CSE 421
Introduction to Algorithms

Lecture 26: Dealing with NP-completeness: Approximation Algorithms

Some other NP-complete examples you should know

Hamiltonian-Cycle: **Given** a directed graph $G = (V, E)$. Is there a cycle in G that visits each vertex in V exactly once?

Hamiltonian-Path: **Given** a directed graph $G = (V, E)$. Is there a path p in G of length $n - 1$ that visits each vertex in V exactly once?

Same problems are also $\mathbf{NP}$-complete for undirected graphs

Note: If we asked about visiting each *edge* exactly once instead of each vertex, the corresponding problems are called **Euler Tour**, **Eulerian-Path** and are polynomial-time solvable.

Travelling-Salesperson Problem (TSP)

Travelling-Salesperson Problem (TSP):

Given: a set of n cities $v_1, \ldots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities

Find the shortest tour that visits all n cities.

DecisionTSP:

Given: a set of n cities $v_1, \ldots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities *and* a distance D

Is there a tour of total length at most D that visits all n cities?

NP-complete problems we've discussed

3SAT → Independent-Set → Clique

↓

Vertex-Cover → 01-Programming → Integer-Programming

↓

Set-Cover

→ 3Color

→ Subset-Sum

→ Hamiltonian-Cycle → DecisionTSP

→ Hamiltonian-Path

Some intermediate problems

Problems reducible to $\mathbf{NP}$ problems not known to be polytime:

Basis for the security of current cryptography:

- **Factoring:** Given an integer N in binary, find its prime factorization.
- **Discrete logarithm:** Given prime p in binary, and g and x modulo p.
  Find y such that $x \equiv g^y \pmod{p}$ if it exists.

Best algorithms known are $2^{\widetilde{\Theta}(n^{1/3})}$ time.

Other famous ones:

- **Graph Isomorphism:** Given graphs G and H, can they be relabelled to be the same?
  Best algorithm now $n^{\Theta(\log^2 n)}$ (recently improved from $2^{\widetilde{\Theta}(n^{1/3})}$) time.
- **Nash equilibrium:** Given a multiplayer game, find randomized strategies for each player so that no player could do better by deviating.

What to do if the problem you want to solve is NP-hard

1st thing to try:

- You might have phrased your problem too generally
  - e.g., In practice, the graphs that actually arise are far from arbitrary
    - Maybe they have some special characteristic that allows you to solve the problem in your special case
      - For example the **Independent-Set** problem is easy on "interval graphs"
        - Exactly the case for the Interval Scheduling problem!
  - Search the literature to see if special cases already solved

What to do if the problem you want to solve is NP-hard

2nd thing to try if your problem is a minimization or maximization problem

- Try to find a polynomial-time worst-case approximation algorithm
  - For a minimization problem
    - Find a solution with value $\leq \boldsymbol{K}$ times the optimum
  - For a maximization problem
    - Find a solution with value $\geq \mathbf{1}/\boldsymbol{K}$ times the optimum

Want $\boldsymbol{K}$ to be as close to $\mathbf{1}$ as possible.

Greedy Approximation for Vertex-Cover

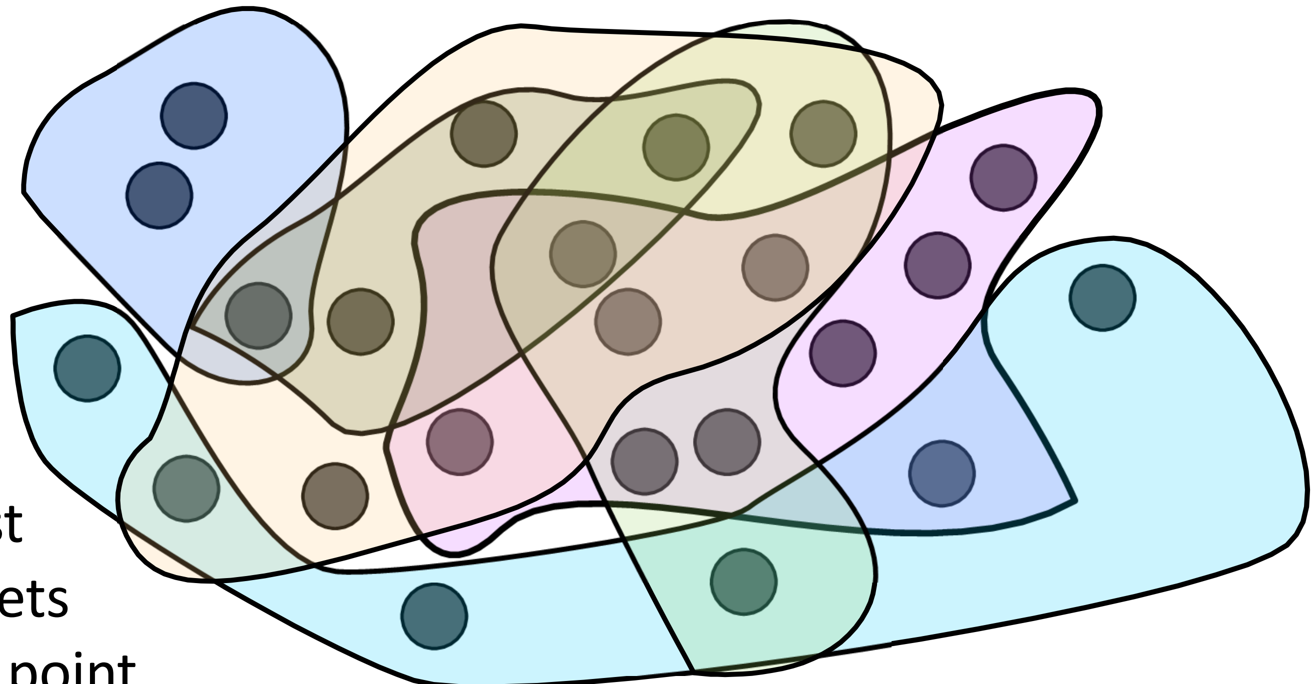
```
On input G = (V, E)
W ← ∅
E' ← E
while E' ≠ ∅
    select any e = (u, v) ∈ E'
    W ← W ∪ {u, v}
    E' ← E' \ {edges e ∈ E' that touch u or v}
```

This actually a better approximation factor than the greedy algorithm that repeatedly chooses the highest degree vertex remaining that you considered on Homework 3.

Claim: At most a factor $\mathbf{2}$ larger than the optimal vertex-cover size.

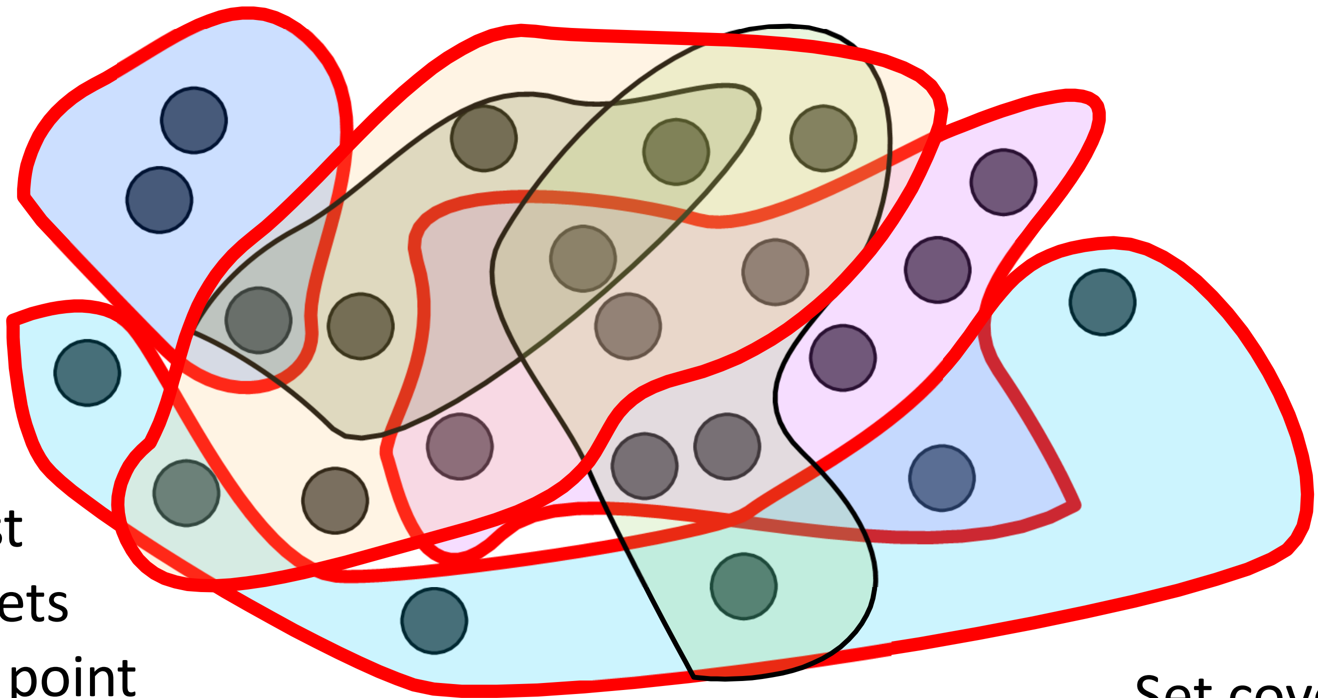
Proof: Edges selected don't share any vertices so any vertex-cover must choose at least one of $\boldsymbol{u}$ or $\boldsymbol{v}$ each time.

Set-Cover



Find smallest collection of sets containing every point

Set-Cover

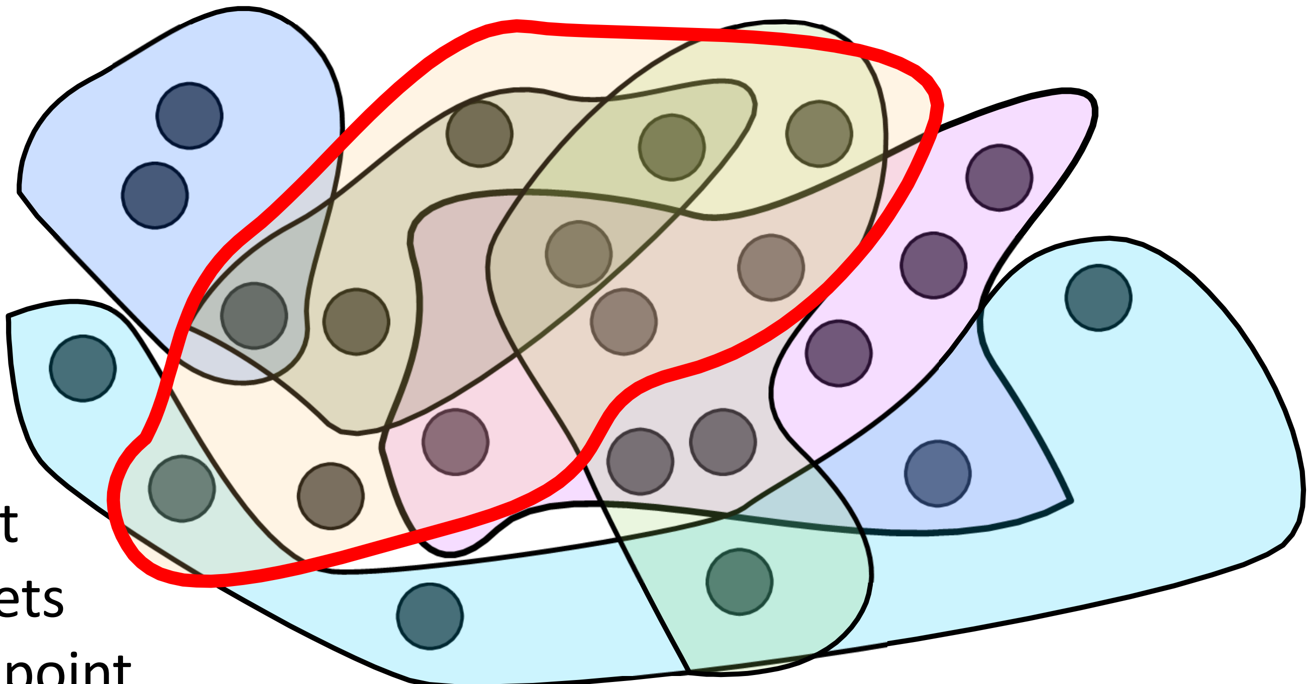


Find smallest
collection of sets
containing every point

Set cover size **4**

Set-Cover

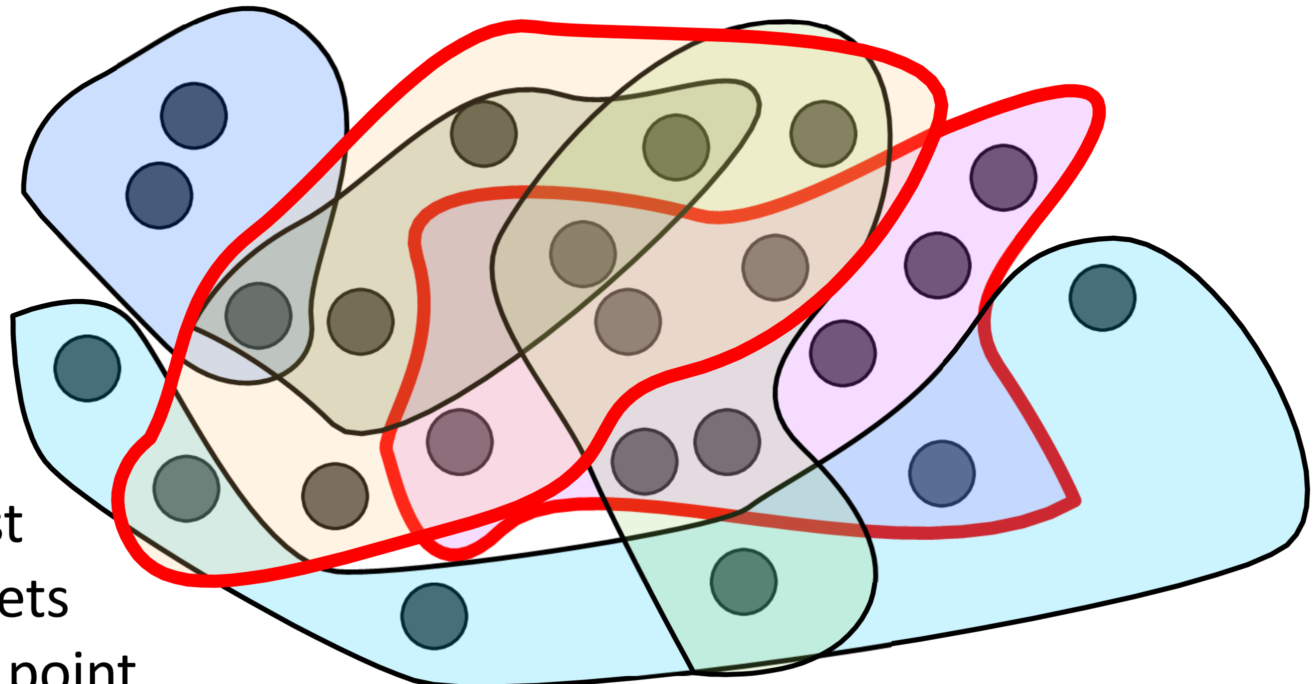
Greedy Set Cover: Repeatedly choose the set that covers the most # of new elements



Find smallest collection of sets containing every point

Set-Cover

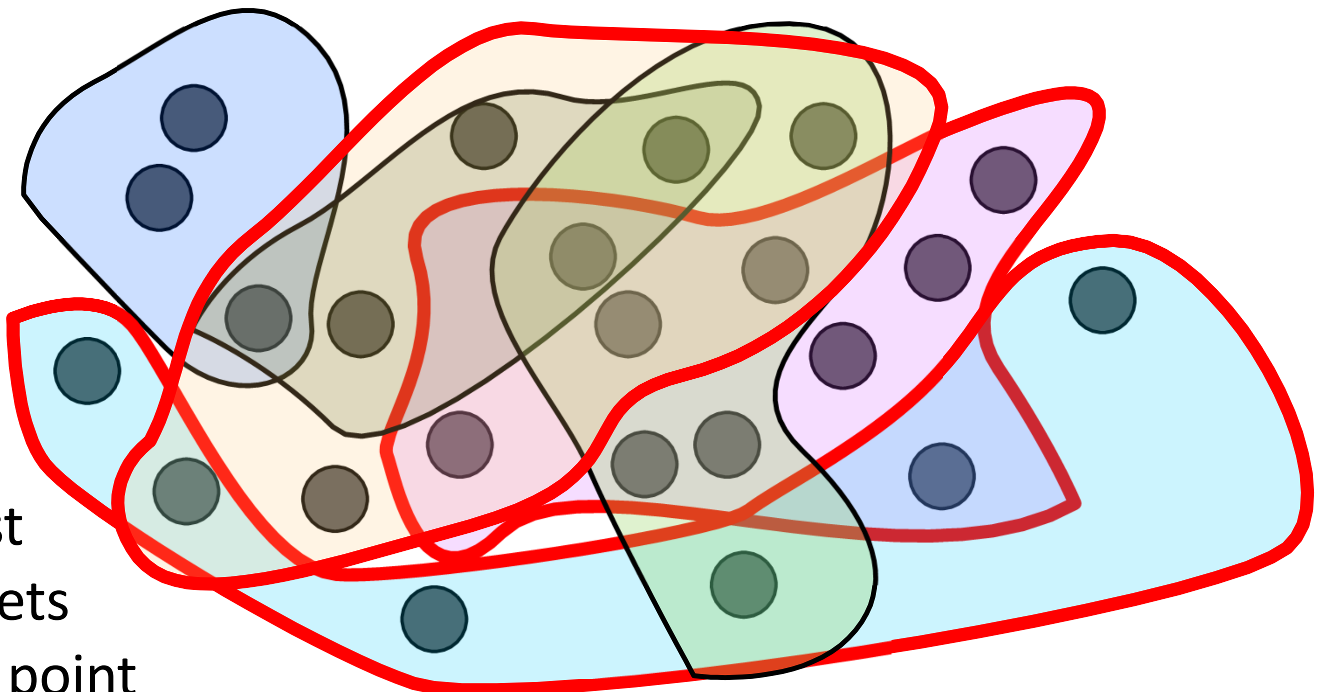
Greedy Set Cover: Repeatedly choose the set that covers the most # of new elements



Find smallest collection of sets containing every point

Set-Cover

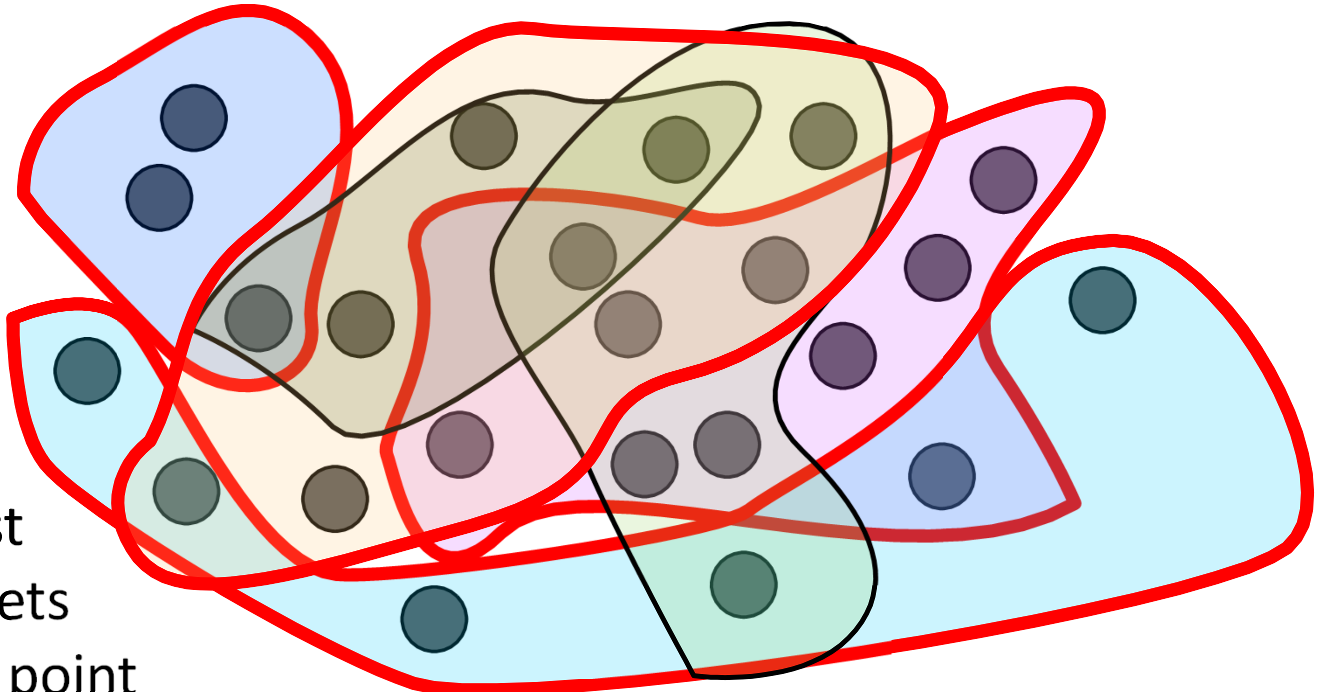
Greedy Set Cover: Repeatedly choose the set that covers the most # of new elements



Find smallest collection of sets containing every point

Set-Cover

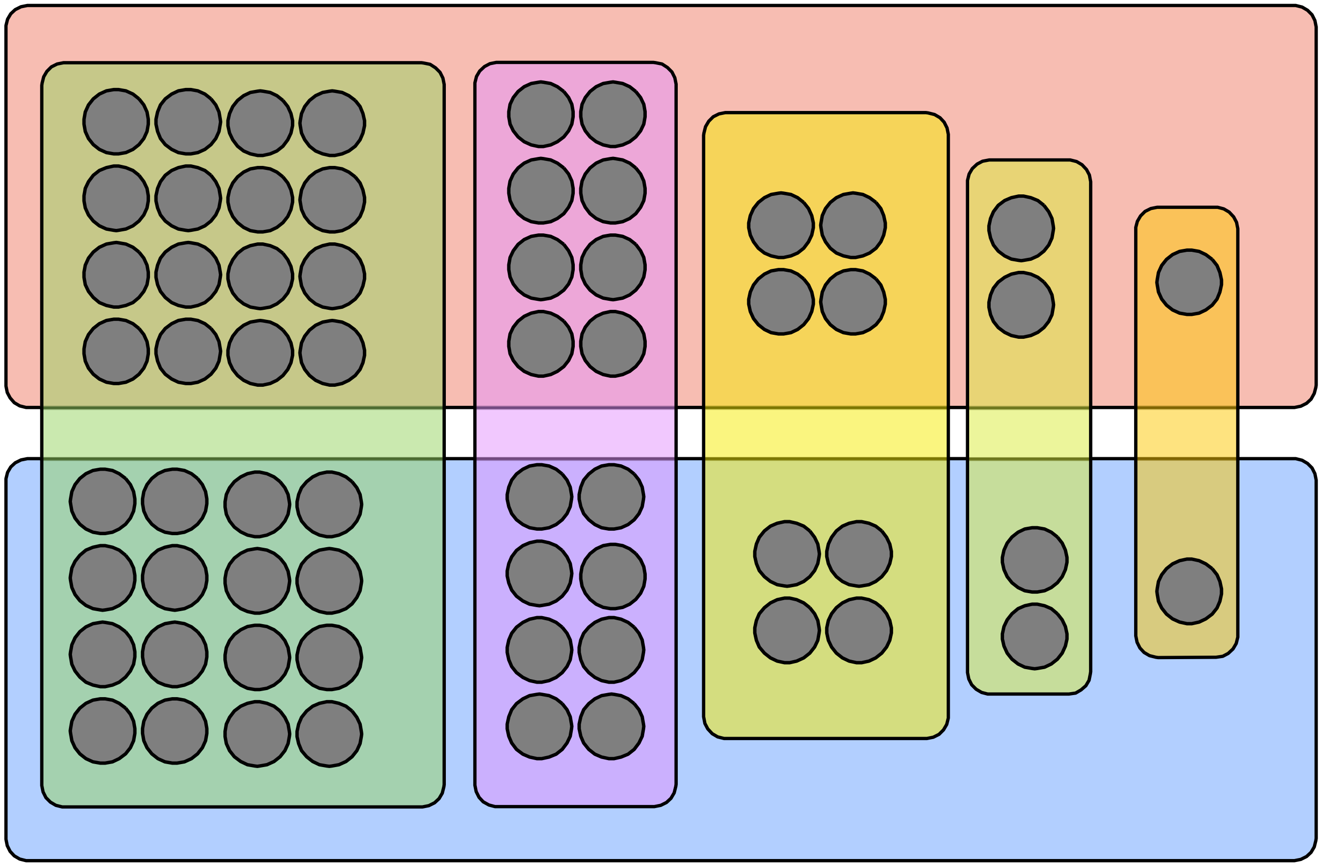
Greedy Set Cover: Repeatedly choose the set that covers the most # of new elements



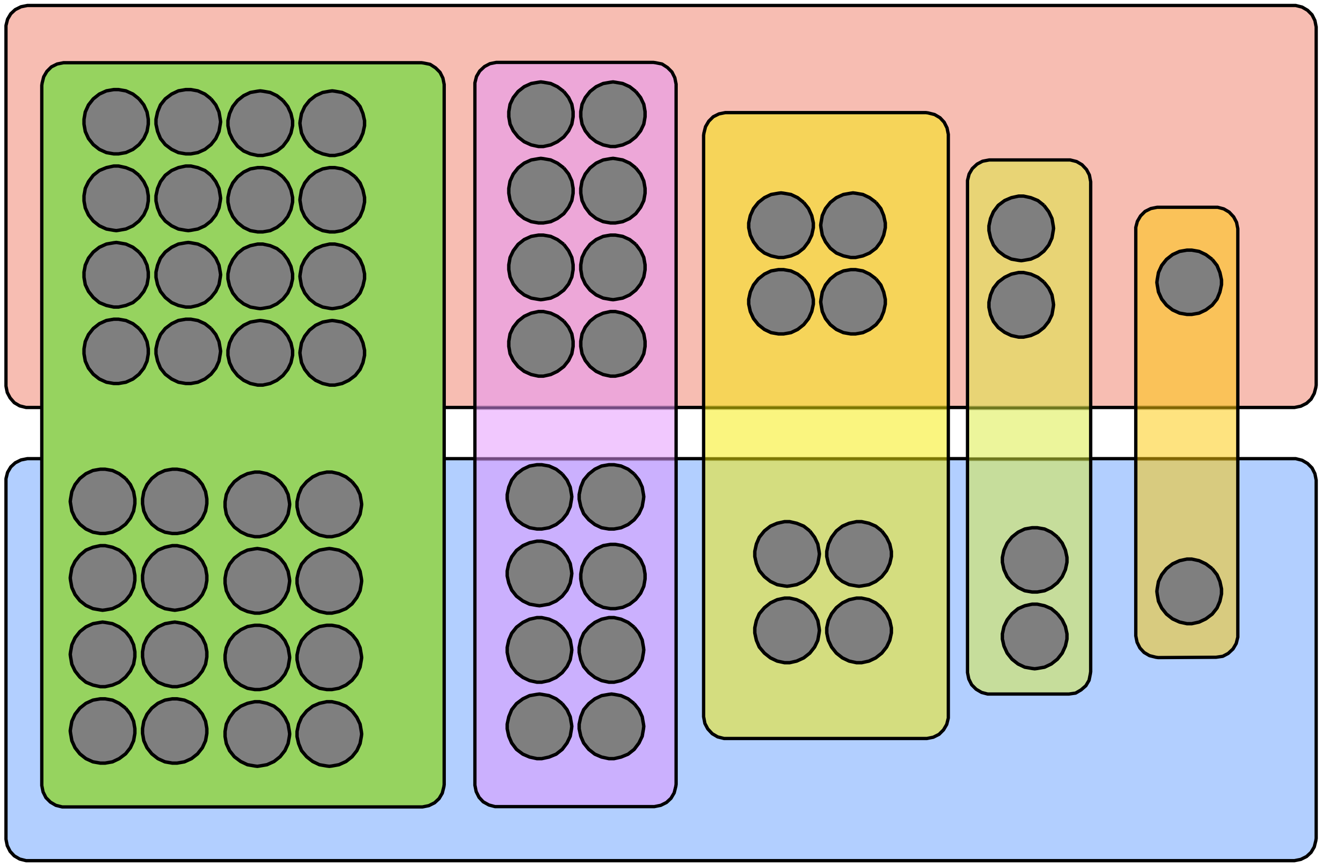
Find smallest collection of sets containing every point

Theorem: Greedy finds best cover up to a factor of $\ln n$.

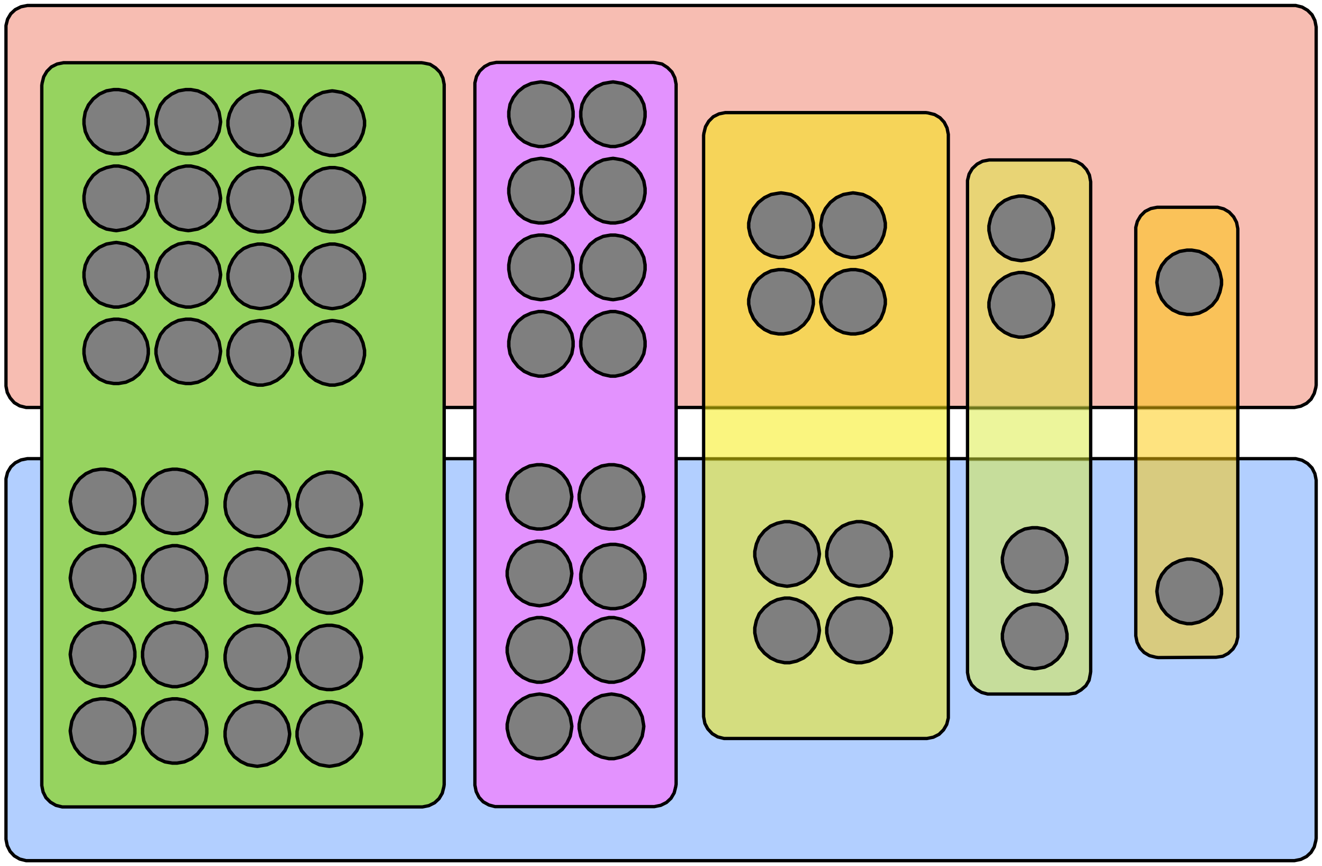
Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



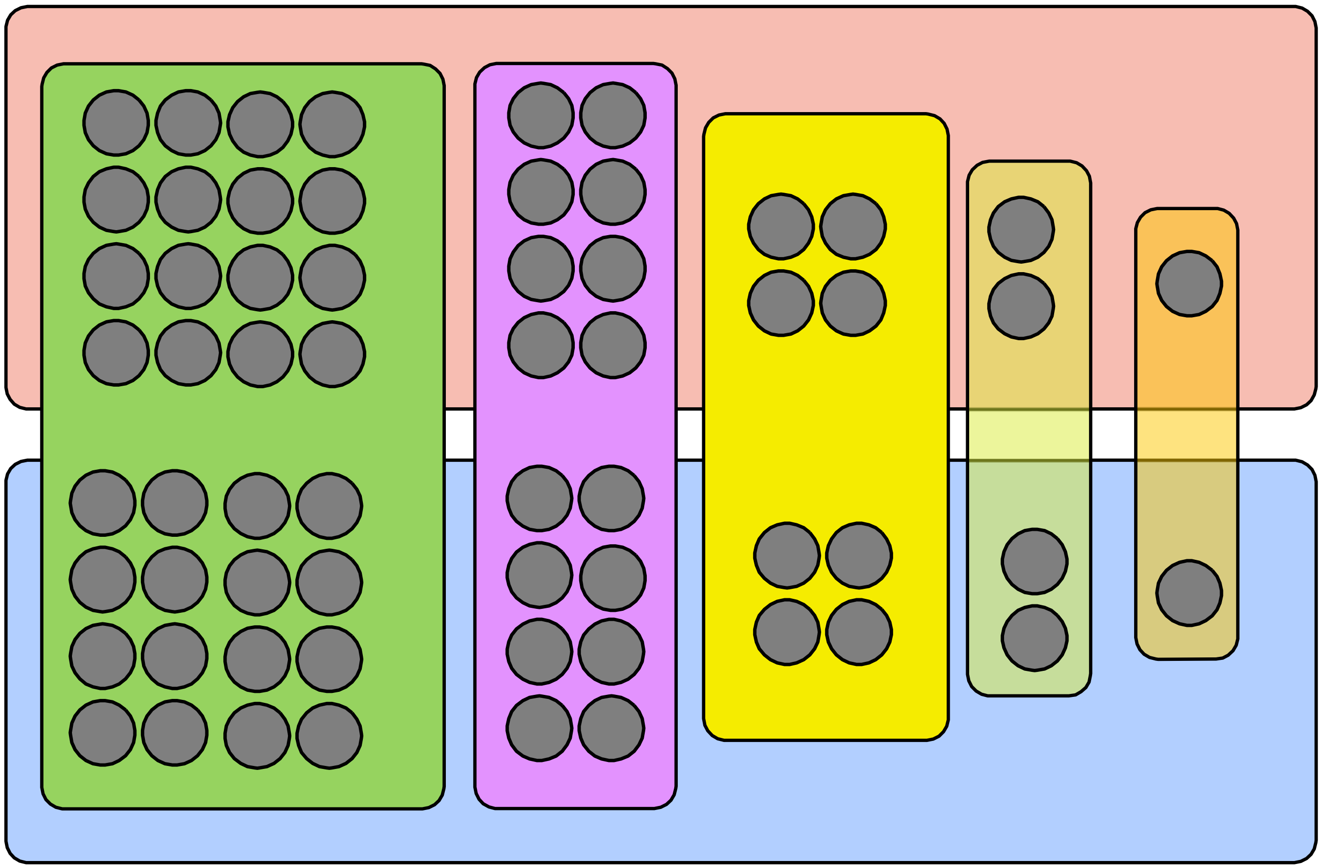
Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



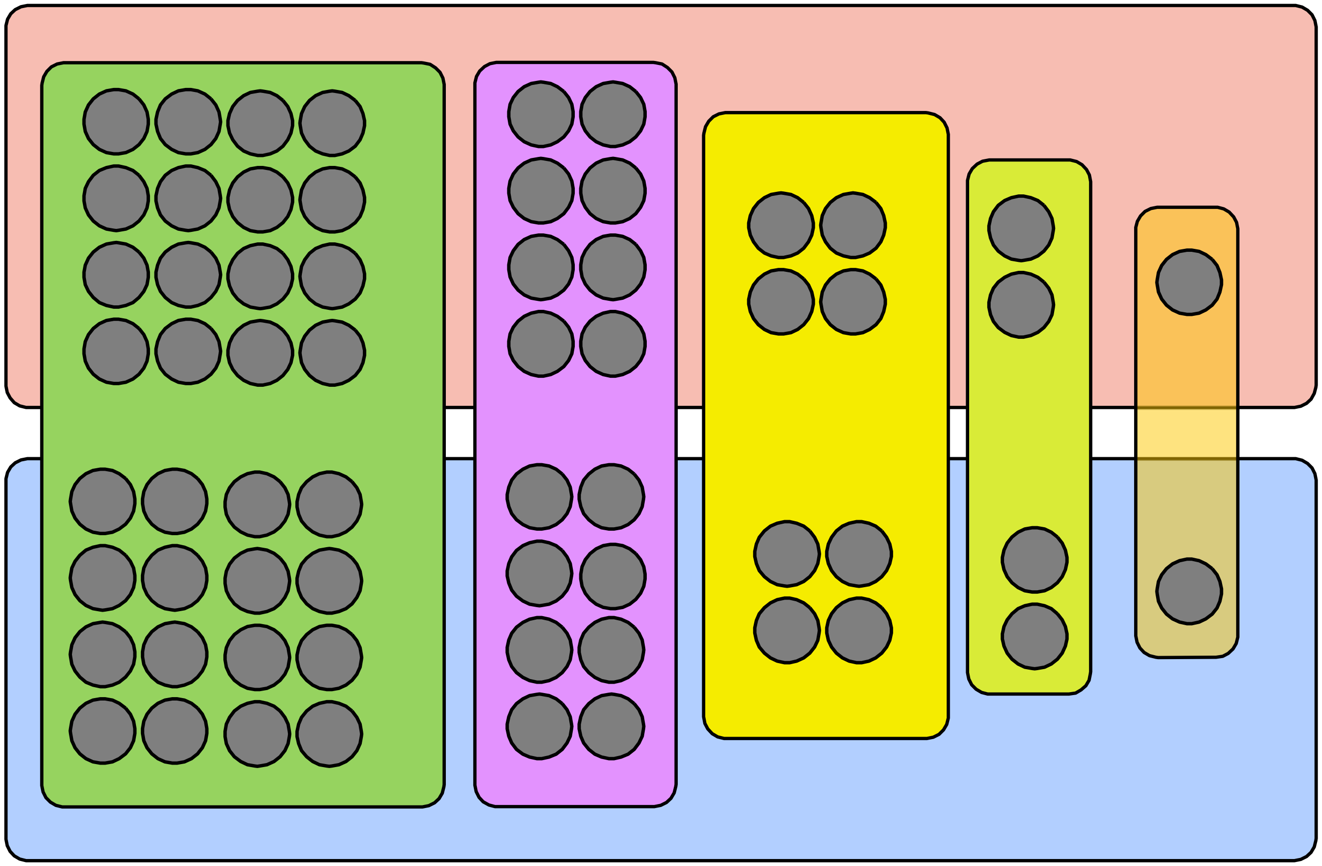
Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



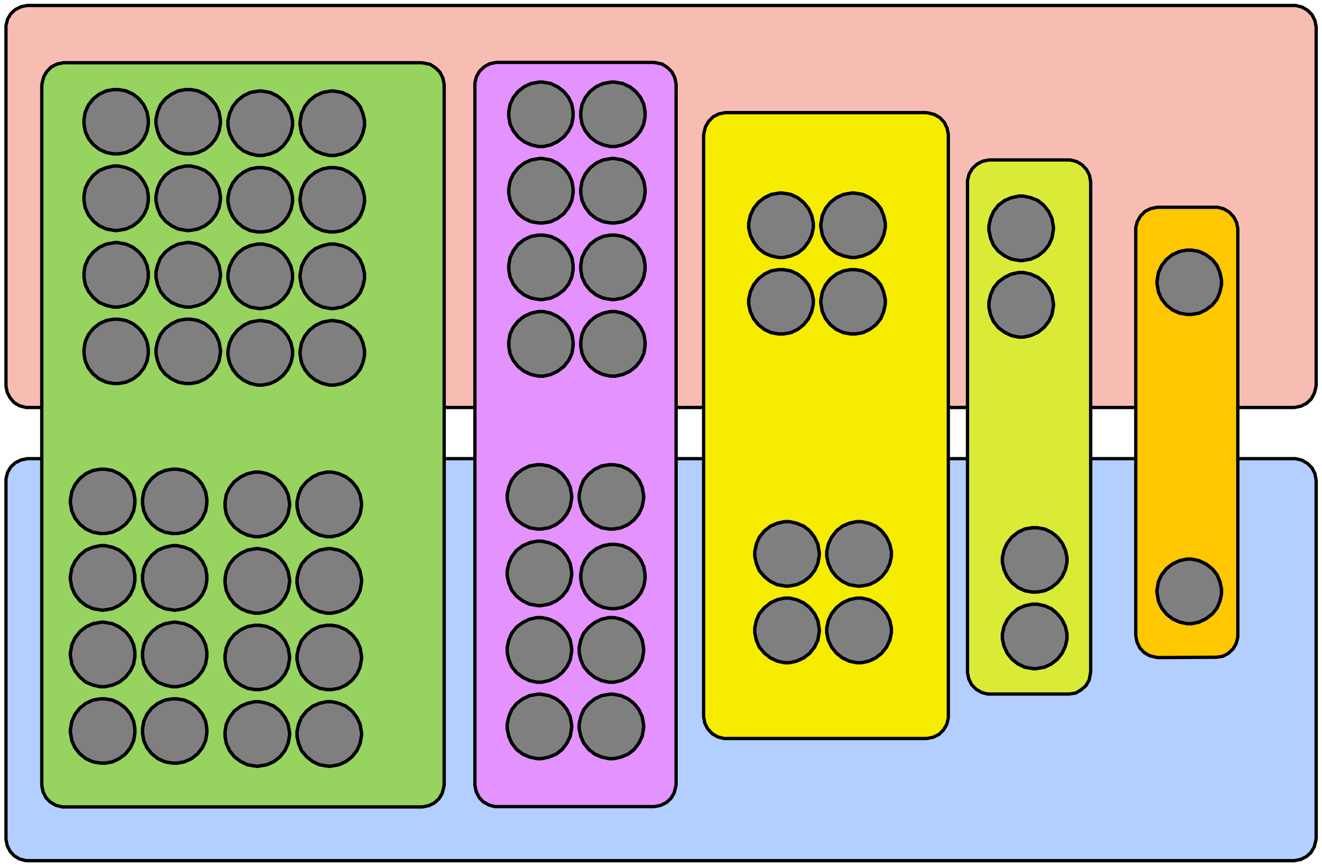
Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



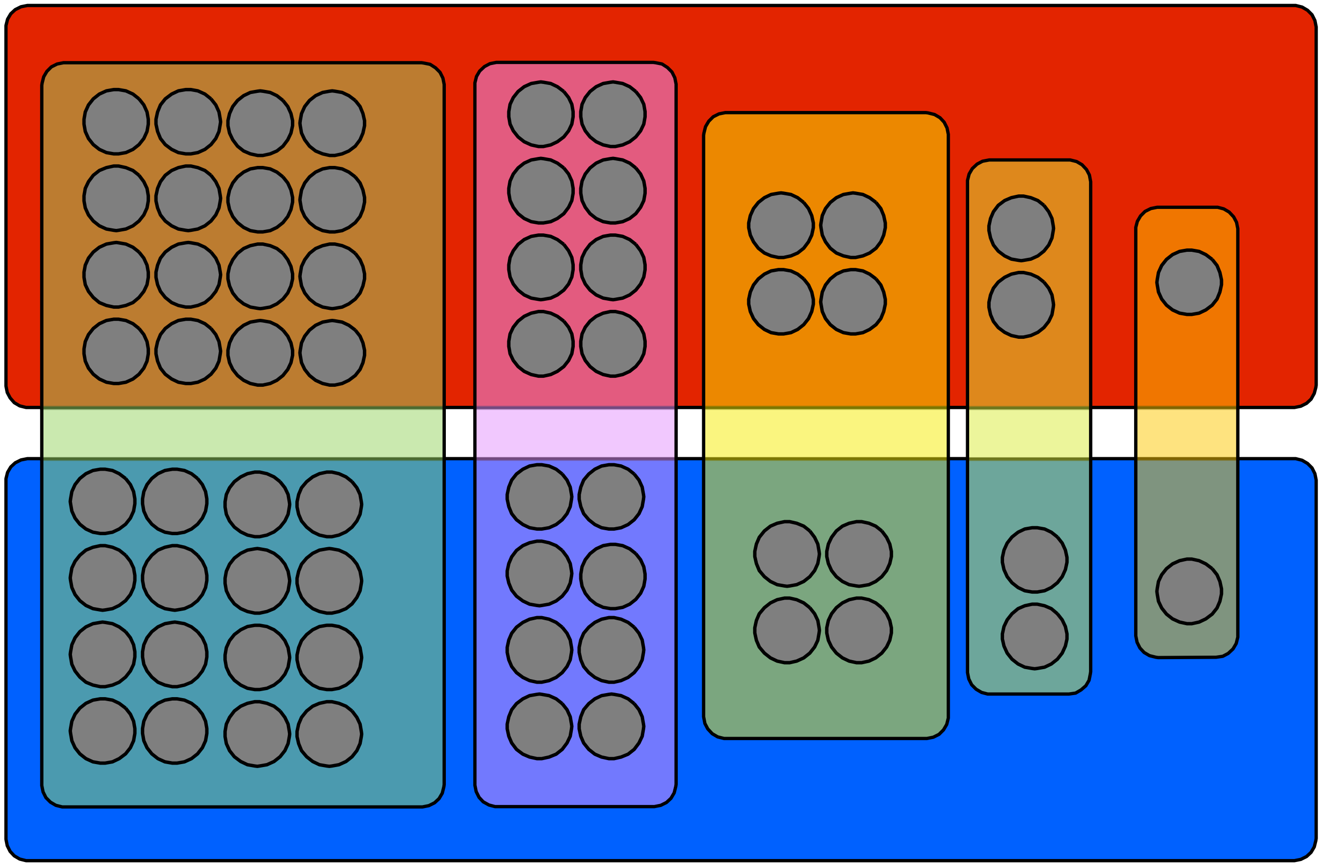
Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



Greedy solution: **5** sets

Greedy solution: $\sim \log_2 n$ sets

Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



Optimal:
2 sets

Greedy Approximation to Set-Cover

Theorem: If there is a set cover of size $\boldsymbol{k}$ then the greedy set cover has size $\leq \boldsymbol{k} \ln \boldsymbol{n}$.

Proof: Suppose that there is a set cover of size $\boldsymbol{k}$.

At each step all elements remaining are covered by these $\boldsymbol{k}$ sets.

So always a set available covering $\geq \mathbf{1}/\boldsymbol{k}$ fraction of remaining elts.

So # of uncovered elts after $\boldsymbol{i}$ sets $\leq \left(\mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{k}}\right) \times$ # after $\boldsymbol{i} - \mathbf{1}$ sets.

Total after $\boldsymbol{t}$ sets $\leq \boldsymbol{n}\left(\mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{k}}\right)^{\boldsymbol{t}} < \boldsymbol{n} \cdot \boldsymbol{e}^{-\boldsymbol{t}/\boldsymbol{k}} = \mathbf{1}$ for $\boldsymbol{t} = \boldsymbol{k} \ln \boldsymbol{n}$. ■

$\boxed{\mathbf{1} - \boldsymbol{x} < \boldsymbol{e}^{-\boldsymbol{x}} \text{ for } \boldsymbol{x} > \mathbf{0}}$

Travelling-Salesperson Problem (TSP)

Travelling-Salesperson Problem (TSP):

Given: a set of n cities $v_1, \dots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities

Find the shortest tour that visits all n cities.

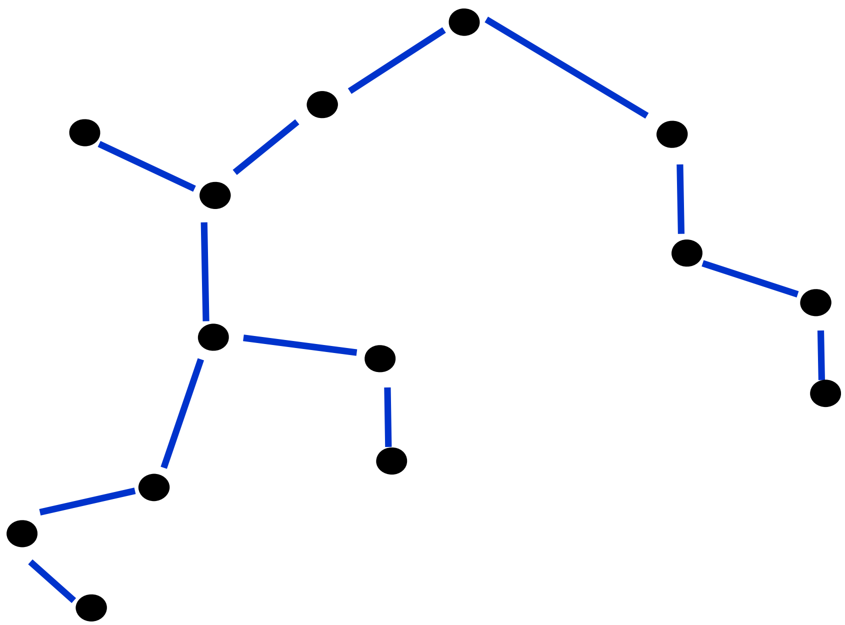
MetricTSP:

The distance function d satisfies the triangle inequality:

$$d(u, w) \le d(u, v) + d(v, w)$$

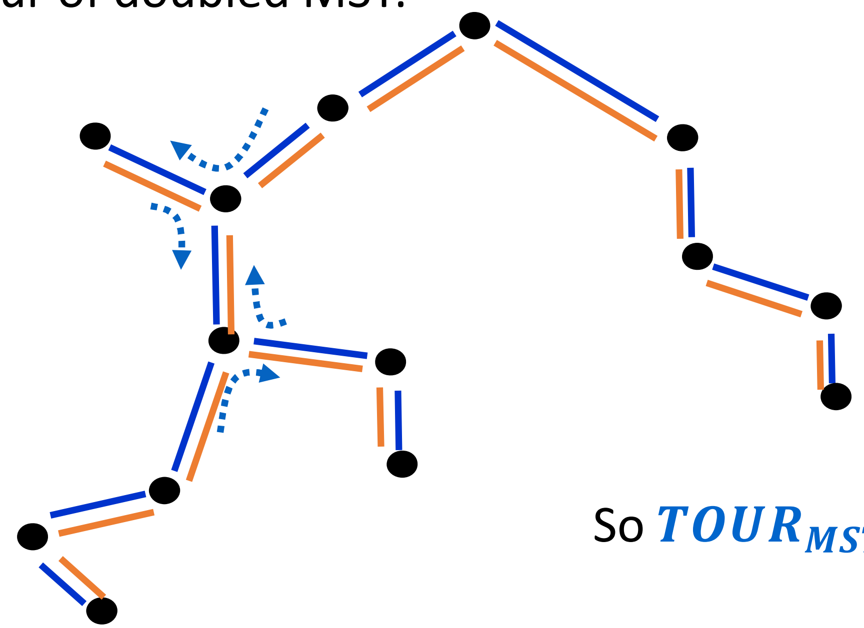
Proper tour: visit each city exactly once.

Minimum Spanning Tree Approximation: Factor of 2



TSP: Minimum Spanning Tree Factor 2 Approximation

Euler Tour of doubled MST:



Euler tour covers each edge twice
so $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G)}$

Any tour contains a spanning tree
so $\boldsymbol{MST(G) \leq TOUR_{OPT}(G)}$

So $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G) \leq 2\ TOUR_{OPT}(G)}$

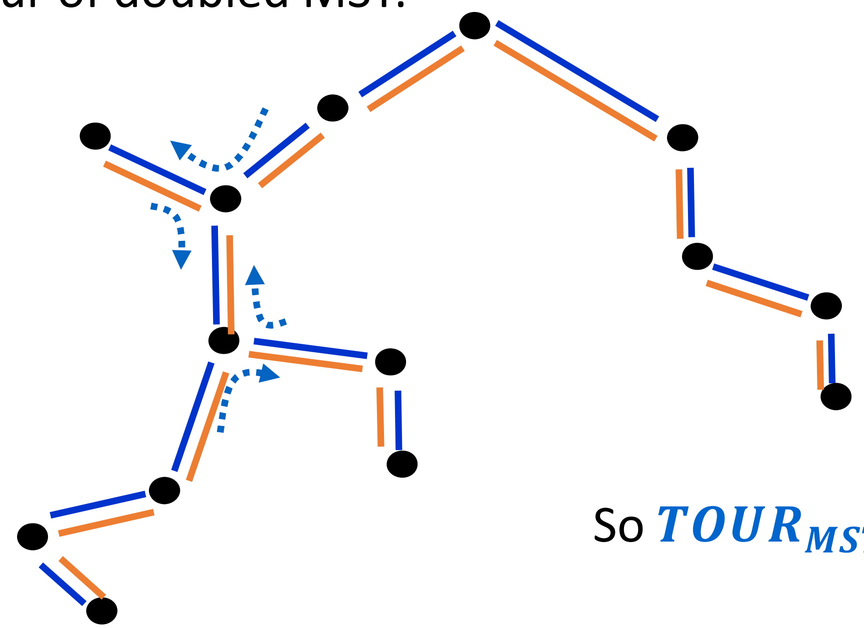
This visits each node more than once, so not a proper tour.

Why did this work?

- We found an Euler tour on a graph that used the edges of the original graph (possibly repeated).
- The weight of the tour was the total weight of the new graph.
- Suppose now
  - All edges possible
  - Weights satisfy the triangle inequality (MetricTSP)

MetricTSP: Minimum Spanning Tree Factor 2 Approximation

Euler Tour of doubled MST:



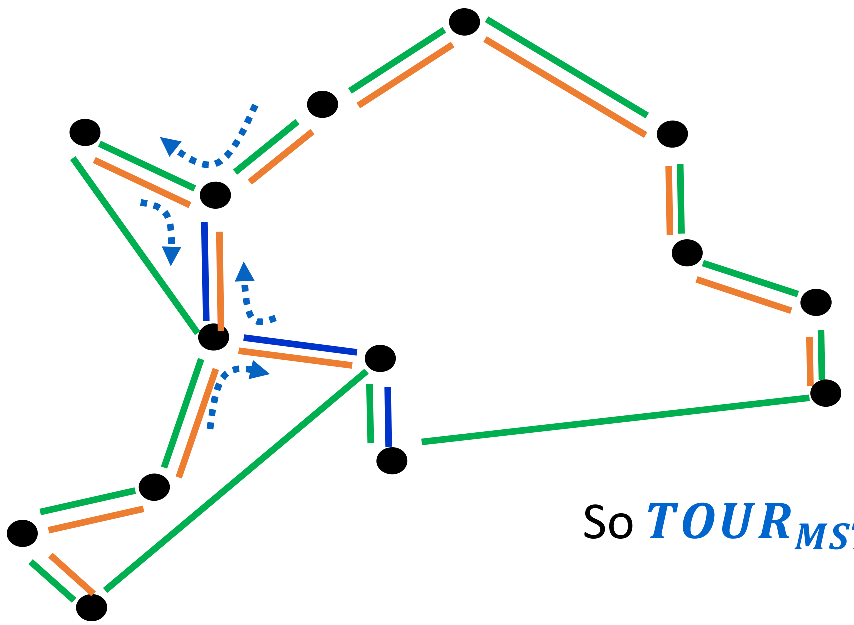
Euler tour covers each edge twice
so $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G)}$

Any tour contains a spanning tree
so $\boldsymbol{MST(G) \le TOUR_{OPT}(G)}$

So $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G) \le 2\ TOUR_{OPT}(G)}$

Instead: take shortcut to next unvisited vertex on the Euler tour
By triangle inequality this can only be shorter.

MetricTSP: Minimum Spanning Tree Factor 2 Approximation



Euler tour covers each edge twice
so $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G)}$

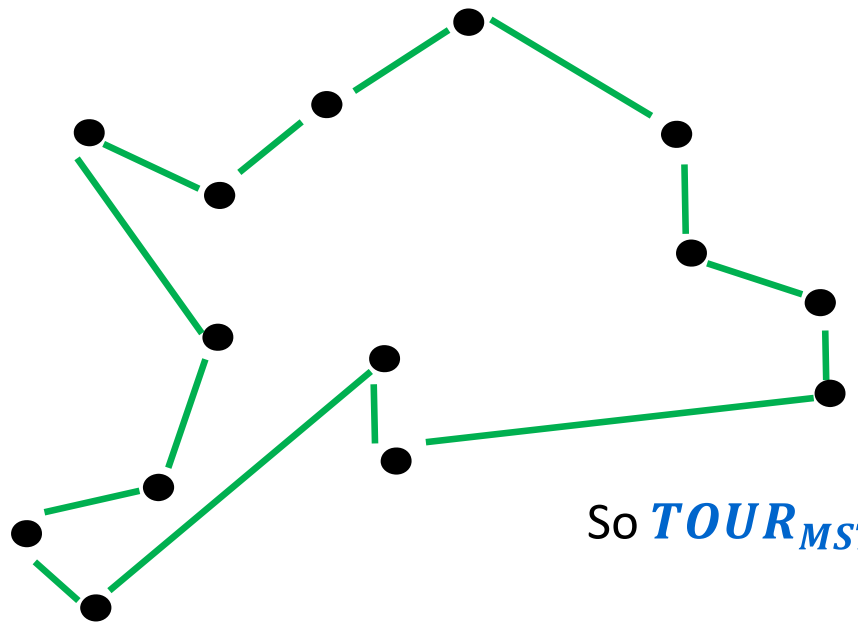
Any tour contains a spanning tree
so $\boldsymbol{MST(G) \leq TOUR_{OPT}(G)}$

So $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G) \leq 2\ TOUR_{OPT}(G)}$

Instead: take shortcut to next unvisited vertex on the Euler tour
By triangle inequality this can only be shorter.

MetricTSP: Minimum Spanning Tree Factor 2 Approximation

Final:



Euler tour covers each edge twice
so $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G)}$

Any tour contains a spanning tree
so $\boldsymbol{MST(G) \leq TOUR_{OPT}(G)}$

So $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G) \leq 2\ TOUR_{OPT}(G)}$

Instead: take shortcut to next unvisited vertex on the Euler tour
By triangle inequality this can only be shorter.

Christofides Algorithm: A factor 3/2 approximation

Any subgraph of the weighted complete graph that has an Euler Tour will work also!

Fact: To have an Euler Tour it suffices to have all degrees even.

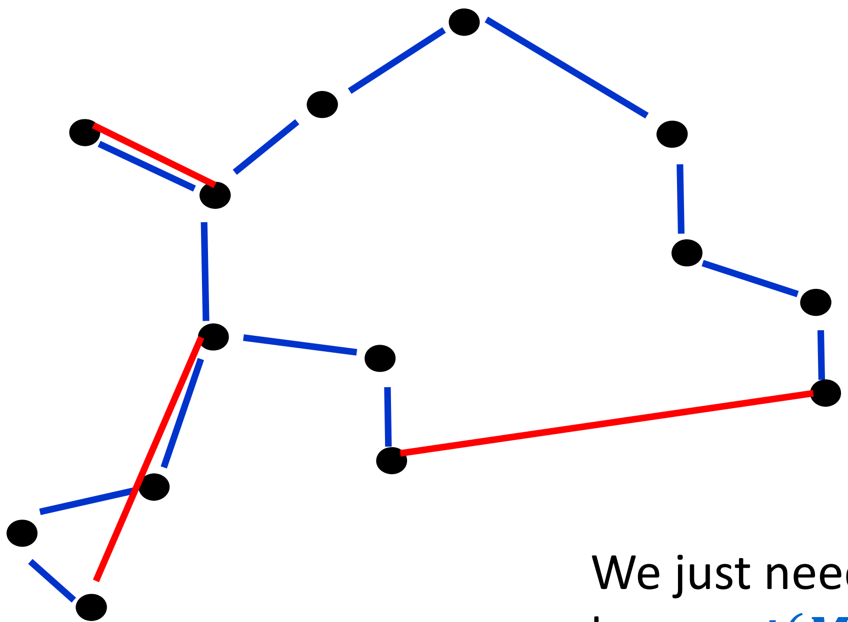
Christofides Algorithm:

- Compute an MST $\boldsymbol{T}$
- Find the set $\boldsymbol{O}$ of odd-degree vertices in $\boldsymbol{T}$
- Add a minimum-weight perfect matching* $\boldsymbol{M}$ on the vertices in $\boldsymbol{O}$ to $\boldsymbol{T}$ to make every vertex have even degree
  - There are an even number of odd-degree vertices!
- Use an Euler Tour $\boldsymbol{E}$ in $\boldsymbol{T} \cup \boldsymbol{M}$ and then shortcut as before

Theorem: $\boldsymbol{Cost(E) \leq 1.5\, TOUR_{OPT}}$

*Requires finding optimal matchings in general graphs, not just bipartite ones

Christofides Approximation

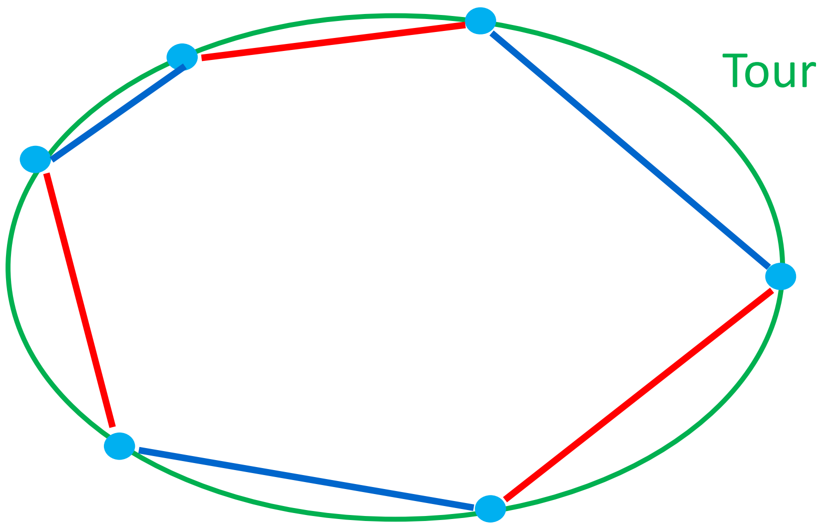


Any tour contains a spanning tree so $\boldsymbol{MST \leq TOUR_{OPT}}$

We just need to show that the matching $\boldsymbol{M}$ has $\boldsymbol{cost(M) \leq TOUR_{OPT}/2}$

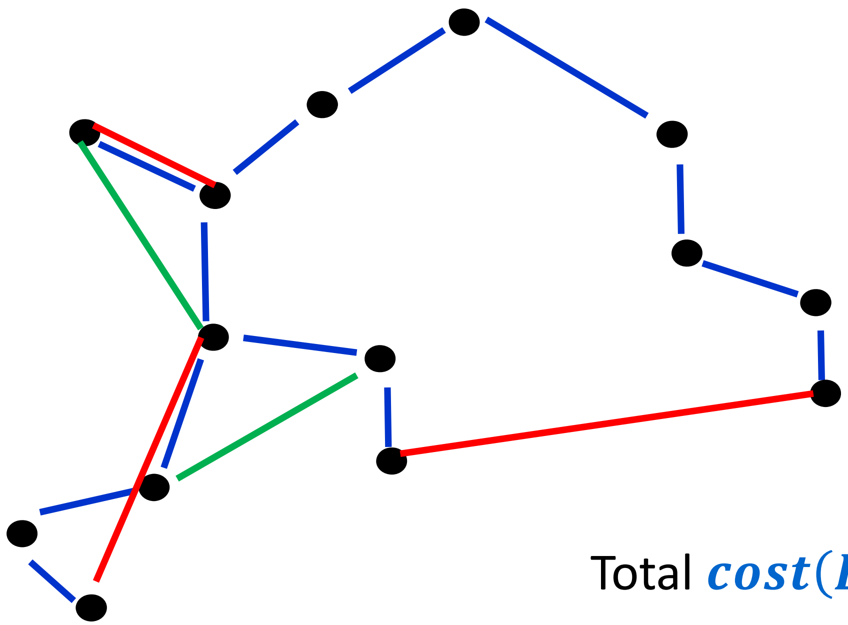
Christofides Approximation

Any tour costs at least the cost of two matchings M_1 and M_2 on O



$$2\,cost(M) \leq cost(M_1) + cost(M_2) \leq TOUR_{OPT}$$

Christofides Approximation Final Tour



Total $\boldsymbol{cost(E) \leq 3\,TOUR_{OPT}/2}$

Max-3SAT Approximation

Max-3SAT: Given a 3CNF formula F find a truth assignment that satisfies the maximum possible # of clauses of F.

Observation: A single clause on 3 variables only rules out $1/8$ of the possible truth assignments since each literal has to be false to be ruled out.

$\Rightarrow$ a random truth assignment will satisfy the clause with probability $7/8$.

So in expectation, if F has m clauses, a random assignment satisfies $7m/8$ of them.

A greedy algorithm can achieve this: Choose most frequent literal appearing in clauses that are not yet satisfied and set it to true.

If $\mathbf{P} \neq \mathbf{NP}$ no better approximation is possible

Knapsack Problem

Each item has a value v_i and a weight w_i.

Maximize $\sum_{i \in S} v_i$ with $\sum_{i \in S} w_i \leq W$.

Theorem: For any $\varepsilon > 0$ there is an algorithm that produces a solution within $(1 + \varepsilon)$ factor of optimal for the Knapsack problem with running time $O(n^2/\varepsilon^2)$

"Polynomial-Time Approximation Scheme" or PTAS

Algorithm: Maintain the high order bits in the dynamic programming solution.

Hardness of Approximation

Polynomial-time approximation algorithms for $\mathbf{NP}$-hard optimization problems can sometimes be ruled out unless $\mathbf{P} = \mathbf{NP}$.

Easy example:

Coloring: Given a graph $G = (V, E)$ find the smallest k such that G has a k-coloring.

Because $\mathbf{3}$-coloring is $\mathbf{NP}$-hard, no approximation ratio better than $4/3$ is possible unless $\mathbf{P} = \mathbf{NP}$ because you would have to be able to figure out if a $\mathbf{3}$-colorable graph can be colored in < 4 colors. i.e. if it can be $\mathbf{3}$-colored.

- We now know a huge amount about the hardness of approximating $\mathbf{NP}$ optimization problems if $\mathbf{P} \neq \mathbf{NP}$.
- Approximation factors are very different even for closely related problems like **Vertex-Cover** and **Independent-Set**.

Approximation Algorithms/Hardness of Approximation

Research has classified many problems based on what kinds of polytime approximations are possible if $\mathbf{P \neq NP}$

- **Best:** $(\mathbf{1} + \boldsymbol{\varepsilon})$ factor for any $\boldsymbol{\varepsilon} > \mathbf{0}$. (PTAS)
  - packing and some scheduling problems, TSP in plane
- Some fixed constant factor $> \mathbf{1}$. e.g. $\mathbf{2}, \mathbf{3/2}, \mathbf{8/7}, \mathbf{100}$
  - Vertex Cover, Max-3SAT, MetricTSP, other scheduling problems
  - Exact best factors or very close upper/lower bounds known for many problems.
- $\Theta(\log \boldsymbol{n})$ factor
  - Set Cover, Graph Partitioning problems
- **Worst:** $\Omega(\boldsymbol{n}^{\mathbf{1}-\boldsymbol{\varepsilon}})$ factor for every $\boldsymbol{\varepsilon} > \mathbf{0}$.
  - Clique, Independent-Set, Coloring