CSE 421

Greedy Algorithms

Shayan Oveis Gharan

An Advice on Problem Solving

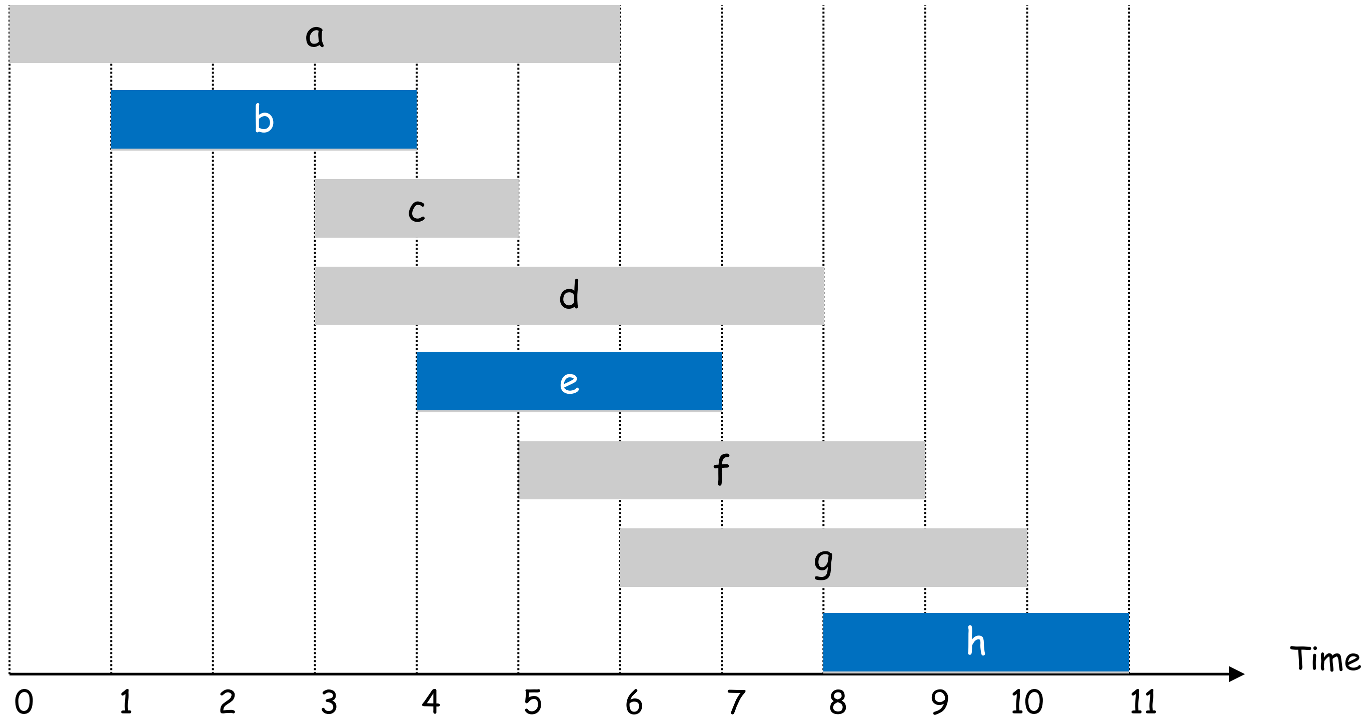
If possible, try not to use arguments of the following type in proofs:

- The Best case is ….
- The worst case is ….

These arguments need rigorous justification, and they are usually the main reason that your proofs can become wrong, or unjustified.

Interval Scheduling

- Job j starts at $s(j)$ and finishes at $f(j)$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Possible Approaches for Inter Sched

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

[Earliest start time] Consider jobs in ascending order of start time s_j.

[Earliest finish time] Consider jobs in ascending order of finish time f_j.

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Greedy Alg: Earliest Finish Time

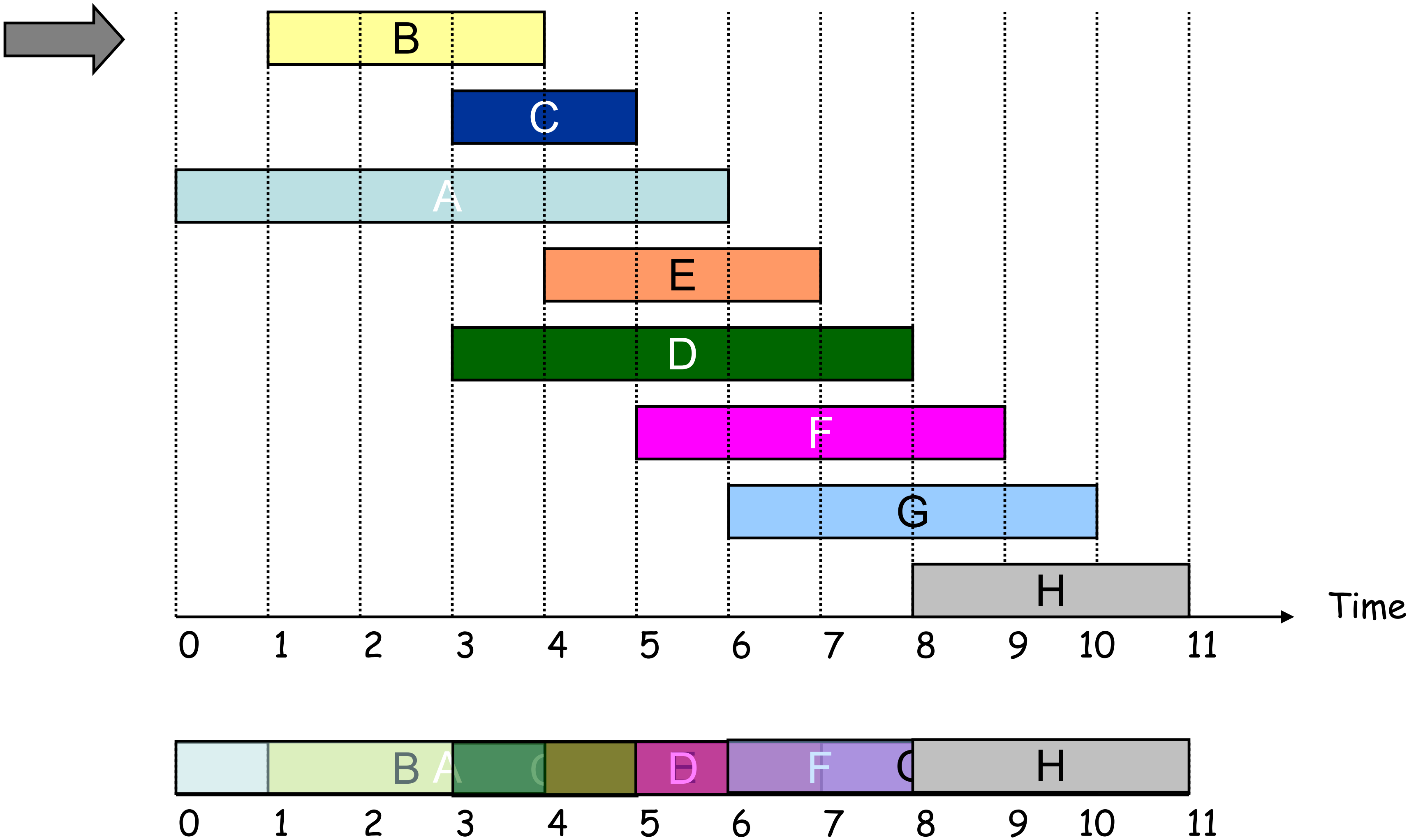
Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f(1) ≤ f(2) ≤ ... ≤ f(n).
A ← ∅
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

- Remember job j* that was added last to A.
- Job j is compatible with A if $s(\mathrm{j}) \geq f(j^*)$.

Greedy Alg: Example



Correctness

Theorem: Greedy algorithm is optimal.

Pf: (technique: "Greedy stays ahead")

Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution in order.

We show $f(i_r) \leq f(j_r)$ for all r, by induction on r.

Base Case: i_1 chosen to have min finish time, so $f(i_1) \leq f(j_1)$.

IH: $f(i_r) \leq f(j_r)$ for some r

IS: Since $f(i_r) \leq f(j_r) \leq s(j_{r+1})$, j_{r+1} is among the candidates considered by greedy when it picked i_{r+1}, & it picks min finish, so $f(i_{r+1}) \leq f(j_{r+1})$

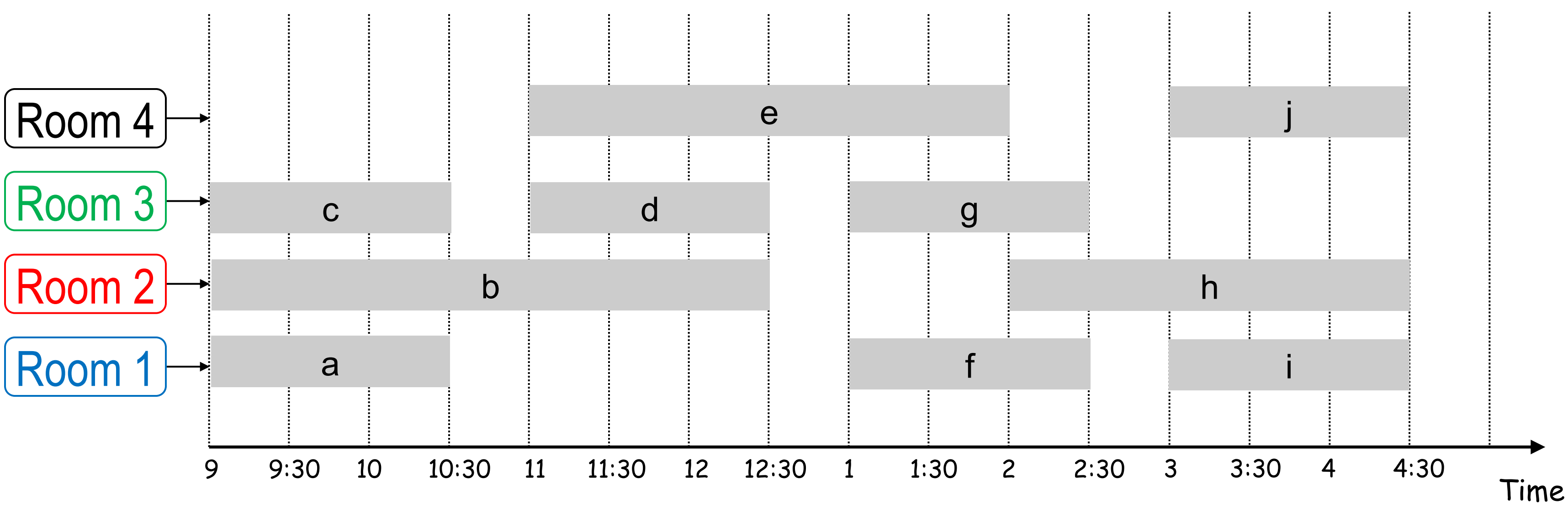
Observe that we must have $k \geq m$, else j_{k+1} is among (nonempty) set of candidates for i_{k+1}

Interval Partitioning
Technique: Structural

Interval Partitioning

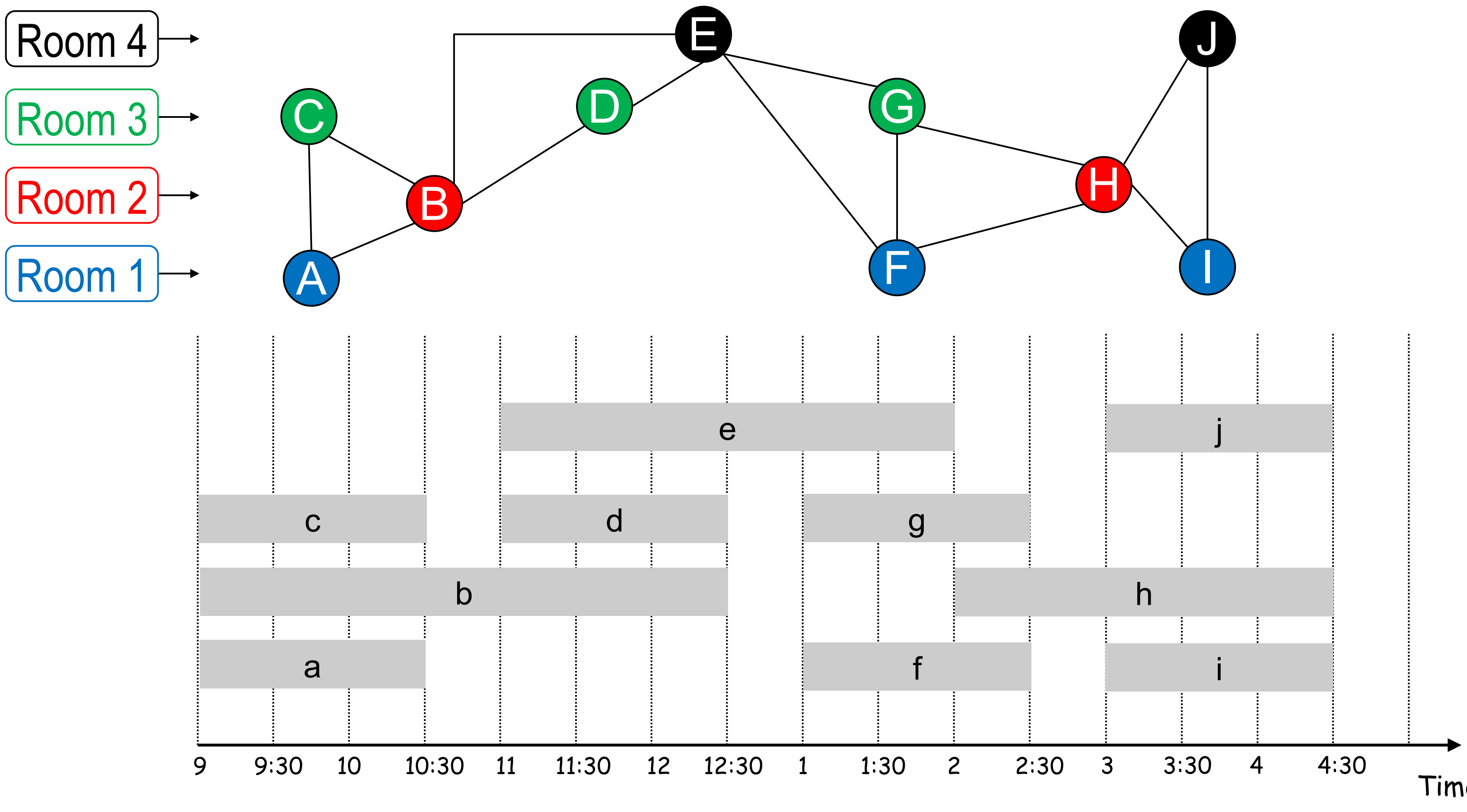
Lecture j starts at s(j) and finishes at f(j).

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.



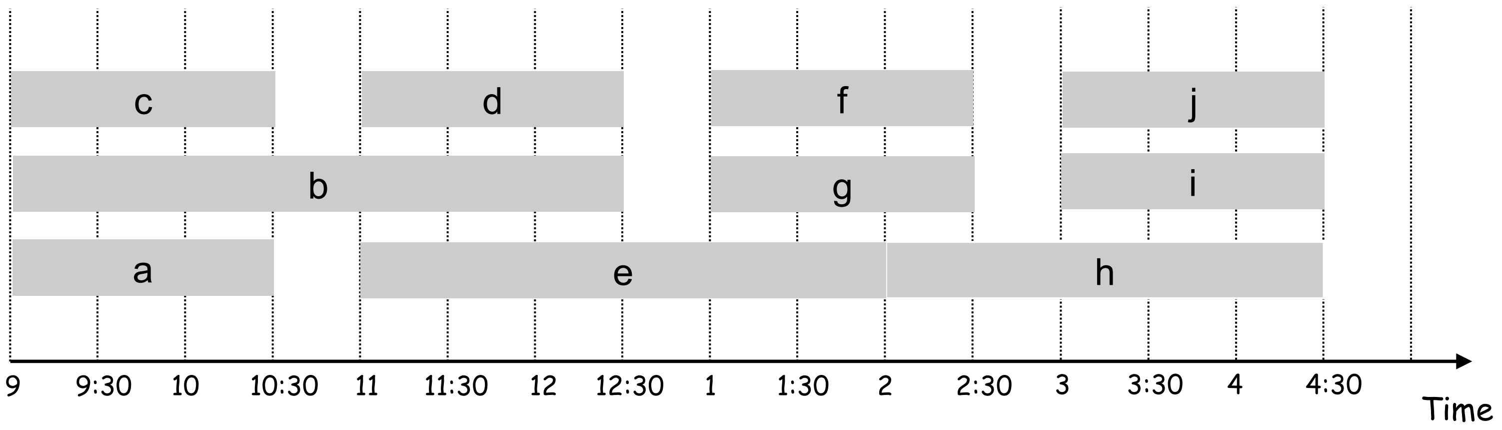
Interval Partitioning

Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are simpler.



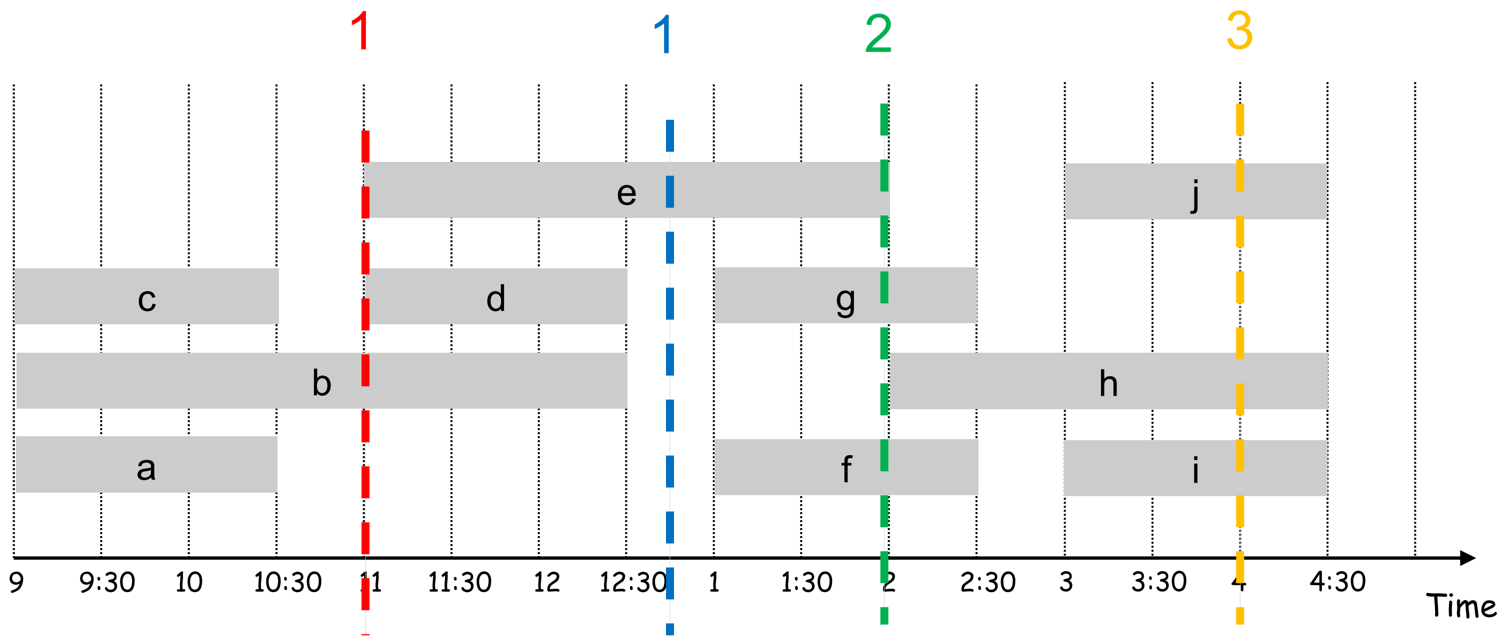
A Better Schedule

This one uses only 3 classrooms



A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.



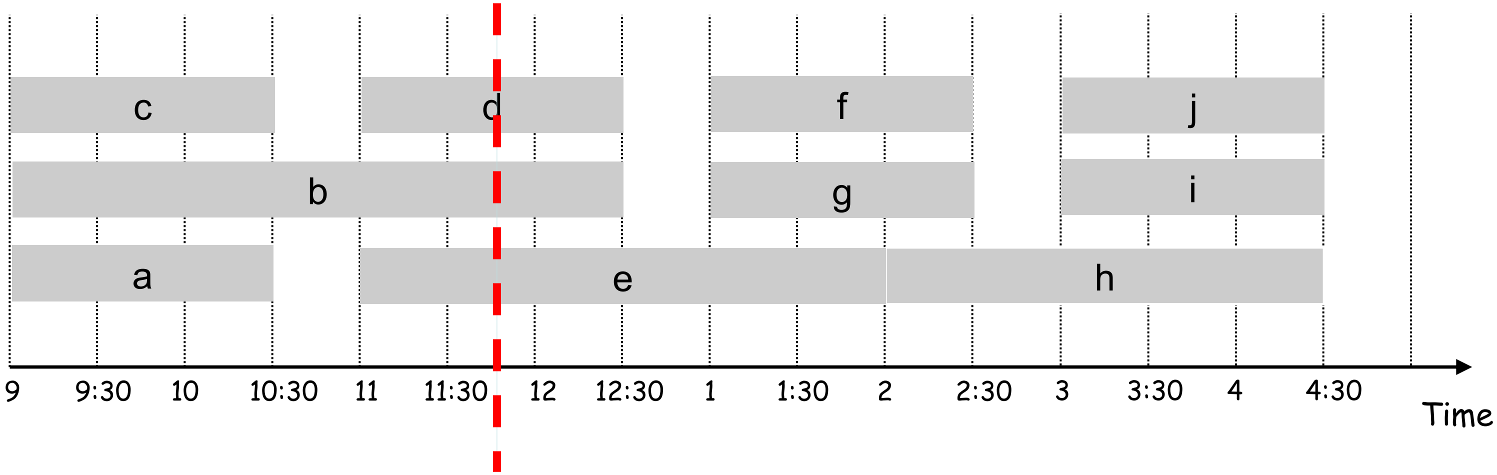
A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

Q. Does there always exist a schedule equal to depth of intervals?



A Greedy Algorithm

Greedy algorithm: Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0

for j = 1 to n {
   if (lect j is compatible with some classroom k, 1 ≤ k ≤ d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation: Exercise!

Correctness

Observation: Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem: Greedy algorithm is optimal.

Pf (exploit structural property).

Let d = number of classrooms that the greedy algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 previously used classrooms.

Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s(j).

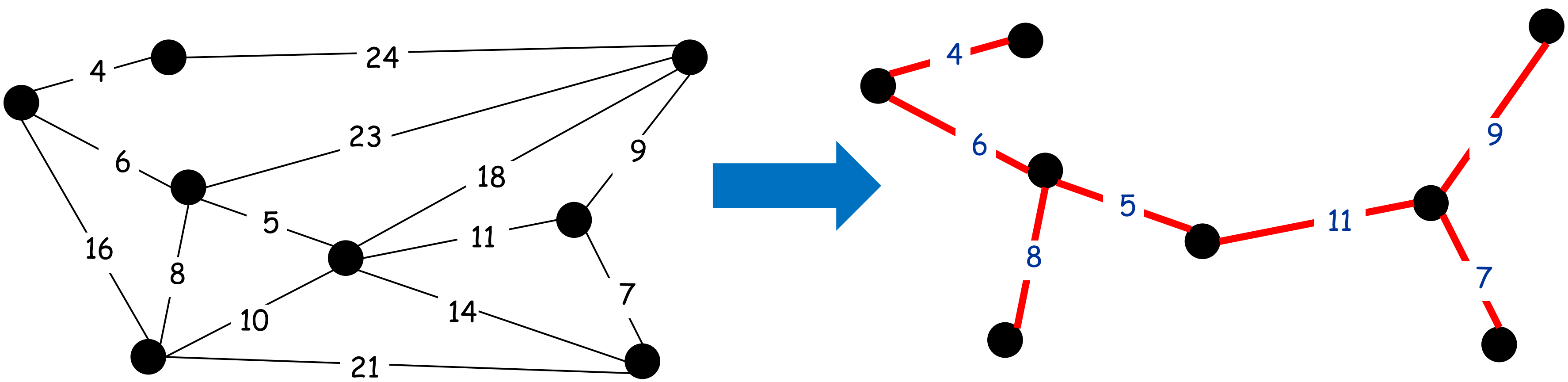
Thus, we have d lectures overlapping at time $s(j) + \epsilon$, i.e. depth $\geq$ d

"OPT Observation" $\Rightarrow$ all schedules use $\geq$ depth classrooms, so d = depth and greedy is optimal ▪

Minimum Spanning Tree Problem

Minimum Spanning Tree (MST)

Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



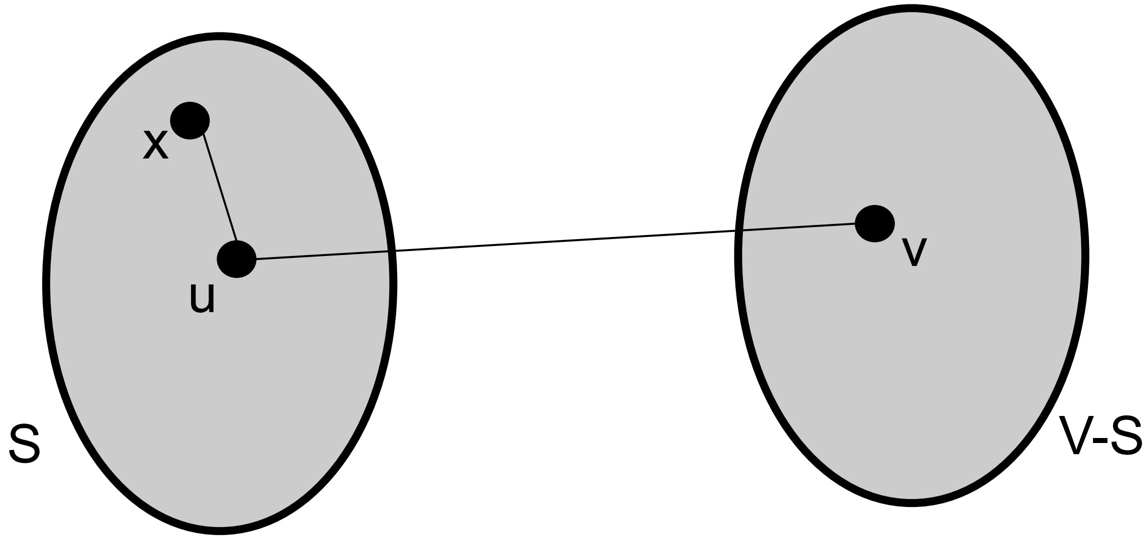
$$G = (V, E)$$

$$c(T) = \sum_{e \in T} c_e = 50$$

Cuts

In a graph $G = (V, E)$ a cut is a bipartition of V into sets $S, V - S$ for some $S \subseteq V$. We show it by $(S, V - S)$

An edge $e = \{u, v\}$ is in the cut $(S, V - S)$ if exactly one of u,v is in S.

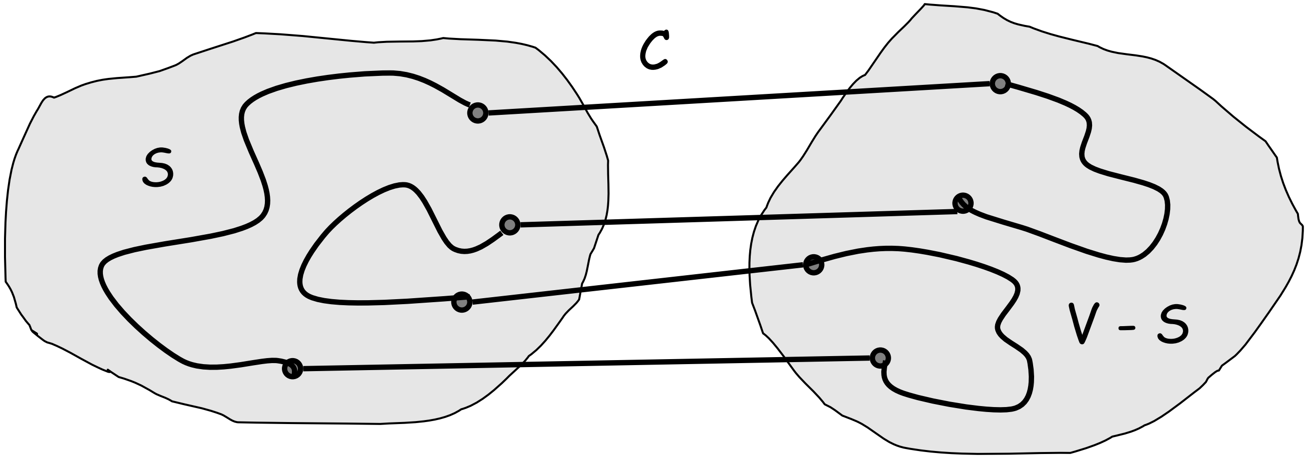


Obs: If G is connected then there is at least one edge in every cut.

Cycles and Cuts

Claim. A cycle crosses a cut (from S to V-S) an even number of times.

Pf. (by picture)

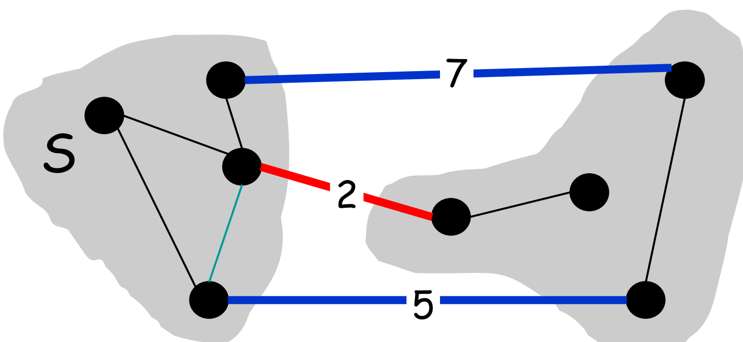


Properties of the OPT

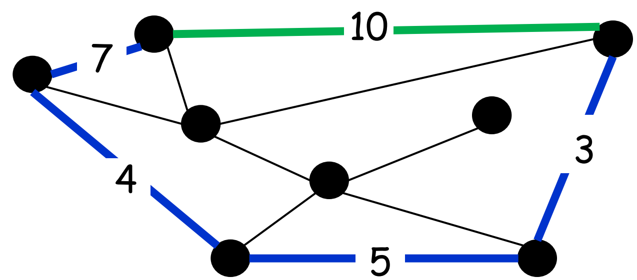
Simplifying assumption: All edge costs c_e are distinct.

Cut property: Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



red edge is in the MST



Green edge is not in the MST

Cut Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then T* contains e.

Pf. By contradiction

Suppose e = {u,v} does not belong to T*.

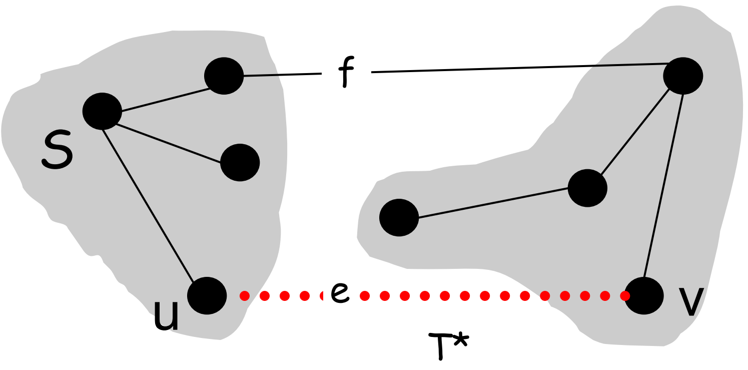
Adding e to T* creates a cycle C in T*.

C crosses S even number of times $\Rightarrow$ there exists another edge, say f, that leaves S.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Cycle Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (By contradiction)

Suppose f belongs to T*.

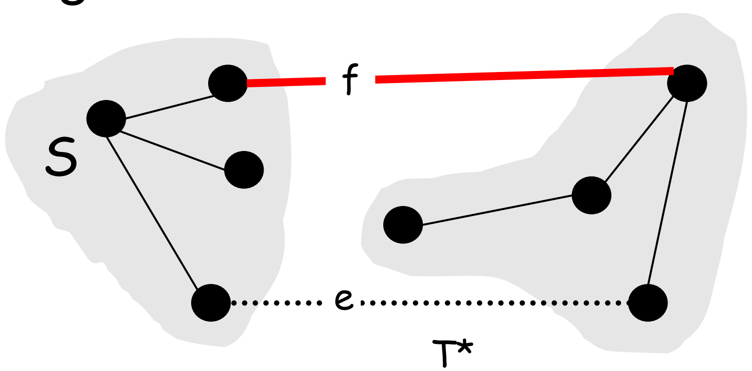
Deleting f from T* cuts T* into two connected components.

There exists another edge, say e, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Kruskal's Algorithm [1956]

```
Kruskal(G, c) {
   Sort edges weights so that c₁ ≤ c₂ ≤ ... ≤ cₘ.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

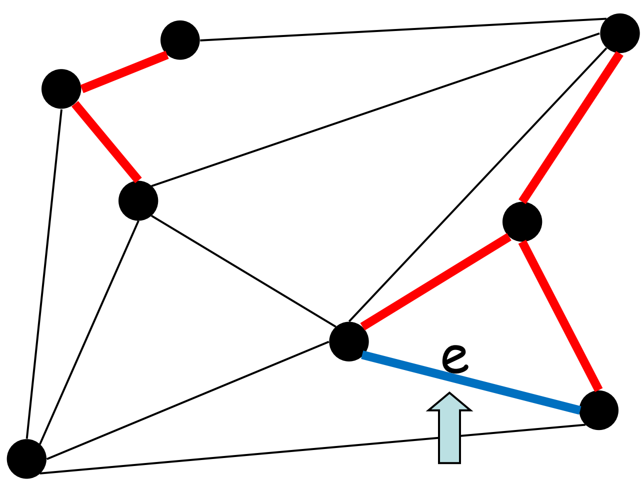
   for i = 1 to m
      Let (u,v) = eᵢ
      if (u and v are in different sets) {
         T ← T ∪ {eᵢ}
         merge the sets containing u and v
      }
   return T
}
```

Kruskal's Algorithm: Pf of Correctness

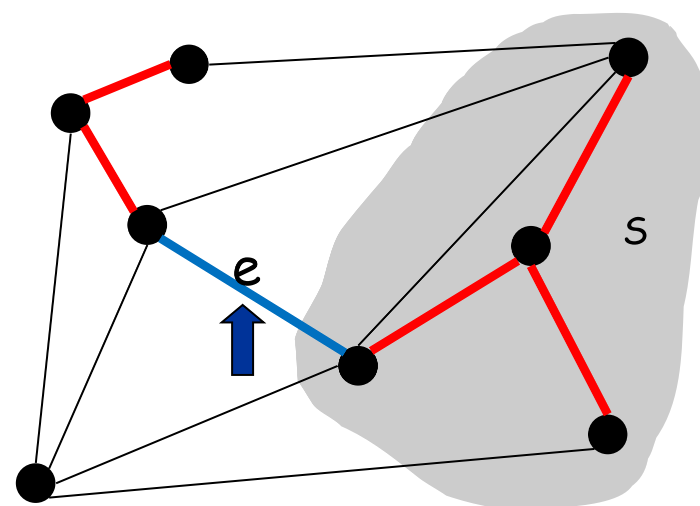
Consider edges in ascending order of weight.

Case 1: If adding e to T creates a cycle, discard e according to cycle property.

Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.



Case 1



Case 2

Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain a set for each connected component.
- O(m log n) for sorting and O(m log n) for union-find

```
Kruskal(G, c) {
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