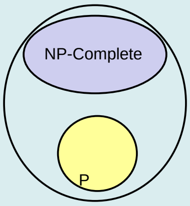


CSE 421
Introduction to Algorithms

Lecture 24

Survey of NP Complete Problems

Announcements

- Homework 9, Due Friday, March 8
- Final exam,
  - Monday, March 11, 2:30-4:20 pm PDT
  - Comprehensive (~60% post midterm, ~40% pre midterm)
  - Old finals / answers on home page

Today

Here are 21 NP-Complete Problems

REDUCIBILITY AMONG COMBINATORIAL PROBLEMS[†]

Richard M. Karp

University of California at Berkeley

Abstract: A large class of computational problems involve the determination of properties of graphs, digraphs, integers, arrays of integers, finite families of finite sets, boolean formulas and elements of other countable domains. Through simple encodings from such domains into the set of words over a finite alphabet these problems can be converted into language recognition problems, and we can inquire into their computational complexity. It is reasonable to consider such a problem satisfactorily solved when an algorithm for its solution is found which terminates within a number of steps bounded by a polynomial in the length of the input. We show that a large number of classic unsolved problems of covering, matching, packing, routing, assignment and sequencing are equivalent, in the sense that either each of them possesses a polynomial-bounded algorithm or none of them does.

1. INTRODUCTION

All the general methods presently known for computing the chromatic number of a graph, deciding whether a graph has a Hamilton circuit, or solving a system of linear inequalities in which the variables are constrained to be 0 or 1, require a combinatorial search for which the worst case time requirement grows exponentially with the length of the input. In this paper we give theorems which strongly suggest, but do not imply, that these problems, as well as many others, will remain intractable perpetually.

[†]This research was partially supported by National Science Foundation Grant GJ-474.

Reducibility Among Combinatorial Problems

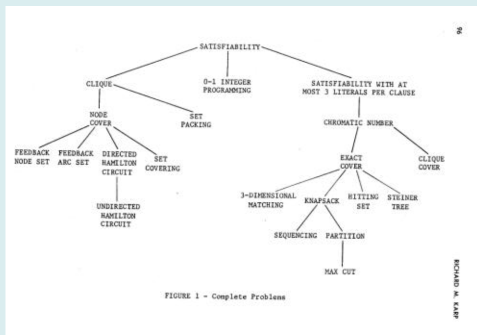


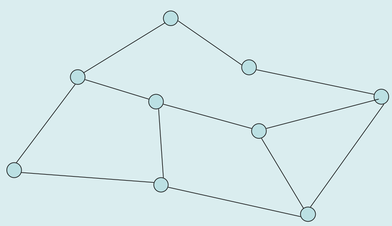
FIGURE 1 - Complete Problems

NP Complete Problems

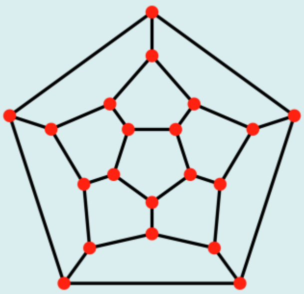
1. Circuit Satisfiability
2. Formula Satisfiability
   a. 3-SAT
3. Graph Problems
   a. Independent Set
   b. Vertex Cover
   c. Clique
4. Path Problems
   a. Hamiltonian cycle
   b. Hamiltonian path
   c. Traveling Salesman
5. Partition Problems
   a. Three dimensional matching
   b. Exact cover
6. Graph Coloring
7. Number problems
   a. Subset sum
8. Integer linear programming
9. Scheduling with release times and deadlines

Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph



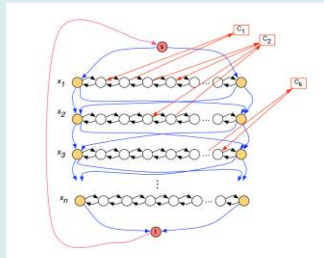
Find the Circuit



7

Thm: Hamiltonian Circuit is NP Complete

- Reduction from 3-SAT

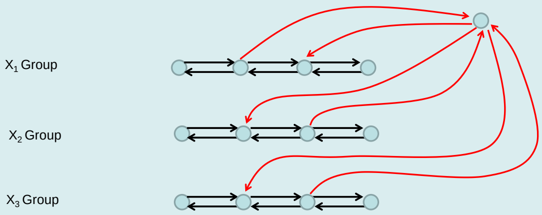


Page 475 in text

8

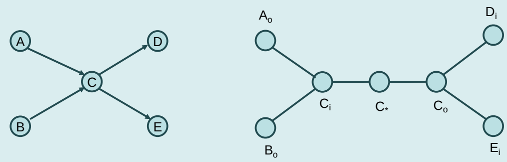
Clause Gadget

$$x_1 \vee \overline{x_2} \vee \overline{x_3}$$



9

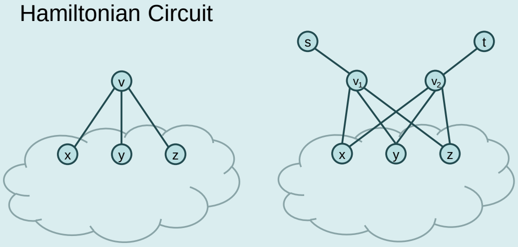
DHC $<_P$ UHC



10

Reduce Hamiltonian Circuit to Hamiltonian Path

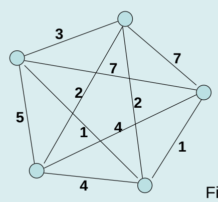
G_2 has a Hamiltonian Path iff G_1 has a Hamiltonian Circuit



11

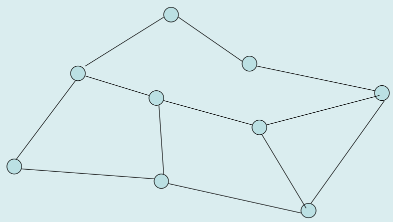
Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)



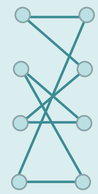
Find the minimum cost tour

Thm: HC $<_P$ TSP

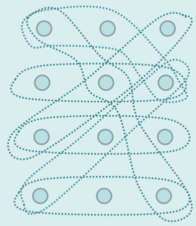


13

Matching



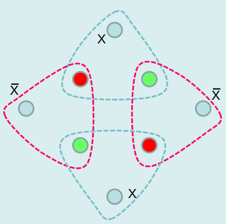
Two dimensional matching



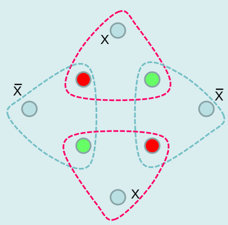
Three dimensional matching (3DM)

14

3-SAT $<_P$ 3DM



X True

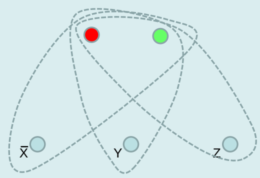


X False

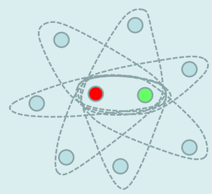
Truth Setting Gadget

15

3-SAT $<_P$ 3DM



Clause gadget for ($\bar{X}$ OR Y OR Z)



Garbage Collection Gadget (Many copies)

16

Exact Cover (sets of size 3) XC3

Given a collection of sets of size 3 of a domain of size 3N, is there a sub-collection of N sets that cover the sets

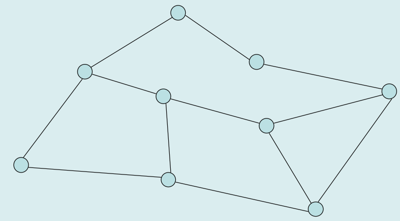
(A, B, C), (D, E, F), (A, B, G),
(A, C, I), (B, E, G), (A, G, I),
(B, D, F), (C, E, I), (C, D, H),
(D, G, I), (D, F, H), (E, H, I),
(F, G, H), (F, H, I)

3DM $<_P$ XC3

17

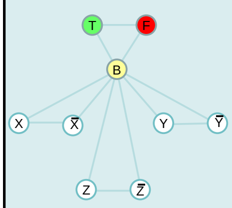
Graph Coloring

- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring

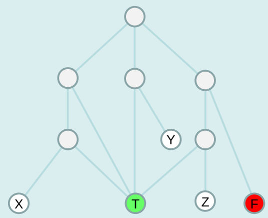


18

3-SAT $<_P$ 3 Colorability



Truth Setting Gadget



Clause Testing Gadget

(Can be colored if at least one input is T)

19

Number Problems

- Subset sum problem
  – Given natural numbers $w_1, \ldots, w_n$ and a target number W, is there a subset that adds up to exactly W?
- Subset sum problem is NP-Complete
- Subset Sum problem can be solved in O(nW) time

20

XC3 $<_P$ SUBSET SUM

Idea: Represent each set as a large integer, where the element x_i is encoded as D^i where D is an integer

$\{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9$

Does there exist a subset that sums to exactly $D^1 + D^2 + D^3 + \ldots + D^{n-1} + D^n$

Detail: How large is D? We need to make sure that we do not have any carries, so we can choose D = m+1, where m is the number of sets.

21

Integer Linear Programming

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

Use 0-1 variables for x_i's

Constraint for clause $x_1 \vee \overline{x_2} \vee \overline{x_3}$

$x_1 + (1 - x_2) + (1 - x_3) > 0$

22

Scheduling with release times and deadlines

- Tasks $T_1, \ldots, T_n$ with release time r_i, deadline d_i, and work w_i
- Reduce from Subset Sum
  - Given natural numbers $w_1, \ldots, w_n$ and a target number K, is there a subset that adds up to exactly K?
  - Suppose the sum $w_1 + \ldots + w_n = W$
- Task T_i has release time 0 and deadline W+1
- Add an additional task with release time K, deadline K+1 and work 1

23