independent-Set → Clique

↓

Vertex-Cover → 01-Programming → Integer-Programming

↓

Set-Cover

→ 3Color

→ Subset-Sum

→ Hamiltonian-Cycle → DecisionTSP

→ Hamiltonian-Path

More Hard Computational Problems

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.