# CSE 421: Introduction to Algorithms

## Greedy Algorithms

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# HW1 Grade

Q: I received low grade in HW1 what should I do?

- Understand what was your mistake. Did you understand the problem statement correctly?
- Show up to office hours and ask for hints or to explain your solution
- Review materials of 311 on proofs/induction
- Do exercises from the book/Problem Solving Sessions

Q: My HW1 grade is low, will I be able to receive 4.0?

- Yes! I look at your progress. Many students are behind at beginning but by practice they catch up and receive 4.0

Q: I have filled out a regrade request, but was not convinced, what should I do?

- Show up to my office hour and discuss your solution

# DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

Pf. (by contradiction)

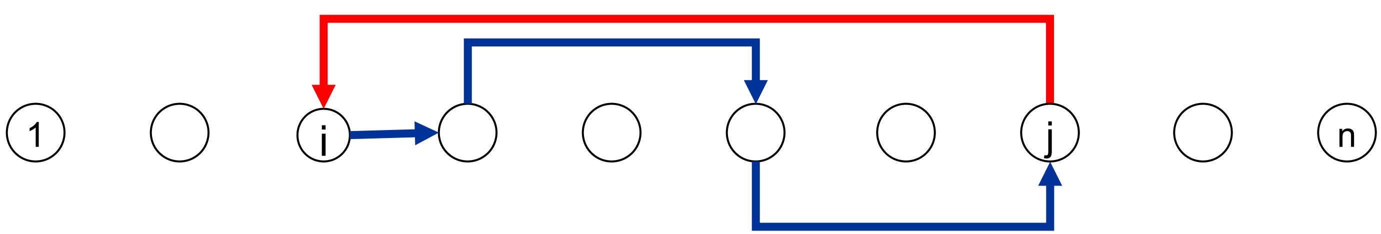
Suppose that G has a topological order $1,2,\ldots,n$ and that G also has a directed cycle C.

Let $i$ be the lowest-indexed node in C, and let $j$ be the node just before $i$; thus $(j,i)$ is an (directed) edge.

By our choice of $i$, we have $i < j$.

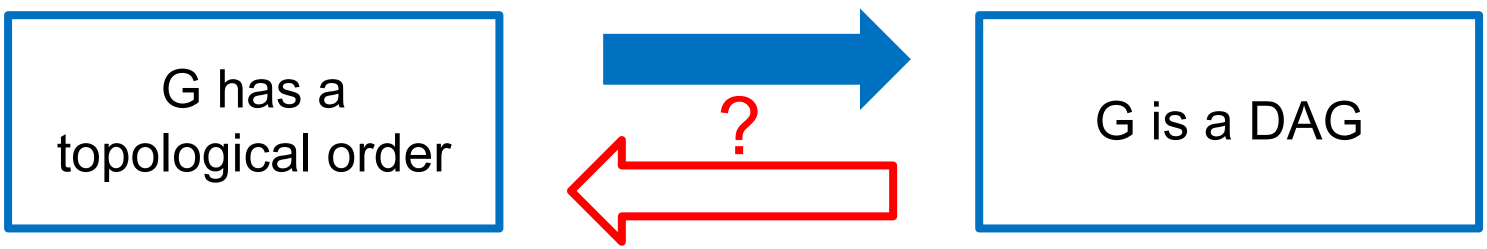
On the other hand, since $(j,i)$ is an edge and $1,\ldots,n$ is a topological order, we must have $j < i$, a contradiction

the directed cycle C

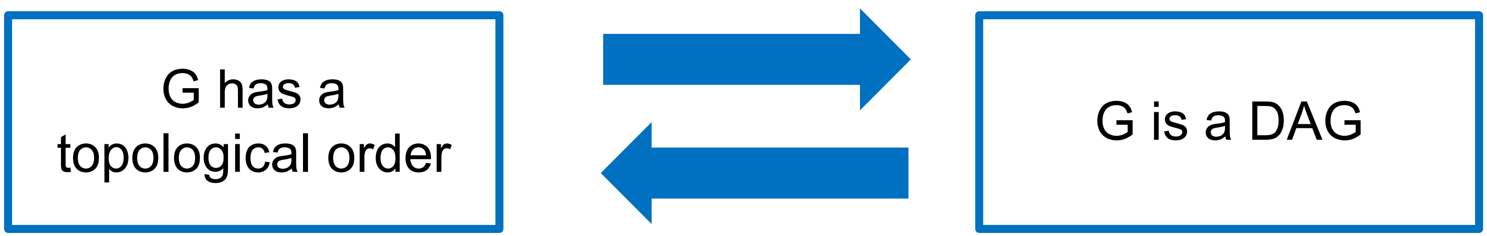



the supposed topological order: 1,2,…,n

# DAGs: A Sufficient Condition



# A Characterization of DAGs



# Every DAG has a source node

Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

Pf. (by contradiction)

Suppose that G is a DAG and and it has no source

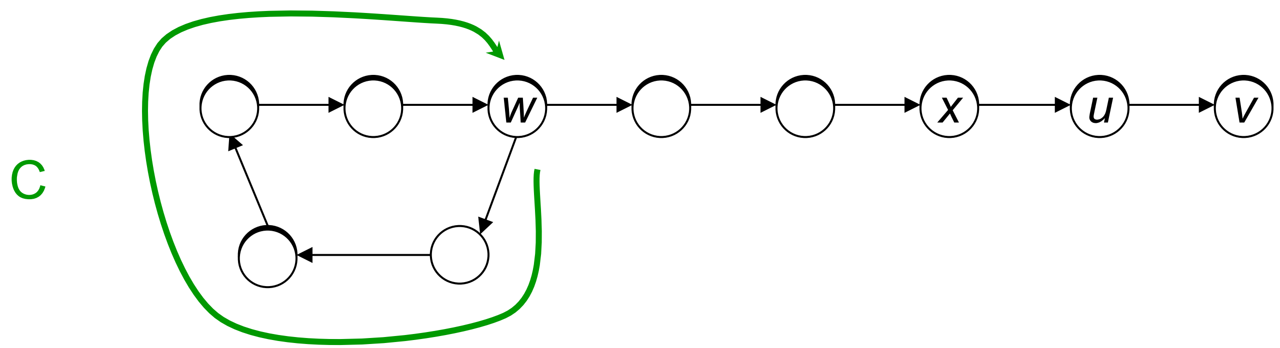
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Is this similar to a previous proof?

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



# DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with $n > 1$ nodes, find a source node v.
$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

Reminder: Always remove vertices/edges to use IH

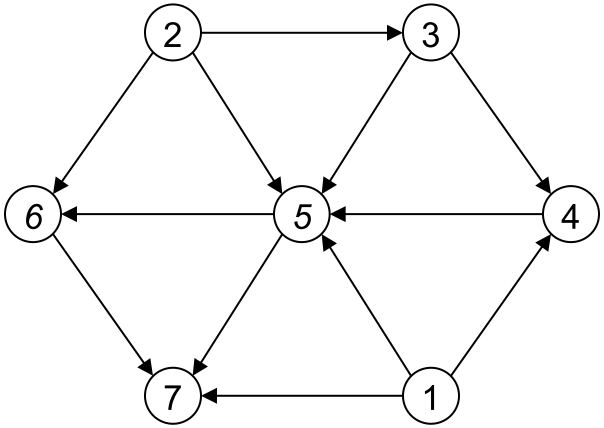
By IH, $G - \{ v \}$ has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges.

# A Characterization of DAGs

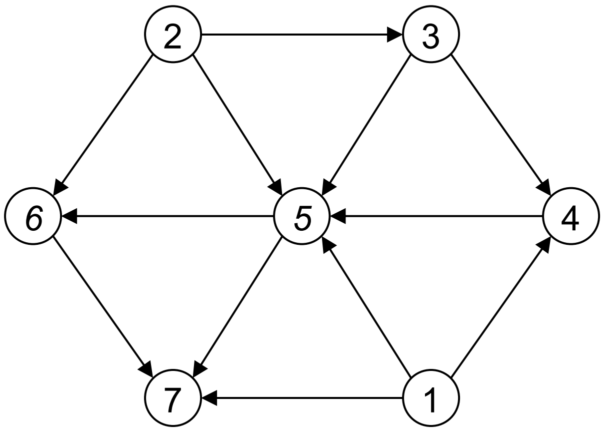
G has a topological order

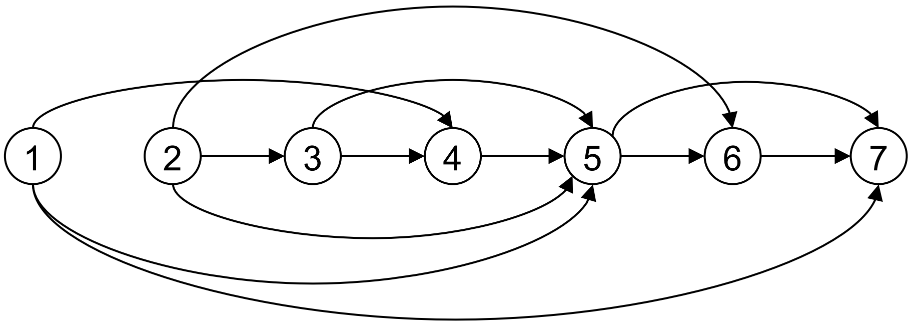
G is a DAG

# Topological Order Algorithm: Example



# Topological Order Algorithm: Example




Topological order: 1, 2, 3, 4, 5, 6, 7

# Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w) — O(m + n)

S = S ∪ {w} for all w with count[w]=0

Main loop:

while S not empty

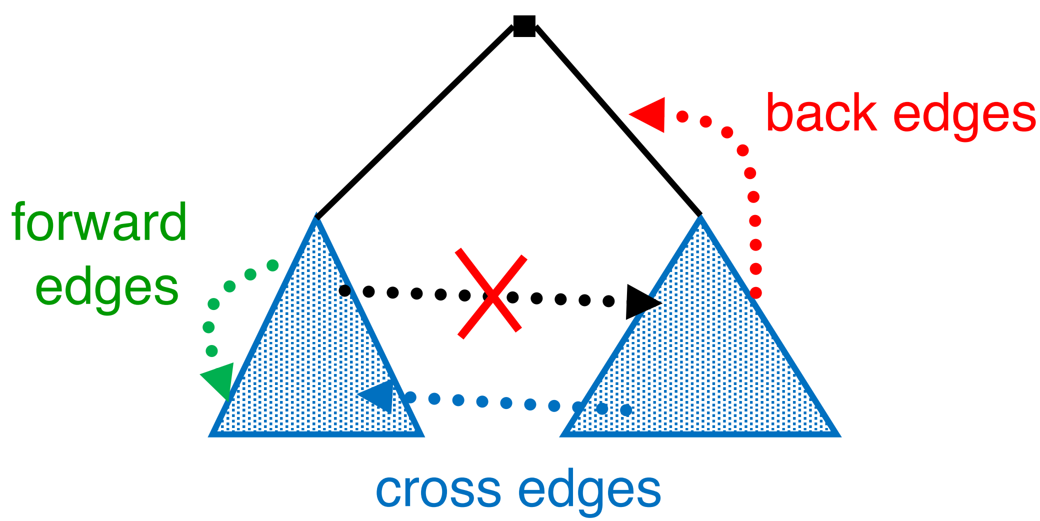
- remove some v from S
- make v next in topo order — O(1) per node
- for all edges from v to some w — O(1) per edge
  - decrement count[w]
  - add w to S if count[w] hits 0

Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)

# DFS on Directed Graphs

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree



# Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: $m = O(n^2)$, often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort

# Greedy Algorithms

# Greedy Strategy

Goal: Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Cashier's algorithm: At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: \$2.89.

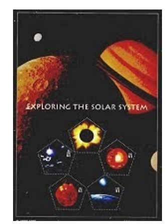
# Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.




Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.

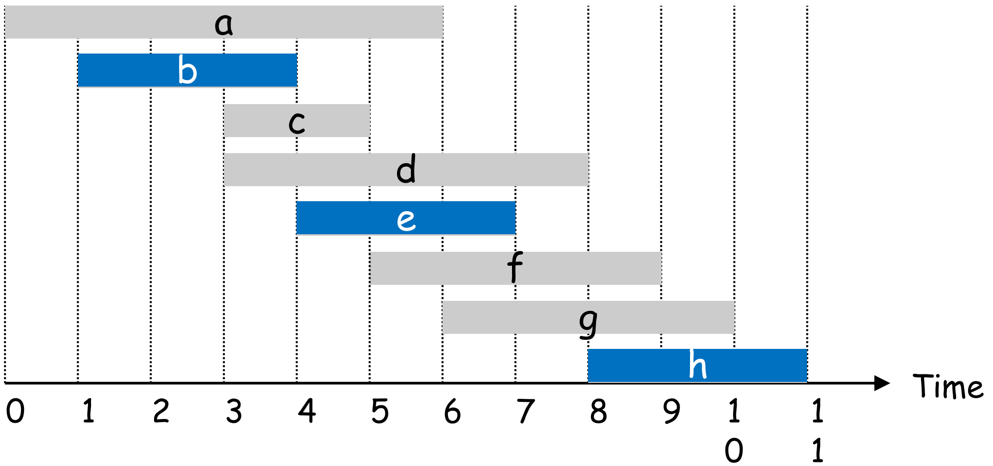
# Greedy Algorithms Outline

Pros
- Intuitive
- Often simple to design (and to implement)
- Often fast

Cons
- Often incorrect!

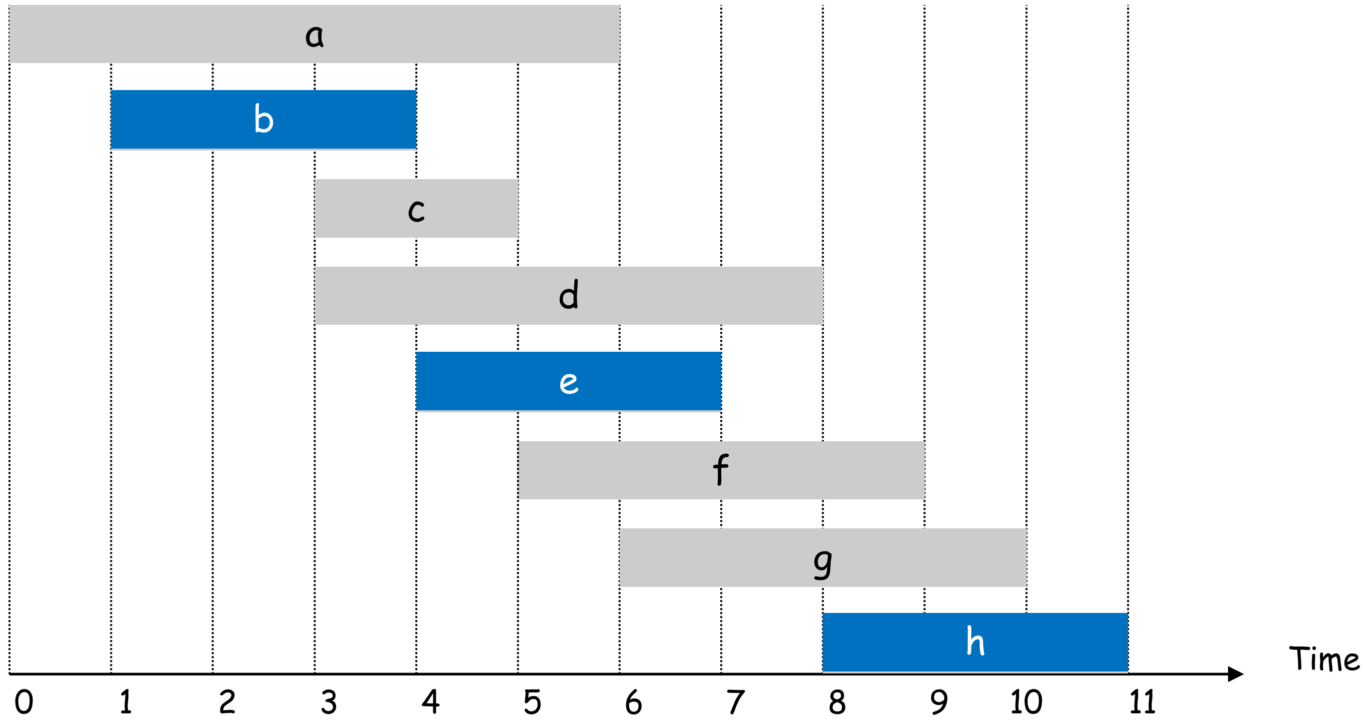
Proof techniques:
- Stay ahead
- Structural
- Exchange arguments

# Interval Scheduling



# Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



# Greedy Strategy

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

Main question:

- What order?
- Does it give the Optimum answer?
- Why?

# Possible Approaches for Inter Sched

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

[Earliest start time]  Consider jobs in ascending order of start time $s_j$.

[Earliest finish time]  Consider jobs in ascending order of finish time $f_j$.

[Shortest interval]  Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts]  For each job, count the number of conflicting jobs $c_j$. Schedule in ascending order of conflicts $c_j$.

# Greedy Alg: Earliest Finish Time

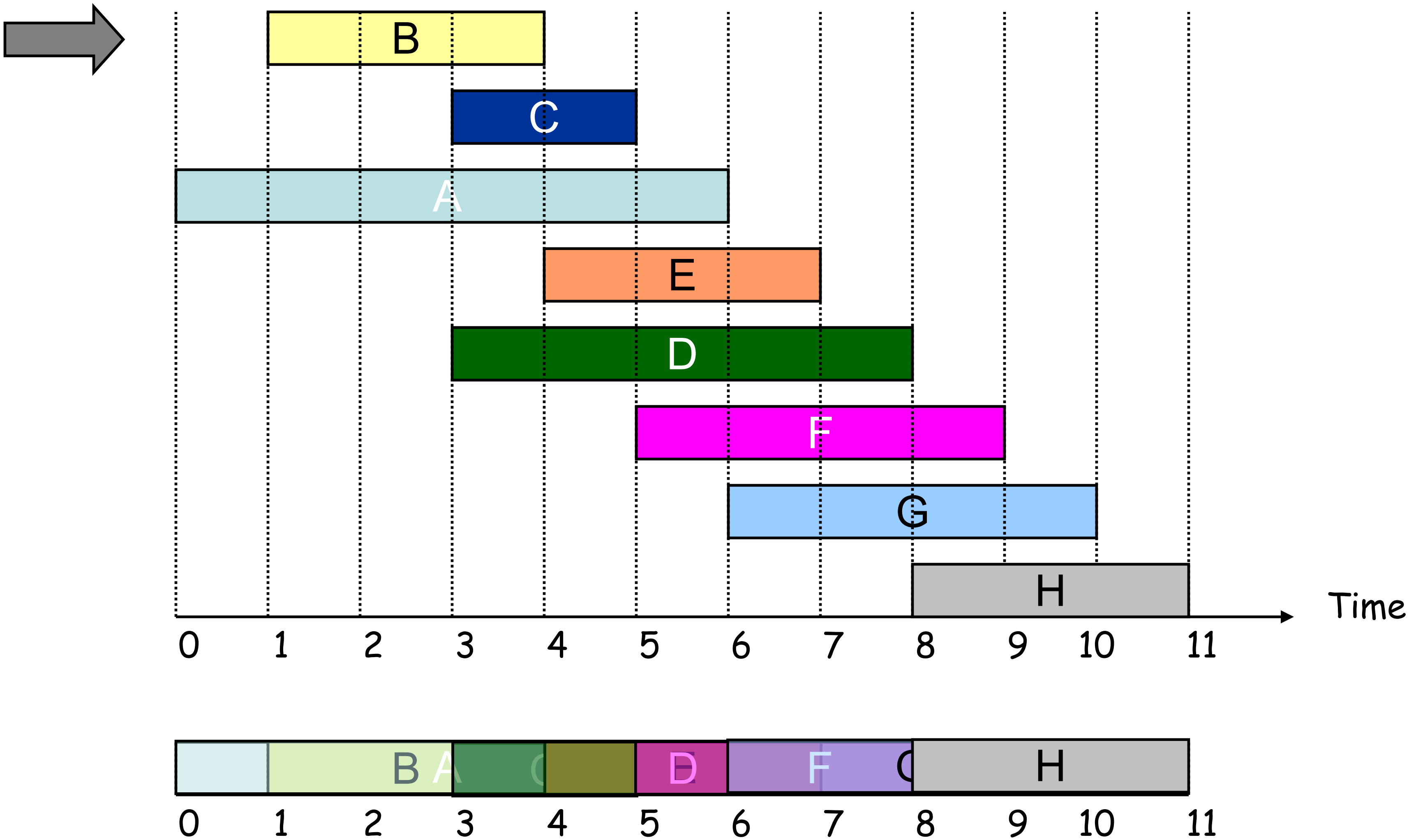
Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f(1) ≤ f(2) ≤ ... ≤ f(n).
A ← ∅
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

- Remember job j* that was added last to A.
- Job j is compatible with A if $s(\mathrm{j}) \geq f(j^*)$.

# Greedy Alg: Example



# Correctness

Theorem: Greedy algorithm is optimal.

Pf: (technique: "Greedy stays ahead")

Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution in order.

We show $f(i_r) \leq f(j_r)$ for all r, by induction on r.

Base Case: $i_1$ chosen to have min finish time, so $f(i_1) \leq f(j_1)$.

IH: $f(i_r) \leq f(j_r)$ for some r

IS: Since $f(i_r) \leq f(j_r) \leq s(j_{r+1})$, $j_{r+1}$ is among the candidates considered by greedy when it picked $i_{r+1}$, & it picks min finish, so $f(i_{r+1}) \leq f(j_{r+1})$

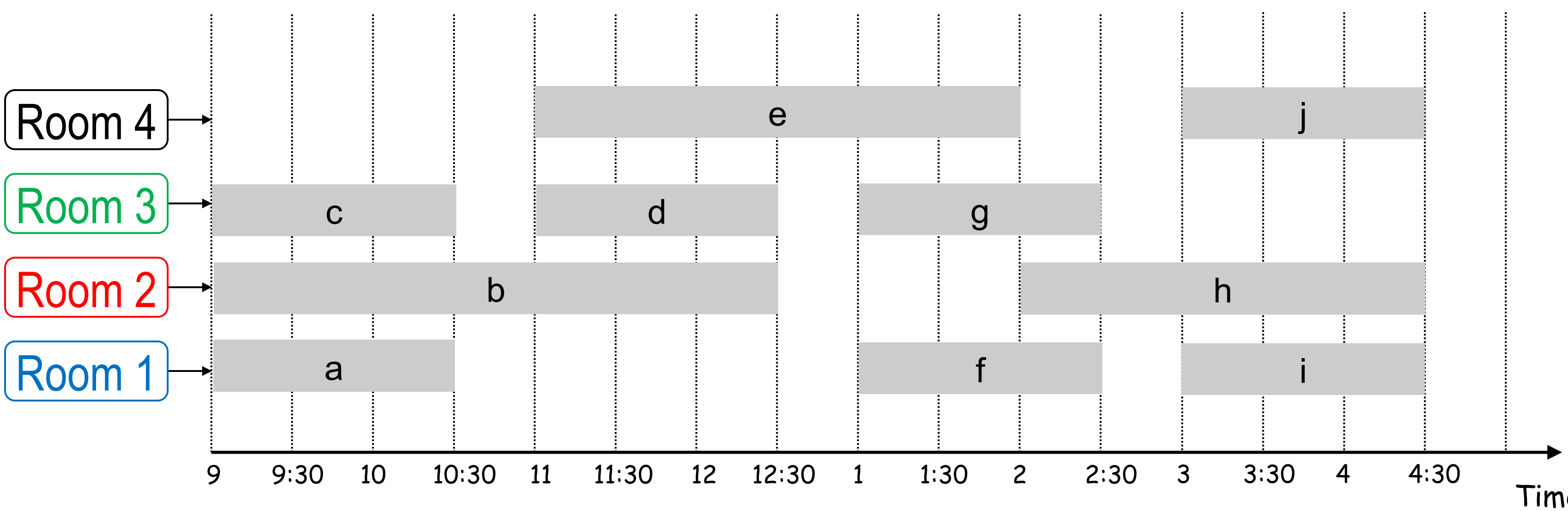
Observe that we must have $k \geq m$, else $j_{k+1}$ is among (nonempty) set of candidates for $i_{k+1}$

# Interval Partitioning
# Technique: Structural

# Interval Partitioning

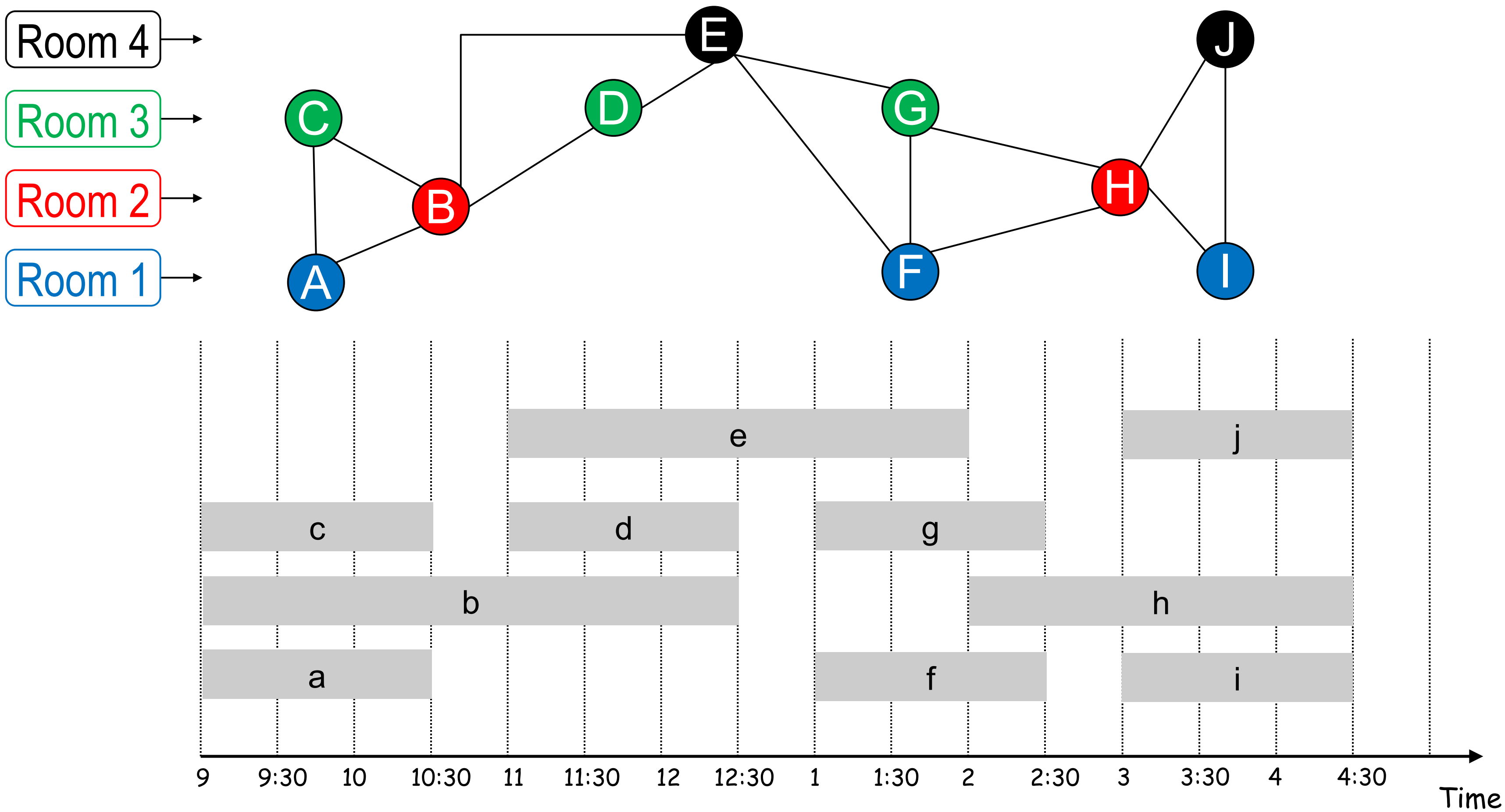
Lecture j starts at s(j) and finishes at f(j).

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.



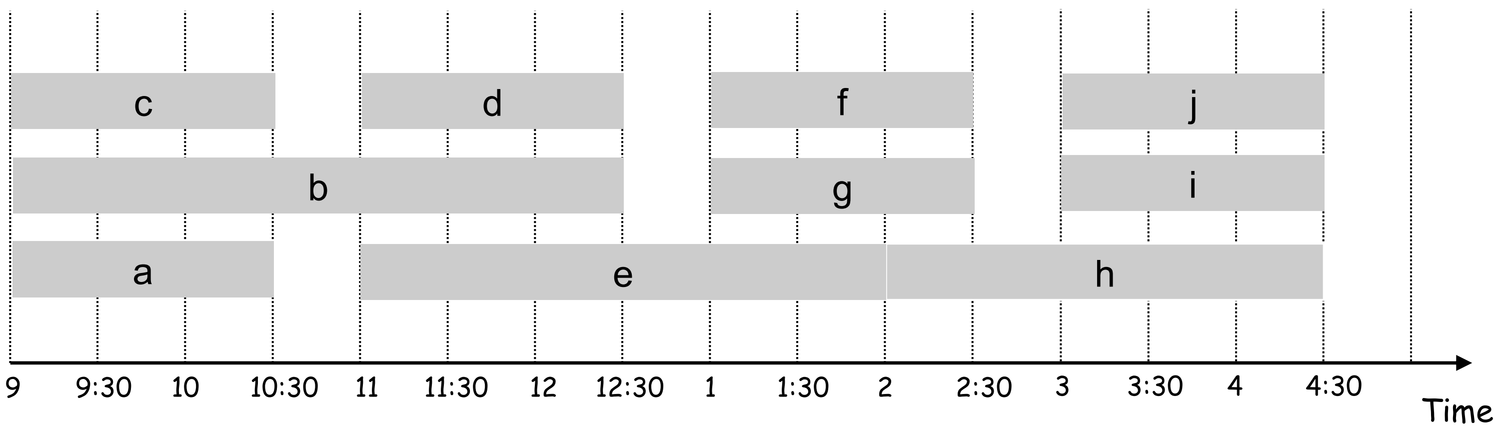
# Interval Partitioning

Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are simpler.



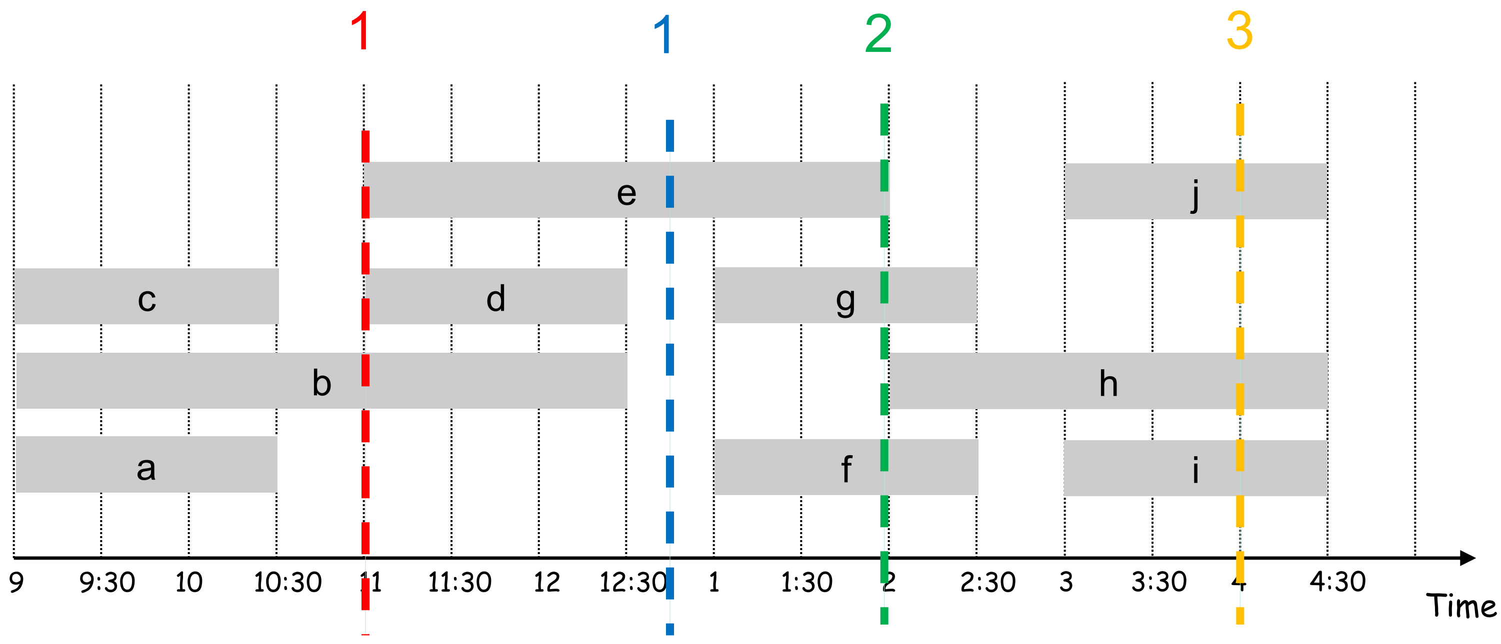
# A Better Schedule

This one uses only 3 classrooms



# A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.



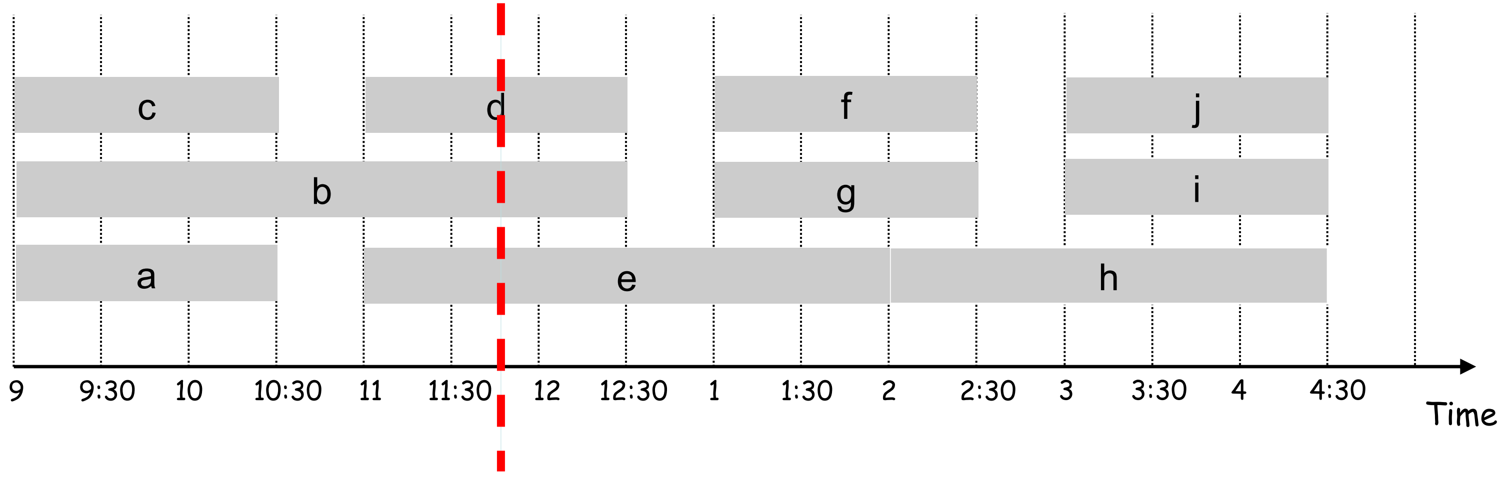
# A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

Q. Does there always exist a schedule equal to depth of intervals?



# A Greedy Algorithm

Greedy algorithm: Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s₁ ≤ s₂ ≤ ... ≤ sₙ.
d ← 0

for j = 1 to n {
   if (lect j is compatible with some classroom k, 1 ≤ k ≤ d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation: Exercise!

# Correctness

Observation: Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem: Greedy algorithm is optimal.

Pf (exploit structural property).

Let d = number of classrooms that the greedy algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 previously used classrooms.

Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s(j).

Thus, we have d lectures overlapping at time $s(j) + \epsilon$, i.e. depth $\geq$ d

"OPT Observation" $\Rightarrow$ all schedules use $\geq$ depth classrooms, so d = depth and greedy is optimal ▪