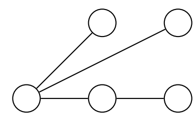
P1) Let G be a tree. Use induction to prove that the number of leaves of G is at least the number of vertices of degree at least 3 in G. For example, the following tree has 3 leaves and 1 vertex of degree at least 3, and $3 \geq 1$.



Solution: Let P(n) denote the statement "The number of leaves of any tree with n vertices is at least the number of vertices of degree at least 3."

Base Case: $P(1)$ and $P(2)$ holds obviously as there is no vertex of degree at least 3.

IH: Suppose $P(n-1)$ holds for some $n \geq 3$.

IS: We prove $P(n)$. Let T be an arbitrary tree with n nodes. Suppose that T has a leaves and b nodes of degree at least three. We need to show that $a \geq b$. Since T is a tree it has a leaf, say x. Let $T' = T - x$ denote the tree T with the vertex x and all its edges removed. As we prove in class when we remove a leaf from a tree the remaining graph, T', is also tree. Suppose T' has a' leaves and b' nodes of degree at least 3. By IH $a' \geq b'$.

Let y be the unique neighbor of x in T. Note that $\deg_{T'}(y) = \deg_T(y) - 1$.

Case 1: $\deg_T(y) = 2$: Then $a = a'$ because y is a leaf in T' which is no longer leaf in T whereas we get a new leaf, x, in T. Also in this case, $b = b'$. Therefore, $a = a' \geq b' = b$ as desired.

Case 2: $\deg_T(y) \geq 3$: In this case, $a = a' + 1$, because y is not a leaf in T' so we have a new leaf, x, in T. And, obviously, $b \leq b' + 1$. Therefore, $b \leq b' + 1 \leq a' + 1 = a$ as desired.

Note that $\deg_T(y) = 1$ cannot happen because in such a case T must have two nodes, i.e., $n = 2$.

P2) Let G be a graph with n vertices and at least n edges. Show that G has a cycle.

Solution: We prove by contradiction! Suppose G has no cycle. Then,

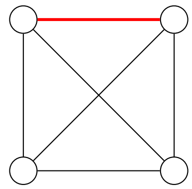
Case 1: G is connected. Then since G has no cycles, G is a tree with n vertices. So it must have $n-1$ edges. But we said it has $\geq n$. That is a contradiction.

Case 2: G is disconnected. Suppose G has ℓ connected components with number of vertices $n_1, n_2, \dots, n_\ell$ and number of edges $m_1, m_2, \dots, m_\ell$.

Claim: For some i we must have $m_i \geq n_i$. **Pf:** For contradiction assume $m_i < n_i$ for all i. Summing up these inequalities we get $m = \sum_i m_i < \sum_i n_i = n$. But that contradicts the assumption that $m \geq n$.

So let i be one of the indices for which $m_i \geq n_i$. But then the i-th component is connected and has no cycles. So similar to Case 1 we get a contradiction.

P3) Given a connected undirected graph $G = (V, E)$ with n vertices and m edges. Design an $O(m + n)$ time algorithm that outputs an edge e of G such that if we delete e, G remains connected. If no such edge exists output "Impossible". For example in the following graph if you delete the red edges the graph remains connected.



Solution: We run the following algorithm: We run BFS from an arbitrary vertex s. In the BFS code, when examining neighbors of u, say we find an already discovered vertex x that is **not** the parent of u. Then we output the edge (u, x) and we end the algorithm. Otherwise, if all edges have been examined without finding such a vertex, we output "Impossible".

Correctness: Let T be the BFS tree. Since G is connected, all vertices are reachable from s; so T has n vertices and $n - 1$ edges.

We consider the following cases: If G has no extra edges other than edges of T, i.e., G has $n - 1$ edges. Then if we remove any edge of G the remaining graph is disconnected. To see this, notice that $G - e$ has no cycles (since G has no cycles) and if in addition it is connected then it must have $n - 1$ edges (not $n - 2$). In such a case since every edge of G is in T our code never finds an already discovered vertex and it outputs "Impossible".

Otherwise, suppose G has extra edges in addition to those contained in T. Then, the algorithm will eventually output some edge $e = (u, x)$ that is not in T while inspecting vertex u. This means that x was previously already marked as discovered, and therefore there is a path in the BFS tree T that connects x to u. Together with the edge $e = (u, x)$, this forms a cycle.

Running time: We are just adding one line to the BFS code, so the algorithm runs in the BFS time, i.e., $O(m + n)$.

We write the psueodo-code below, although the above description is already enough:

```
Function BFS(s)
    Initialize: mark all vertices "undiscovered"
    mark s "discovered", set P[s] = s
    queue = { s }
    while queue not empty do
        u = remove_first(queue)
        for each edge {u, x} do
            if x is "undiscovered" then
                mark x "discovered"
                Set Parent of x to be u, P[x] = u. append x on queue
            end
            else
                If P[u] ≠ x, output {u, x} and end the algorithm
            end
        end
        mark u "fully-explored"
    end
    output "Impossible"
```

Algorithm 1: Algorithm for P3