CSE 421, Introduction to Algorithms

Lecture 15, Winter 2024

Dynamic Programming

Longest Common Subsequence

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Announcements

- Dynamic Programming Reading:
  - Weighted Interval Scheduling, Segmented Least Squares, Knapsack and Subset Sum
  - 6.6 String Alignment
  - 6.8 Shortest Paths (Bellman-Ford)
- Midterm, Friday, Feb 9
  - Material through 6.3 and HW 5
  - Feb 8 Section will be Midterm review
    - Old exam problems on course homepage
  - Homework 6 due Feb 14

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Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

LCS(BARTHOLEMEWSIMPSON, KRUSTYTHECLOWN)
= RTHOWN

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Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

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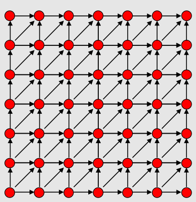
Dynamic Programming Computation



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Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
       Opt[ i, j ] := Opt[ i-1, j ];
    else
       Opt[ i, j ] := Opt[ i, j-1];
```

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Storing the path information

A[1..m], B[1..n]

for i := 1 to m   Opt[i, 0] := 0;

for j := 1 to n   Opt[0,j] := 0;

Opt[0,0] := 0;

for i := 1 to m

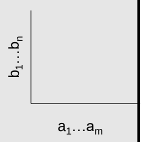
    for j := 1 to n

        if A[i] = B[j] { Opt[i,j] := 1 + Opt[i-1,j-1]; Best[i,j] := Diag; }

        else if Opt[i-1, j] >= Opt[i, j-1]

            { Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }

        else   { Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }



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Reconstructing Path from Distances

```
LCS Arguments
 211031321102033212120000321302
 100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

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How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

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Implementation 1

```
public int ComputeLCS() {
        int n = str1.Length;
        int m = str2.Length;

        int[,] opt = new int[n + 1, m + 1];
        for (int i = 0; i <= n; i++)
            opt[i, 0] = 0;
        for (int j = 0; j <= m; j++)
            opt[0, j] = 0;

         for (int i = 1; i <= n; i++)
            for (int j = 1; j <= m; j++)
                if (str1[i-1] == str2[j-1])
                    opt[i, j] = opt[i - 1, j - 1] + 1;
                else if (opt[i - 1, j] >= opt[i, j - 1])
                    opt[i, j] = opt[i - 1, j];
                else
                    opt[i, j] = opt[i, j - 1];

        return opt[n,m];
    }
```

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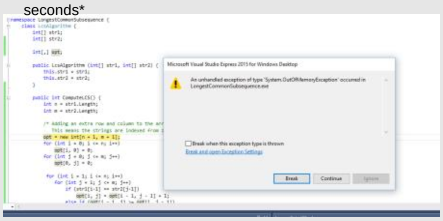
N = 17000

Runtime should be about 5 seconds*



* Personal PC, 10 years old

Manufacturer:	Dell
Model:	Optiplex 990
Processor:	Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM):	8.00 GB (7.88 GB usable)
System type:	64-bit Operating System, x64-based processor

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Implementation 2

```
public int SpaceEfficientLCS() {
        int n = str1.Length;
        int m = str2.Length;
        int[] prevRow = new int[m + 1];
        int[] currRow = new int[m + 1];

        for (int j = 0; j <= m; j++)
            prevRow[j] = 0;

        for (int i = 1; i <= n; i++) {
            currRow[0] = 0;
            for (int j = 1; j <= m; j++) {
                if (str1[i - 1] == str2[j - 1])
                    currRow[j] = prevRow[j - 1] + 1;
                else if (prevRow[j] >= currRow[j - 1])
                    currRow[j] = prevRow[j];
                else
                    currRow[j] = currRow[j - 1];
            }
            for (int j = 0; j <= m; j++)
                prevRow[j] = currRow[j];
        }

        return currRow[m];
    }
```

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N = 300000

N: 10000 Base 2 Length: 8096	Gamma: 0.8096	Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231	Gamma: 0.81155	Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317	Gamma: 0.8105667	Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510	Gamma: 0.81275	Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563	Gamma: 0.81126	Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700	Gamma: 0.8116667	Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824	Gamma: 0.8117715	Runtime:00:01:33.36

N: 300000 Base 2 Length: 243605 Gamma: 0.8120167
Runtime:00:28:07.32

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Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

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Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

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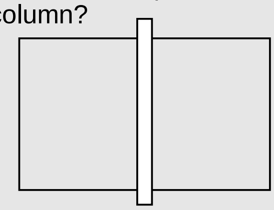
Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

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Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

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A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

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A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

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Computing the middle column

- From the left, compute $LCS(a_1 \ldots a_{m/2}, b_1 \ldots b_j)$
- From the right, compute $LCS(a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n)$
- Add values for corresponding j's

- Note – this is space efficient

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Divide and Conquer

- $A = a_1, \ldots, a_m$ $B = b_1, \ldots, b_n$
- Find j such that
  - $LCS(a_1 \ldots a_{m/2}, b_1 \ldots b_j)$ and
  - $LCS(a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n)$ yield optimal solution
- Recurse

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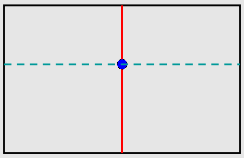
Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$



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Prove by induction that $T(m,n) <= 2cmn$

$T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$

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Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes

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Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The computation requires O(nm) space if we store all of the string information

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String Alignment Problem

- Align sequences with gaps

CAT TGA AT

CAGAT AGGA

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

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Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

Let a_j = x and b_k = y
Express as minimization

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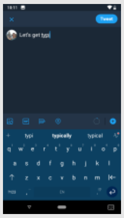
String edit with Typo Distance

- Find closest dictionary word to typed word
- Dist('a', 's') = 1
- Dist('a', 'u') = 6
- Capture the likelihood of mistyping characters



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