CSE 421, Introduction to Algorithms

Lecture 14, Winter 2024

Dynamic Programming

Subset Sum, Lognest Common Subsequence

Announcements

- Dynamic Programming Reading:
  - 6.1-6.2, Weighted Interval Scheduling
  - 6.3 Segmented Least Squares
  - 6.4 Knapsack and Subset Sum
  - 6.6 String Alignment
  - 6.8 Shortest Paths (Bellman-Ford)
- Midterm, Friday, Feb 9
  - Material through 6.3 and HW 5
  - Feb 8 Section will be Midterm review

What is the largest sum you can make of the following integers that is ≤ 20

{4, 5, 8, 10, 13, 14, 17, 18, 21, 23, 28, 31, 37}

What is the largest sum you can make of the following integers that is ≤ 2000

{78, 101, 122, 133, 137, 158, 189, 201, 220, 222, 267, 271, 281, 289, 296, 297, 301, 311, 315, 321, 322, 341, 349, 353, 361, 385, 396 }

Subset Sum Problem

- Given integers $\{w_1,\ldots,w_n\}$ and an integer K
- Find a subset that is as large as possible that does not exceed K

- Dynamic Programming: Express as an optimization over sub-problems.

- New idea: Represent at a sub problems depending on K and n
  - Two dimensional grid

Subset Sum Optimization

Opt[j, K] the largest subset of $\{w_1, \ldots, w_j\}$ that sums to at most K

Opt[j, K] = max(Opt[j – 1, K], Opt[j – 1, K – w_j] + w_j)

Subset Sum Grid

$$Opt[j, K] = \max(Opt[j-1, K], Opt[j-1, K-w_j] + w_j)$$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

{2, 4, 7, 10}

Subset Sum Grid

$Opt[\,j, K] = \max(Opt[\,j - 1, K], Opt[\,j - 1, K - w_j] + w_j)$

4	0	2	2	4	4	6	7	7	9	10	11	12	13	14	14	16	17
3	0	2	2	4	4	6	7	7	9	9	11	11	13	13	13	13	13
2	0	2	2	4	4	6	6	6	6	6	6	6	6	6	6	6	6
1	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

{2, 4, 7, 10}

Subset Sum Code

```
for j = 1 to n
    for k = 1 to W
        Opt[j, k] = max(Opt[j-1, k], Opt[j-1, k-w_j] + w_j)
```

Knapsack Problem

- Items have weights and values
- The problem is to maximize total value subject to a bound on weght
- Items $\{I_1, I_2, \ldots I_n\}$
  - Weights $\{w_1, w_2, \ldots, w_n\}$
  - Values $\{v_1, v_2, \ldots, v_n\}$
  - Bound K
- Find set S of indices to:
  - Maximize $\sum_{i \varepsilon S} v_i$ such that $\sum_{i \varepsilon S} w_i <= K$

Knapsack Recurrence

Subset Sum Recurrence:

$$Opt[\ j, K] = \max(Opt[\ j - 1, K], Opt[\ j - 1, K - w_j] + w_j)$$

Knapsack Recurrence:

Knapsack Grid

$Opt[\ j, K] = \max(Opt[\ j - 1, K], Opt[\ j - 1, K - w_j] + v_j)$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Weights {2, 4, 7, 10} Values: {3, 5, 9, 16}

Knapsack Grid

$$Opt[j, K] = \max(Opt[j-1, K], Opt[j-1, K-w_j] + v_j)$$

4	0	3	3	5	5	8	9	9	12	16	16	18	18	21	21	24	25
3	0	3	3	5	5	8	9	9	12	12	14	14	17	17	17	17	17
2	0	3	3	5	5	8	8	8	8	8	8	8	8	8	8	8	8
1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Weights {2, 4, 7, 10} Values: {3, 5, 9, 16}

Alternate approach for Subset Sum

- Alternate formulation of Subset Sum dynamic programming algorithm
  - Sum[i, K] = true if there is a subset of {w_1,…w_i} that sums to exactly K, false otherwise
  - Sum [i, K] = Sum [i -1, K] **OR** Sum[i - 1, K - w_i]
  - Sum [0, 0] = true; Sum[i, 0] = false for i != 0

- To allow for negative numbers, we need to fill in the array between K_{min} and K_{max}

Run time for Subset Sum

- With n items and target sum K, the run time is O(nK)
- If K is 1,000,000,000,000,000,000,000,000 this is very slow
- Alternate brute force algorithm: examine all subsets: $O(n2^n)$
- Point of confusion: Subset sum is NP Complete

Two dimensional dynamic programming

Subset sum and knapsack

$$Opt[j, K] = \max(Opt[j-1, K], Opt[j-1, K-w_j] + w_j)$$

$$Opt[j, K] = \max(Opt[j-1, K], Opt[j-1, K-w_j] + v_j)$$

4	0																
3	0																
2	0																
1	0																
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Reducing dimensions

- Computing values in the array only requires the previous row
  - Easy to reduce this to just tracking two rows
  - And sometimes can be implemented in a single row
- Space savings is significant in practice
- Reconstructing values is harder

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

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Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- Opt[j, k] is the length of LCS($a_1a_2\ldots a_j$, $b_1b_2\ldots b_k$)

Optimization recurrence

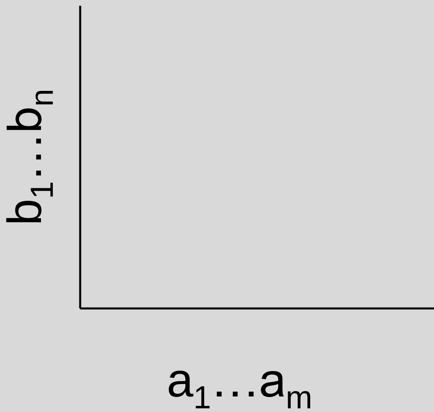
If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
       Opt[ i, j ] := Opt[ i-1, j ];
    else
       Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information




```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else    {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

Reconstructing Path from Distances

```
LCS Arguments
 211031321102033212120000321302
 100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

How good is this algorithm?

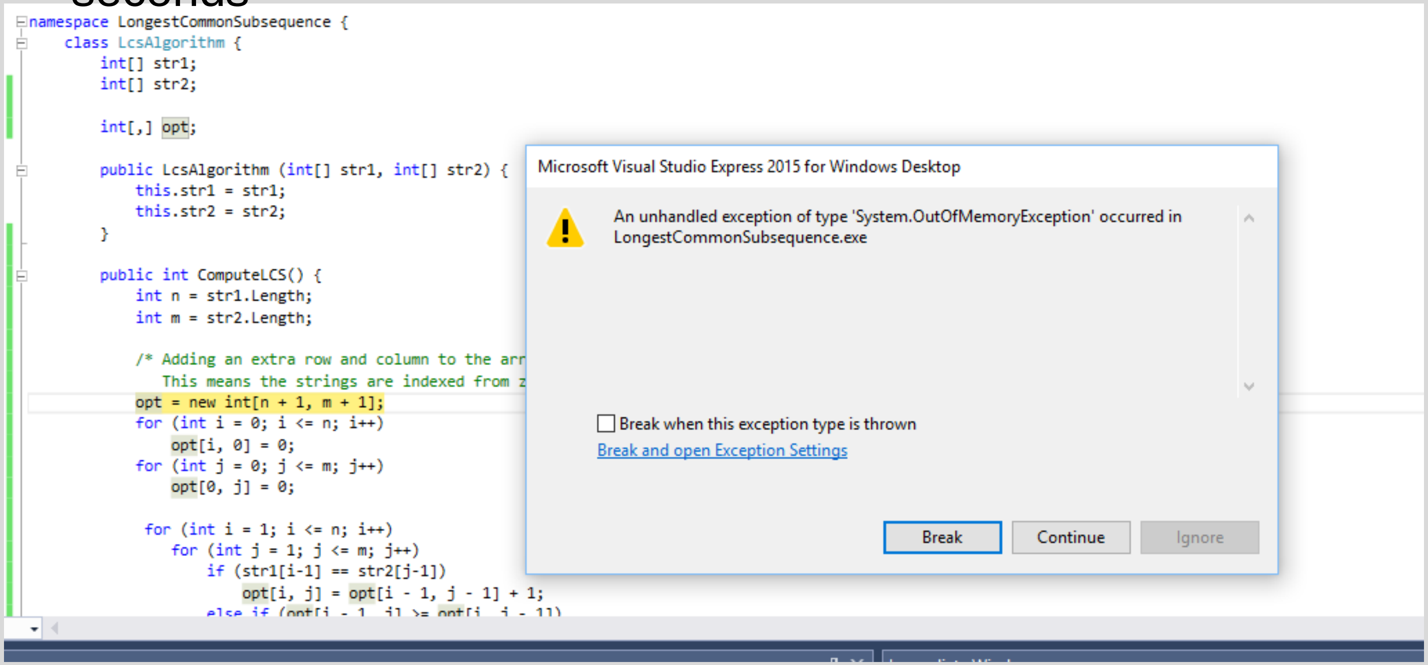
- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Implementation 1

```
public int ComputeLCS() {
            int n = str1.Length;
            int m = str2.Length;

            int[,] opt = new int[n + 1, m + 1];
            for (int i = 0; i <= n; i++)
                opt[i, 0] = 0;
            for (int j = 0; j <= m; j++)
                opt[0, j] = 0;

             for (int i = 1; i <= n; i++)
                for (int j = 1; j <= m; j++)
                    if (str1[i-1] == str2[j-1])
                        opt[i, j] = opt[i - 1, j - 1] + 1;
                    else if (opt[i - 1, j] >= opt[i, j - 1])
                        opt[i, j] = opt[i - 1, j];
                    else
                        opt[i, j] = opt[i, j - 1];

            return opt[n,m];
        }
```

N = 17000

Runtime should be about 5 seconds*



* Personal PC, 10 years old

Manufacturer:	Dell
Model:	Optiplex 990
Processor:	Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM):	8.00 GB (7.88 GB usable)
System type:	64-bit Operating System, x64-based processor

Implementation 2

```
public int SpaceEfficientLCS() {
          int n = str1.Length;
          int m = str2.Length;
          int[] prevRow = new int[m + 1];
          int[] currRow = new int[m + 1];

          for (int j = 0; j <= m; j++)
              prevRow[j] = 0;

          for (int i = 1; i <= n; i++) {
              currRow[0] = 0;
              for (int j = 1; j <= m; j++) {
                  if (str1[i - 1] == str2[j - 1])
                      currRow[j] = prevRow[j - 1] + 1;
                  else if (prevRow[j] >= currRow[j - 1])
                      currRow[j] = prevRow[j];
                  else
                      currRow[j] = currRow[j - 1];
              }
              for (int j = 1; j <= m; j++)
                  prevRow[j] = currRow[j];
          }

          return currRow[m];
      }
```

N = 300000

N: 10000 Base 2 Length: 8096	Gamma: 0.8096	Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231	Gamma: 0.81155	Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317	Gamma: 0.8105667	Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510	Gamma: 0.81275	Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563	Gamma: 0.81126	Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700	Gamma: 0.8116667	Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824	Gamma: 0.8117715	Runtime:00:01:33.36

N: 300000 Base 2 Length: 243605 Gamma: 0.8120167
Runtime:00:28:07.32

Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values

- The computation requires O(nm) space if we store all of the string information