CSE 421
Introduction to Algorithms

Winter 2024

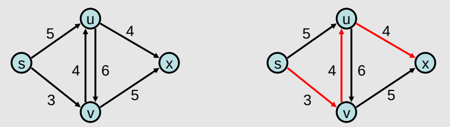
Lecture 8

Minimum Spanning Trees

1

Bottleneck Shortest Path

- Define the bottleneck distance for a path P, $Len_B(P)$ to be the maximum cost edge along the path



$Len_B(x,y) = \text{Min }\{P \text{ from } x \text{ to } y \mid Len_B(P)\}$

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Dijkstra's Algorithm for Bottleneck Shortest Paths

S = { }; d[s] = negative infinity; d[v] = infinity for v != s

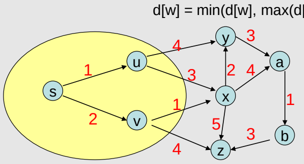
While S != V

  Choose v in V-S with minimum d[v]

  Add v to S

  For each w in the neighborhood of v

    d[w] = min(d[w], max(d[v], c(v, w)))



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Minimum Spanning Tree

- Introduce Problem
- Demonstrate Prim's and Kruskal's algorithms
- Provide proofs that the algorithms work

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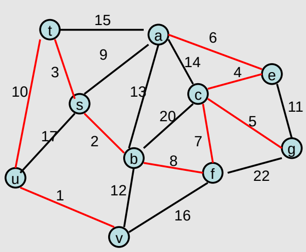
Minimum Spanning Tree



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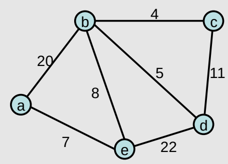
Greedy Algorithms for Minimum Spanning Tree

- Extend a tree by including the cheapest out going edge
- Add the cheapest edge that joins disjoint components
- ~~Delete the most expensive edge that does not disconnect the graph~~



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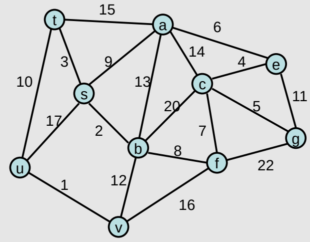
Greedy Algorithm 1
Prim's Algorithm

- Extend a tree by including the cheapest out going edge

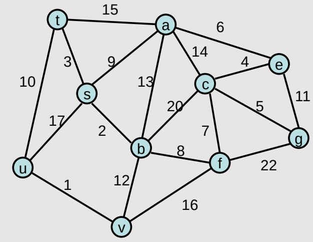


Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion

Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components

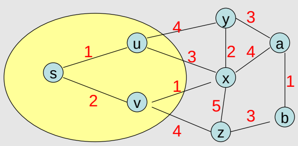


Construct the MST with Kruskal's algorithm

Label the edges in order of insertion

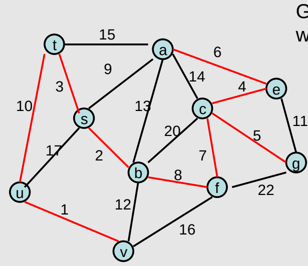
Dijkstra's Algorithm
for Minimum Spanning Trees

S = { }; d[s] = 0; d[v] = infinity for v != s
While S != V
    Choose v in V-S with minimum d[v]
    Add v to S
    For each w in the neighborhood of v
        d[w] = min(d[w], c(v, w))



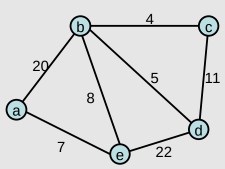
Minimum Spanning Tree

Undirected Graph G=(V,E) with edge weights



Greedy Algorithms for Minimum Spanning Tree

- [Prim] Extend a tree by including the cheapest out going edge
- [Kruskal] Add the cheapest edge that joins disjoint components

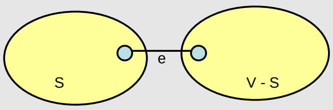


Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct

Edge inclusion lemma

- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



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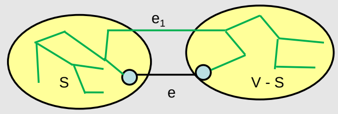
Proof

e is the minimum cost edge between S and V-S

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

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Optimality Proofs

- Prim's Algorithm computes a MST
- Kruskal's Algorithm computes a MST

- Show that when an edge is added to the MST by Prim or Kruskal, the edge is the minimum cost edge between S and V-S for some set S.

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Prim's Algorithm

S = { }; T = { };

while S != V

    choose the minimum cost edge
    e = (u,v), with u in S, and v in V-S

    add e to T

    add v to S

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Prove Prim's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

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Kruskal's Algorithm

Let $C = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$; T = { }

while |C| > 1

    Let e = (u, v) with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

    Replace C_i and C_j by C_i U C_j

    Add e to T

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Prove Kruskal's algorithm computes an MST

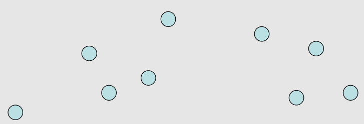
- Show an edge e is in the MST when it is added to T

Dealing with the assumption of no equal weight edges

- Force the edge weights to be distinct
  - Give a tie breaking rule for equal weight edges

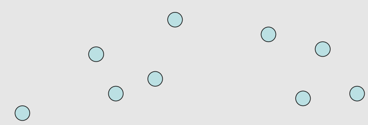
Application: Clustering

- Given a collection of points in an r-dimensional space and an integer K, divide the points into K sets that are closest together

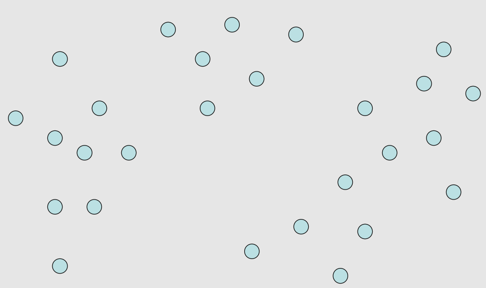


Distance clustering

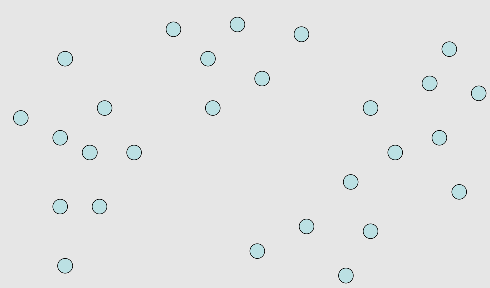
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - dist (S_1, S_2) = min {dist(x, y) | x in S_1, y in S_2}



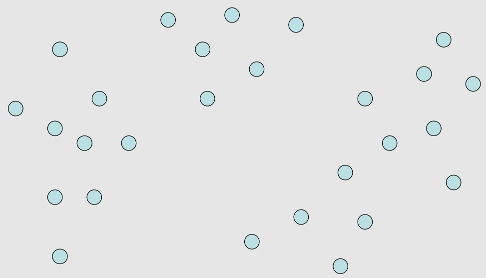
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

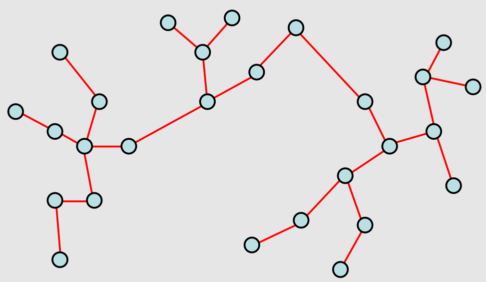
Let $C = \{\{v_1\}, \{v_2\}, . . ., \{v_n\}\}$; T = { }

while |C| > K

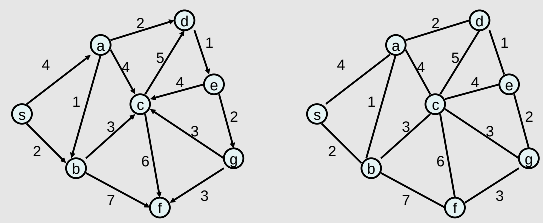
Let e = (u, v) with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by C_i U C_j

K-clustering

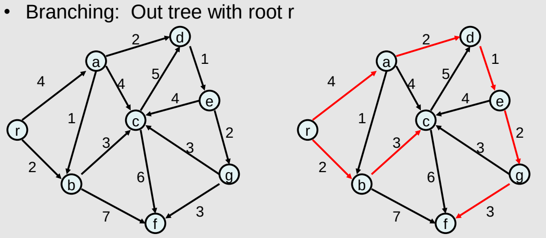


Shortest paths in directed graphs vs undirected graphs



What about the minimum spanning tree of a directed graph?

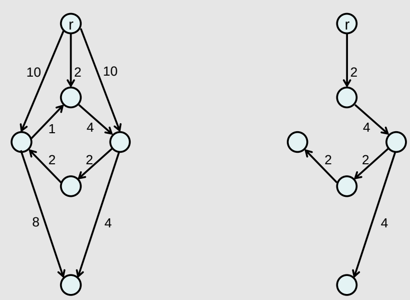
- Must specify the root r
- Branching: Out tree with root r



Assume all vertices reachable from r

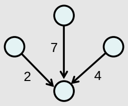
Also called an arborescence

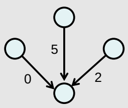
Finding a minimum branching



Finding a minimum branching

- Remove all edges going into r
- Normalize the edge weights, so the minimum weight edge coming into each vertex has weight zero





This does not change the edges of the minimum branching

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Finding a minimum branching

- Consider the graph that consists of the minimum cost edge coming in to each vertex
  - If this graph is a branching, then it is the minimum cost branching
  - Otherwise, the graph contains one or more cycles
    - Collapse the cycles in the original graph to super vertices
    - Reweight the graph and repeat the process

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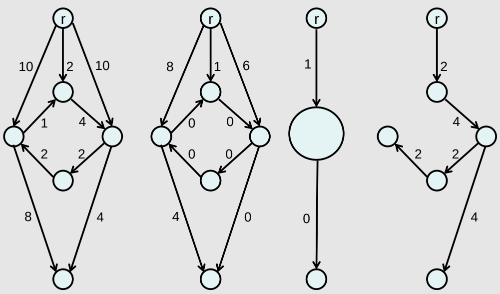
Finding a minimum branching



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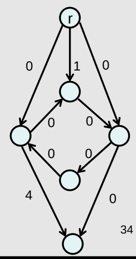
Correctness Proof

Lemma 4.38 Let C be a cycle in G consisting of edges of cost 0 with r not in C. There is an optimal branching rooted at r that has exactly one edge entering C.

- The lemma justifies using the edges of the cycle in the branching
- An induction argument is used to cover the multiple levels of compressing cycles



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