CSE 421: Introduction to Algorithms

Trees, BFS

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Degree 1 vertices

Claim: If G has no cycle, then it has a vertex of degree ≤ 1

(So, every tree has a leaf)

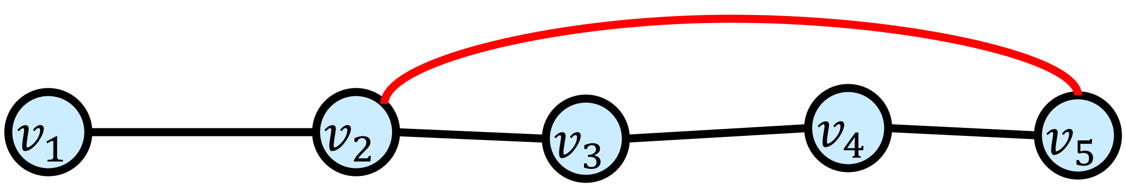
Pf: (By contradiction)

Suppose every vertex has degree ≥ 2.

Start from a vertex v_1 and follow a path, $v_1, \ldots, v_i$ when we are at v_i we choose the next vertex to be different from v_{i-1}. We can do so because $\deg(v_i) \geq 2$.

The first time that we see a repeated vertex ($v_j = v_i$) we get a cycle.

We always get a repeated vertex because G has finitely many vertices



Trees and Induction

Claim: Show that every tree with n vertices has n-1 edges.

Pf: By induction.

Base Case: n=1, the tree has no edge

IH: Suppose every tree with n-1 vertices has n-2 edges

IS: Let T be a tree with n vertices.

So, T has a vertex v of degree 1.

Remove v and the neighboring edge, and let T' be the new graph.

We claim T' is a tree: It has no cycle, and it must be connected.

So, T' has n-2 edges and T has n-1 edges.

Trees and Properties

Thm: Any graph G with n vertices having two of the following three properties is a tree and has the the third property:

- G has n-1 edges
- G is connected
- G has no cycle

#edges

Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges.

Claim: $0 \leq m \leq \binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$

Pf: Since every edge connects two distinct vertices (i.e., G has no loops)

and no two edges connect the same pair of vertices (i.e., G has no multi-edges)

It has at most $\binom{n}{2}$ edges.

Sparse Graphs

A graph is called sparse if $m \ll n^2$ and it is called dense otherwise.

edges

Sparse graphs are very common in practice

- Friendships in social network
- Planar graphs
- Web braph

Q: Which is a better running time $O(n + m)$ vs $O(n^2)$?

A: $O(n + m) = O(n^2)$, but $O(n + m)$ is usually much better.

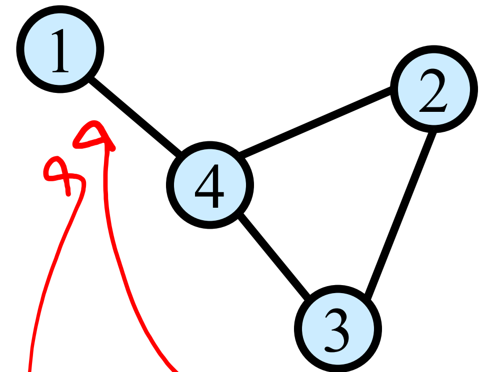
linear

Storing Graphs (Internally in ALG)

Vertex set $V = \{v_1, \ldots, v_n\}$.

Adjacency Matrix: A

- For all, $i, j, A[i,j] = 1$ iff $(v_i, v_j) \in E$
- Storage: n^2 bits



	1	2	3	4
1	0	0	0	1
2	0	0	1	1
3	0	1	0	1
4	1	1	1	0

Advantage:

- $O(1)$ test for presence or absence of edges

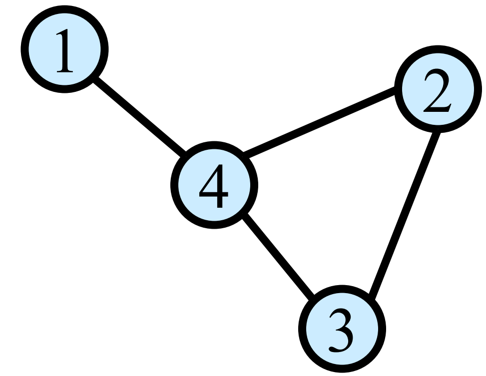
Disadvantage:

- Inefficient for sparse graphs both in storage and edge-access

Storing Graphs (Internally in ALG)

Adjacency List:

O(n+m) words

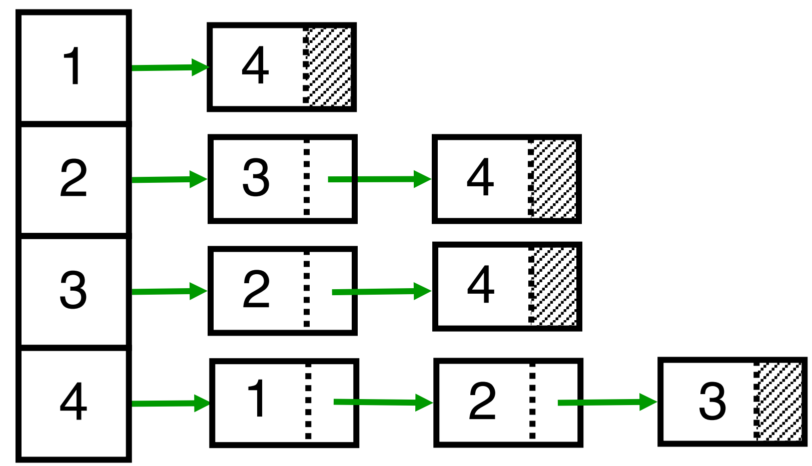


Advantage

- Compact for sparse
- Easily see all edges

Disadvantage

- No O(1) edge test
- More complex data structure

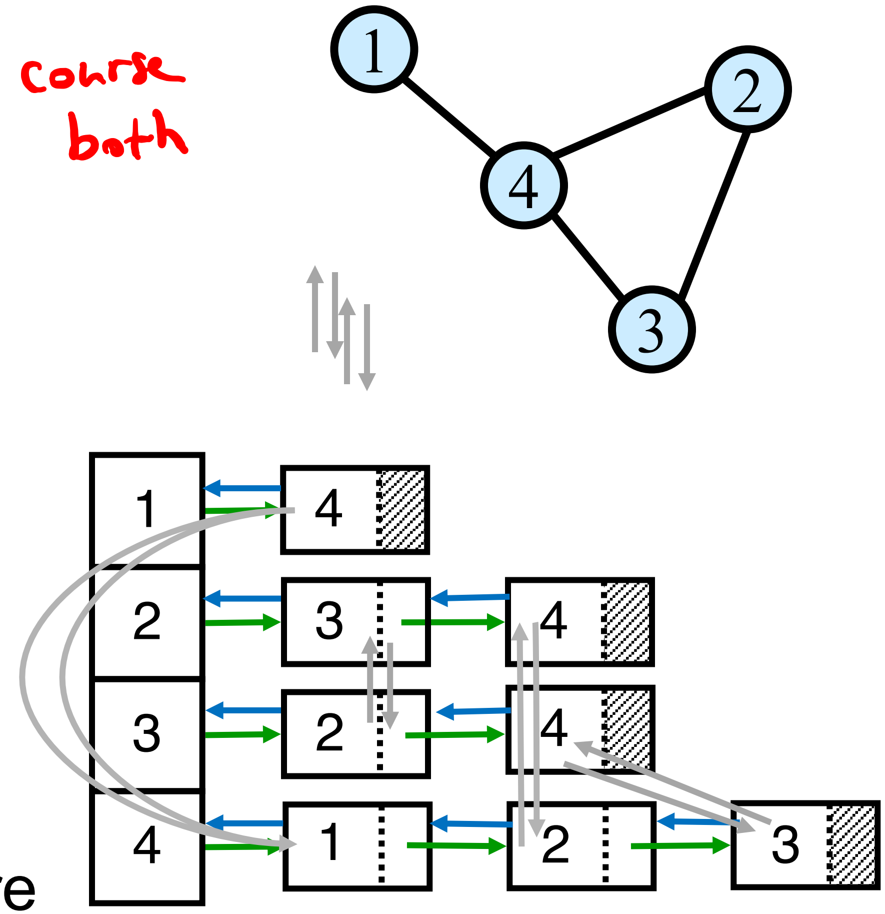


Storing Graphs (Internally in ALG)

Adjacency List:

O(n+m) words

In this course we have both



Advantage

- Compact for sparse
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Disadvantage

- No O(1) edge test
- More complex data structure

Graph Traversal

Walk (via edges) from a fixed starting vertex s to all vertices reachable from s.

- Breadth First Search (BFS): Order nodes in successive layers based on distance from s
- Depth First Search (DFS): More natural approach for exploring a maze; many efficient algs build on it.

Applications:

- Finding Connected components of a graph
- Testing Bipartiteness
- Finding Aritculation points

Breadth First Search (BFS)

Completely explore the vertices in order of their distance from s.

Three states of vertices:

- Undiscovered
- Discovered
- Fully-explored

Naturally implemented using a queue

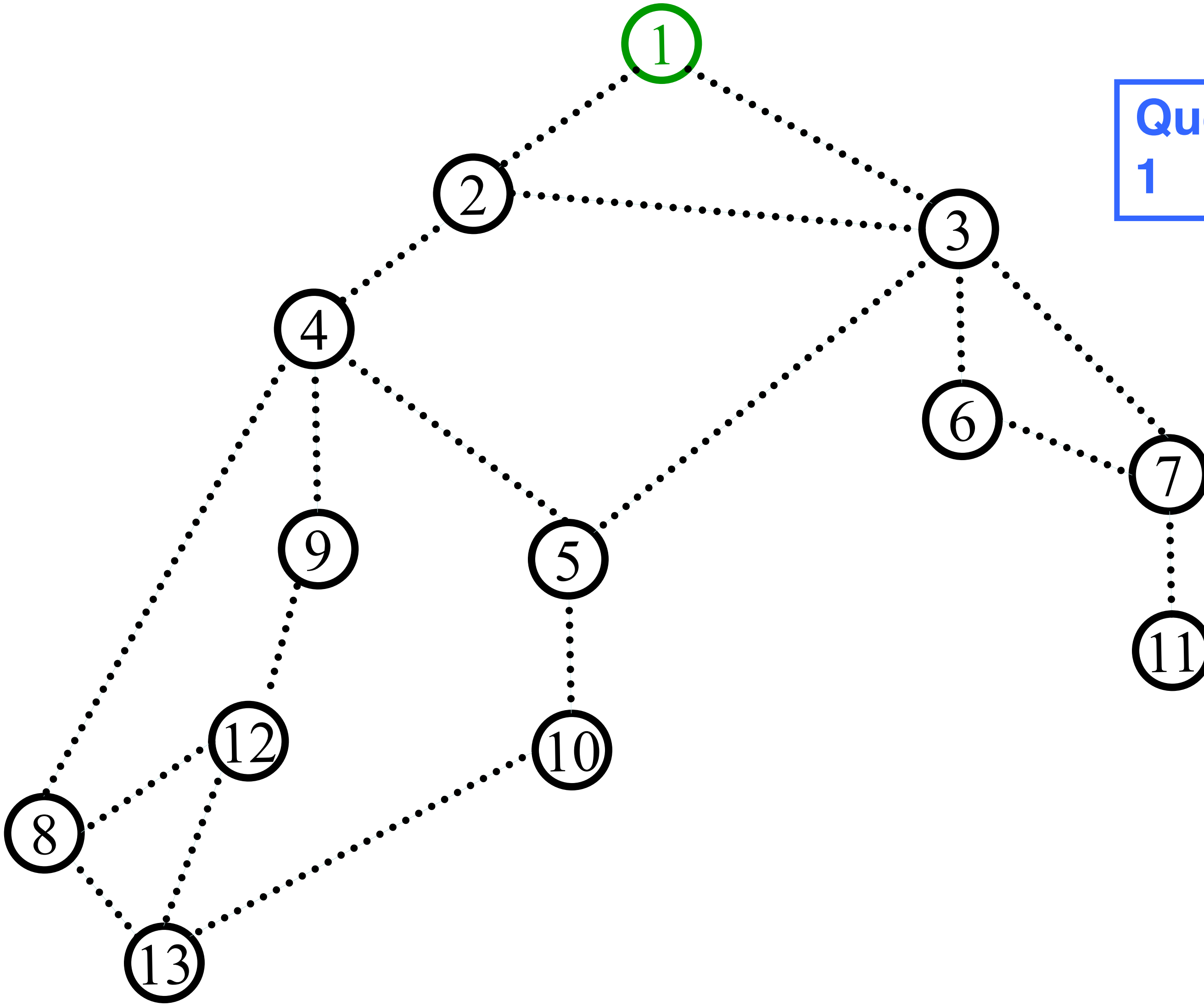
The queue will always have the list of Discovered vertices

BFS implementation

Global initialization: mark all vertices "undiscovered"

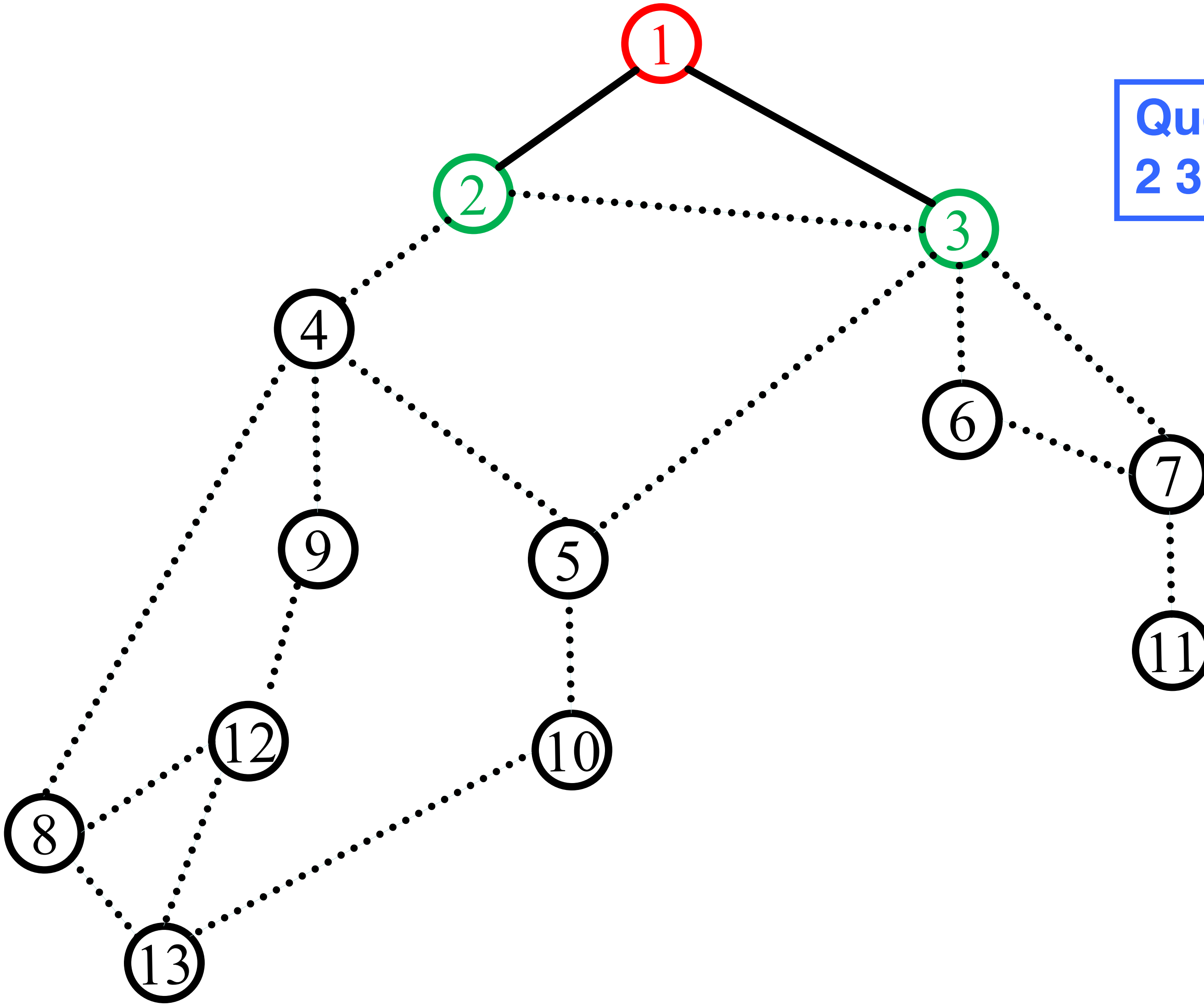
```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully-explored
```

BFS(1)



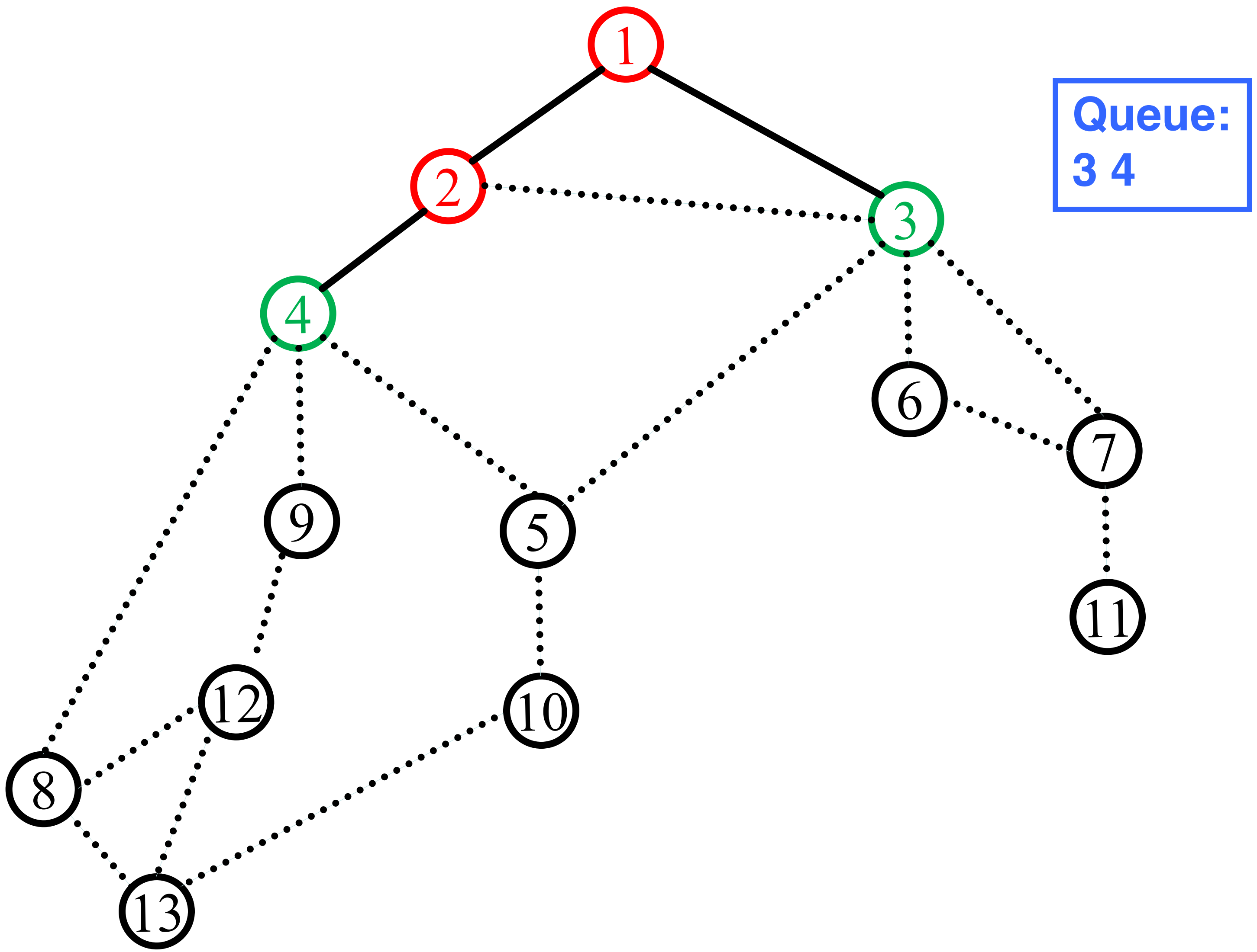
Queue:
1

BFS(1)

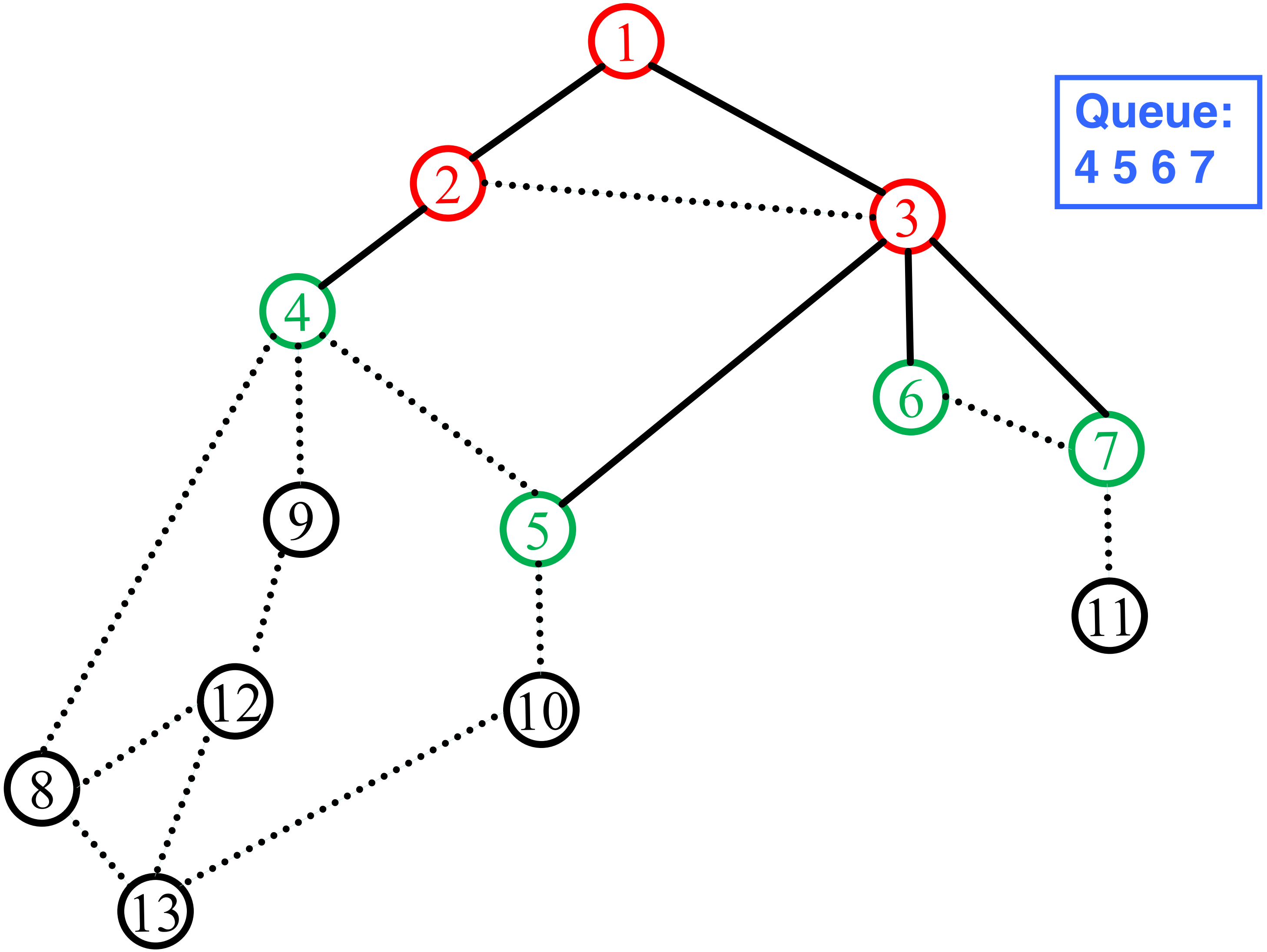


Queue:
2 3

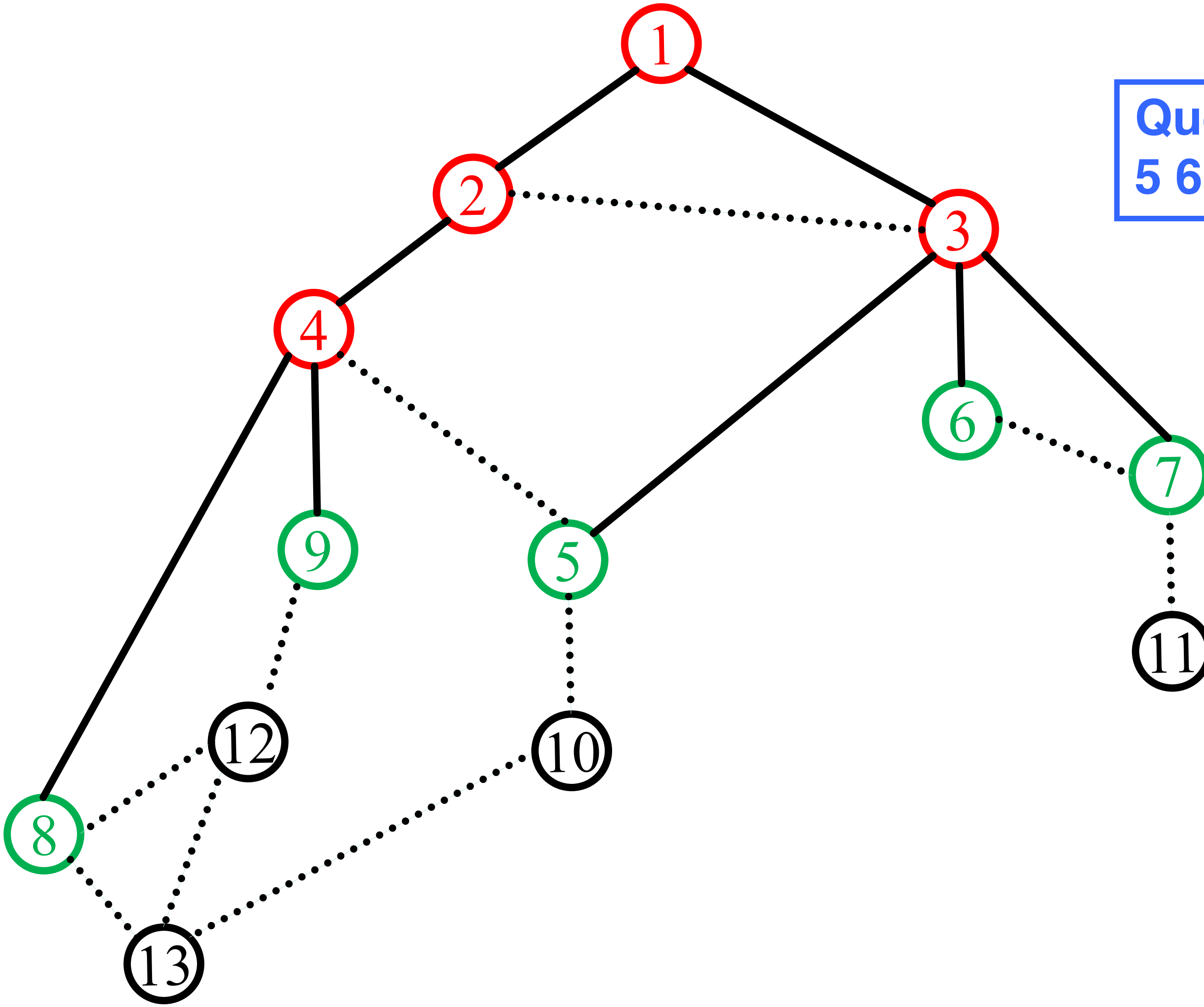
BFS(1)



BFS(1)

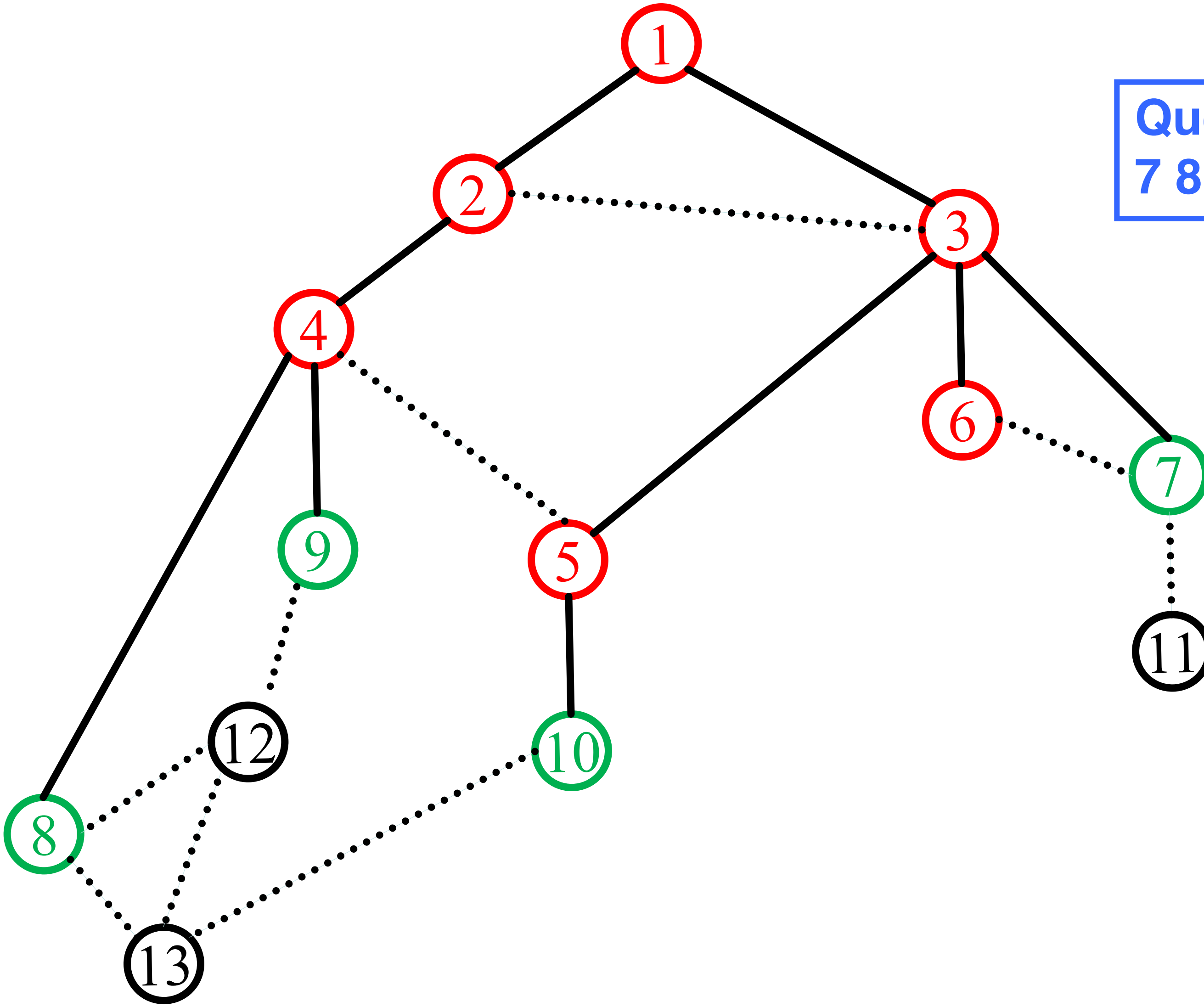


BFS(1)



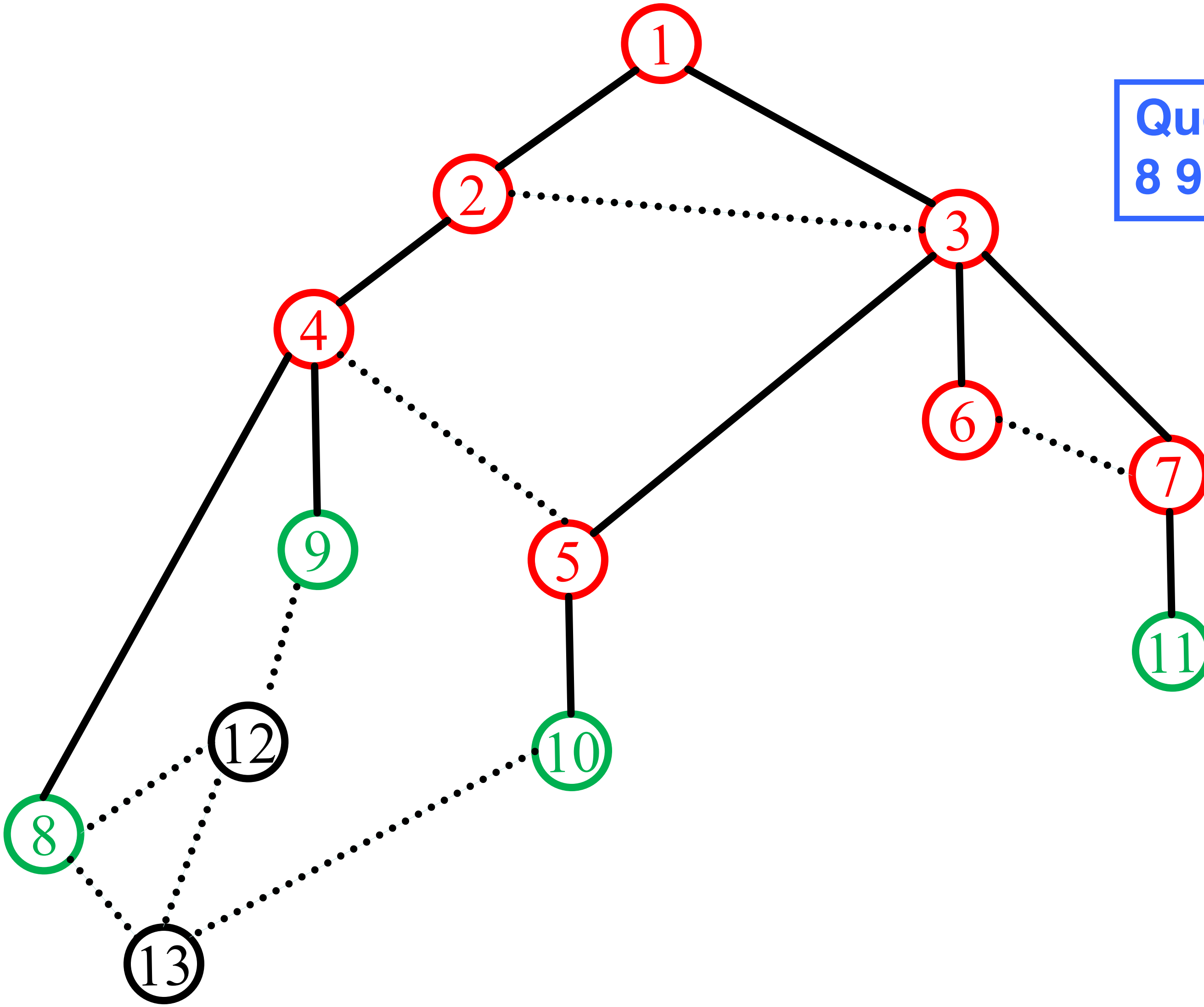
Queue:
5 6 7 8 9

BFS(1)



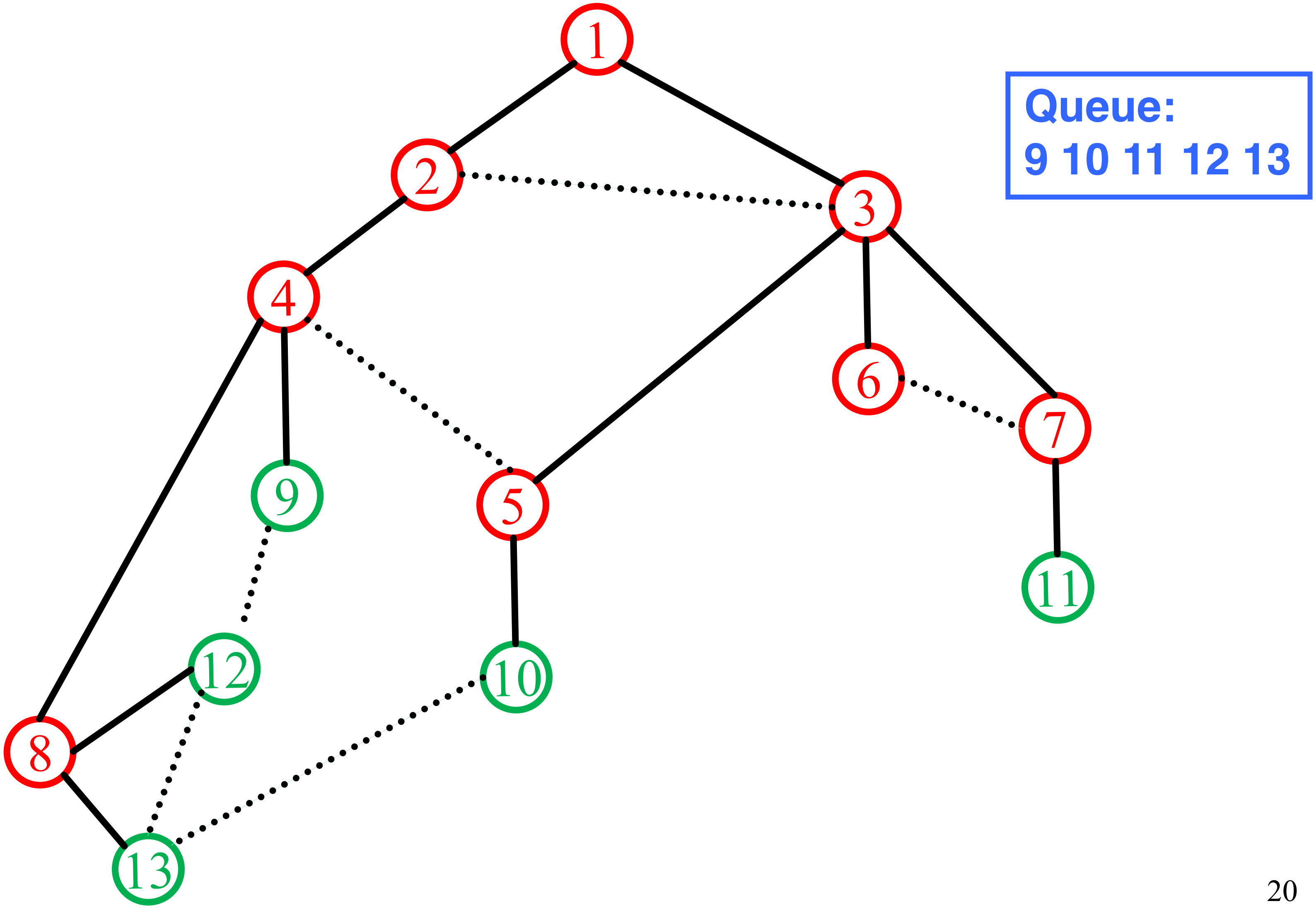
Queue:
7 8 9 10

BFS(1)

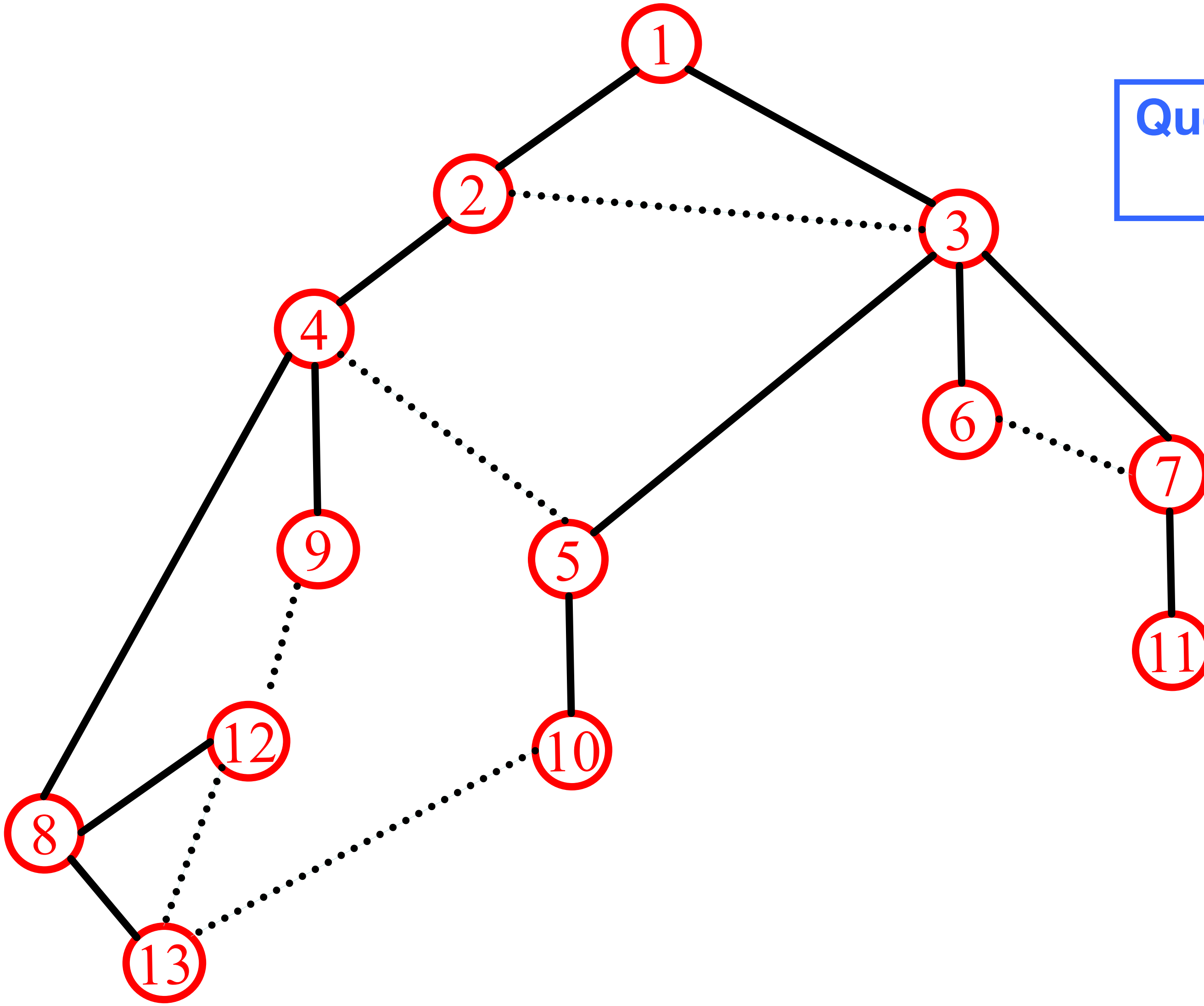


Queue:
8 9 10 11

BFS(1)



BFS(1)



Queue:

BFS Analysis

Global initialization: mark all vertices "undiscovered"

BFS(s)

- mark s discovered
- queue = { s }
- while queue not empty
  - u = remove_first(queue)
  - for each edge {u,x}
    - if (x is undiscovered)
      - mark x discovered
      - append x on queue
  - mark u fully-explored

O(n) times: Once from every vertex if G is connected

$\mathbf{deg}(u) \leq O(n)$ **times**

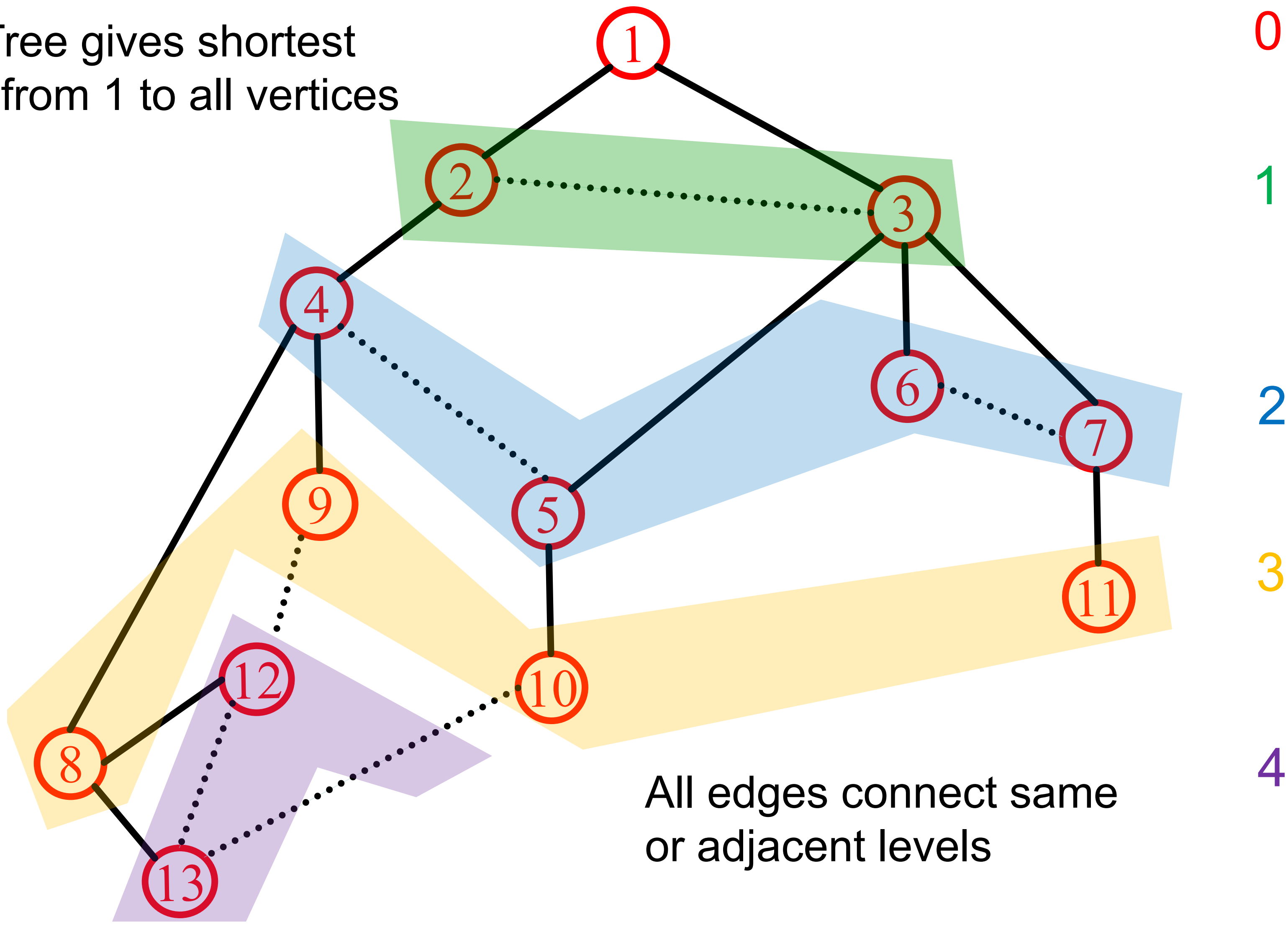
If we use adjacency list: $O(n) + O(\sum_v \deg(v)) = O(m + n)$

Properties of BFS

- BFS(s) visits a vertex v if and only if there is a path from s to v
- Edges into then-undiscovered vertices define a tree – the “Breadth First spanning tree” of G
- Level i in the tree are exactly all vertices v s.t., the shortest path (in G) from the root s to v is of length i
- All nontree edges join vertices on the same or adjacent levels of the tree

BFS Application: Shortest Paths

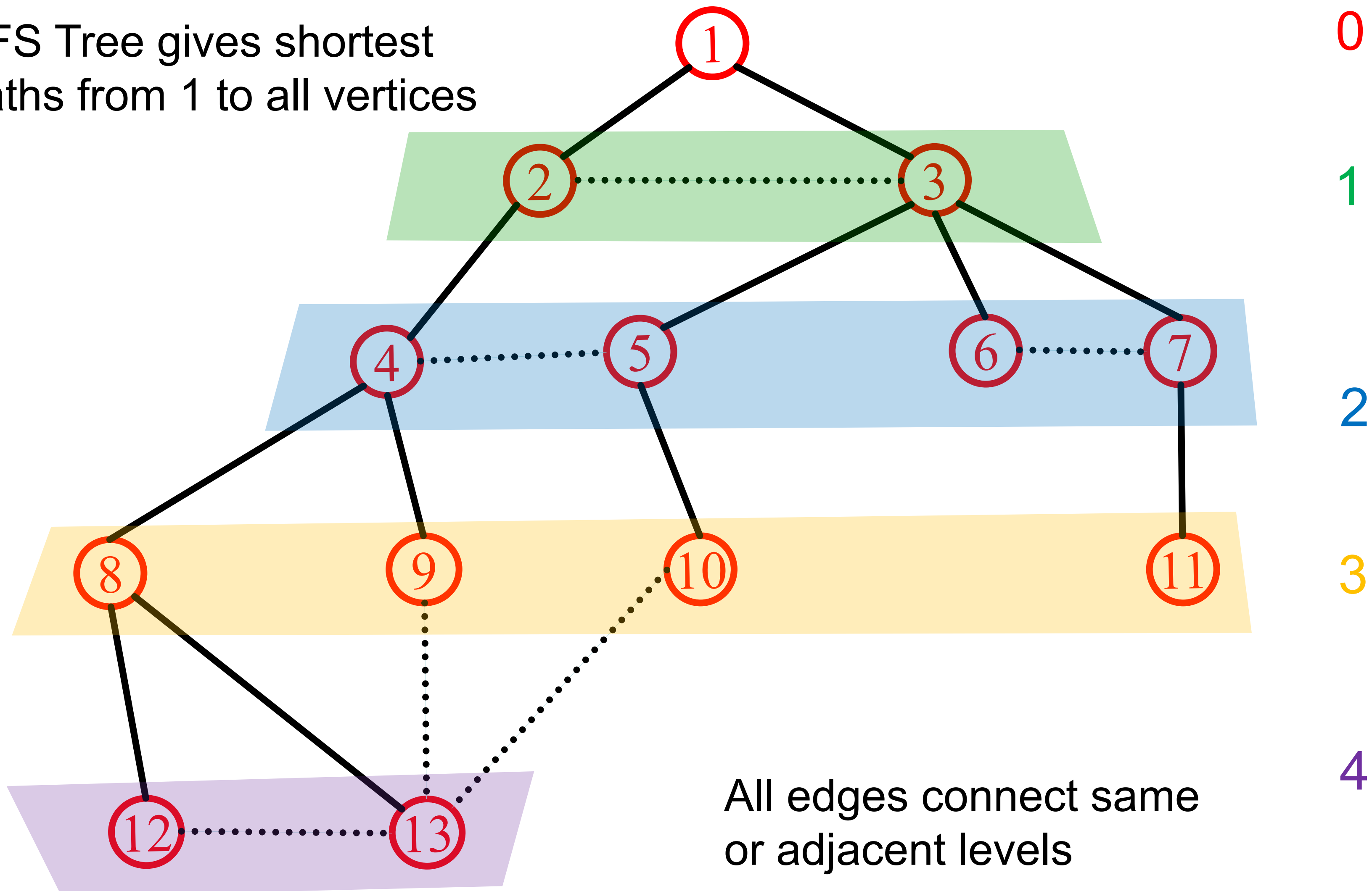
BFS Tree gives shortest paths from 1 to all vertices



All edges connect same or adjacent levels

BFS Application: Shortest Paths

BFS Tree gives shortest paths from 1 to all vertices



All edges connect same or adjacent levels

Properties of BFS

Claim: All nontree edges join vertices on the same or adjacent levels of the tree

Pf: Consider an edge {x,y}

Say x is first discovered and it is added to level i.

We show y will be at level i or $i + 1$

This is because when vertices incident to x are considered in the loop, if y is still undiscovered, it will be discovered and added to level $i + 1$.

Properties of BFS

Lemma: All vertices at level i of BFS(s) have shortest path distance i to s.

Claim: If $L(v) = i$ then shortest path $\leq i$

Pf: Because there is a path of length i from s to v in the BFS tree

Claim: If shortest path $= i$ then $L(v) \leq i$

Pf: If shortest path $= i$, then say $s = v_0, v_1, \ldots, v_i = v$ is the shortest path to v.

By previous claim,

$$L(v_1) \leq L(v_0) + 1$$
$$L(v_2) \leq L(v_1) + 1$$
$$\ldots$$
$$L(v_i) \leq L(v_{i-1}) + 1$$

So, $L(v_i) \leq i$.

This proves the lemma.

Why Trees?

Trees are simpler than graphs

Many statements can be proved on trees by induction

So, computational problems on trees are simpler than general graphs

This is often a good way to approach a graph problem:

- Find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure
- Solve the problem on the tree
- Use the solution on the tree to find a “good” solution on the graph

Graph Search App: Connected Comp

We want to answer the following type questions (fast):
Given vertices u,v is there a path from u to v in G?

Idea: Create an array A such that
For all u, A[u] is the label of the connected component that contains u

Therefore, question reduces to

If A[u] = A[v]?

Connected Components Implementation

Initial State: All vertices undiscovered, c $\leftarrow$0

for v = 1 to n do

If state(v) != fully-explored then

BFS(v): setting A[u] $\leftarrow$c for each u found (and marking u discovered/fully-explored)

c$\leftarrow$c+1

Note: We no longer initialize to undiscovered in the BFS subroutine

Total Cost: O(m+n)

In every connected component with n_i vertices and m_i edges BFS takes time $O(m_i + n_i)$.

Connected Components

Lesson: We can execute any algorithm on disconnected graphs by running it on each connected component.

We can use the previous algorithm to detect connected components.

There is no overhead, because the algorithm runs in time O(m+n).

So, from now on, we can (almost) always assume the input graph is connected.