
Affine Transformations

CSE 457, Autumn 2003
Graphics

<http://www.cs.washington.edu/education/courses/457/03au/>

Readings and References

Readings

- Section 1.1, *3D Computer Graphics*, Watt

Other References

- Sections 5.1-5.5, *Computer graphics : principles and practice*, James D. Foley, et al.
 - » on reserve in the Engineering Library

Geometric transformations

- Geometric transformations will map points in one space to points in another: $(x',y',z') = f(x,y,z)$.
- These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.
- We'll focus on transformations that can be represented easily with matrix operations.
- We'll start in 2D...

Representation : point

- We can represent a **point**, $\mathbf{p} = (x,y)$, in the plane

» as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$

» as a row vector $[x \quad y]$

Representation : matrix

- Represent a **2-D transformation** M by a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- If \mathbf{p} is a column vector, M goes on the left: $\mathbf{p}' = M\mathbf{p}$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
- If \mathbf{p} is a row vector, M^T goes on the right: $\mathbf{p}' = \mathbf{p}M^T$
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
- We will use **column vectors**.

Two-dimensional transformations

- Here's what you get with a 2 x 2 transformation matrix M :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- And so:

$$x' = ax + by$$

$$y' = cx + dy$$

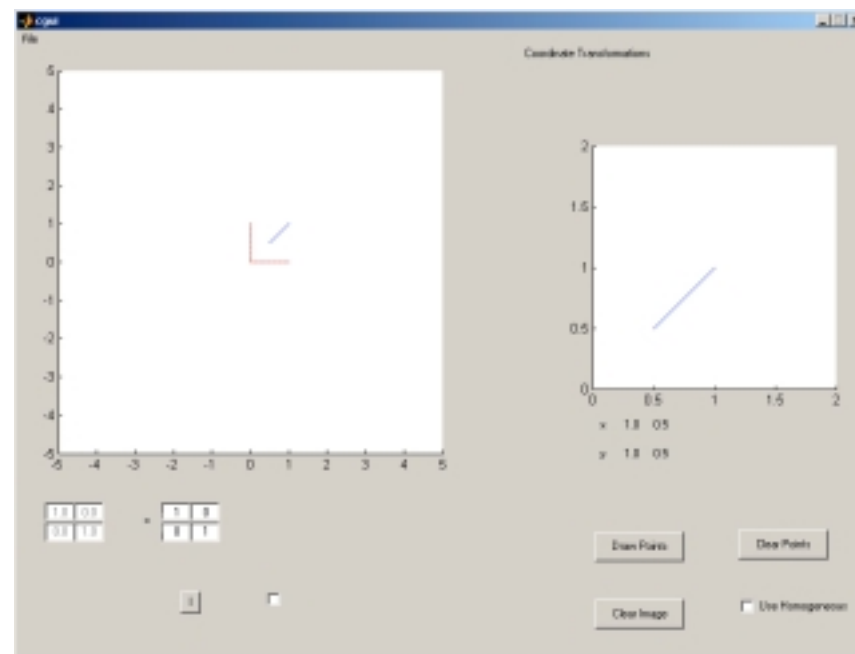
- We will develop some intimacy with the elements a, b, c, d, \dots

Identity

- Suppose we choose $a=d=1, b=c=0$:
 - » Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- » Doesn't move the points at all



Scaling

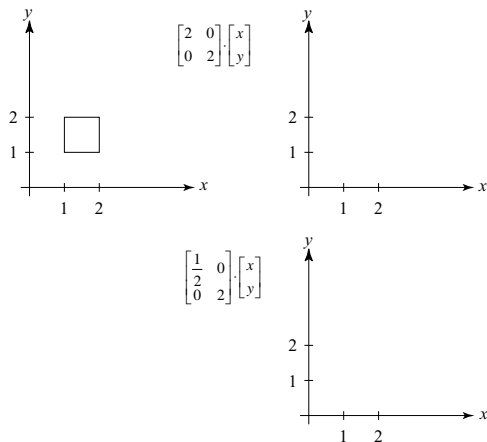
- Suppose we set $b=c=0$, but let a and d take on any *positive* value:

Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Provides **uniform scaling** or **differential scaling** in x and y :

$$\begin{aligned} x' &= ax \\ y' &= dy \end{aligned}$$



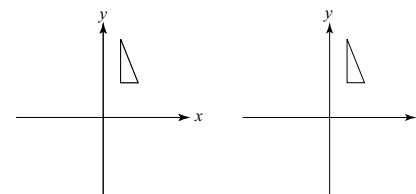
$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflections

- Suppose we keep $b=c=0$, but let either a or d go negative.
- Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



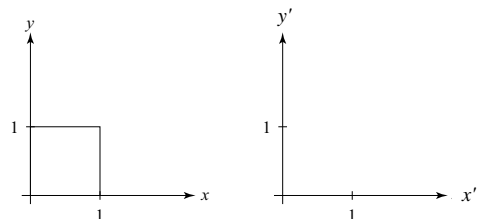
Shear

- Now let's leave $a=d=1$ and experiment with b

- The matrix $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

- gives:

$$\begin{aligned} x' &= x + by \\ y' &= y \end{aligned}$$



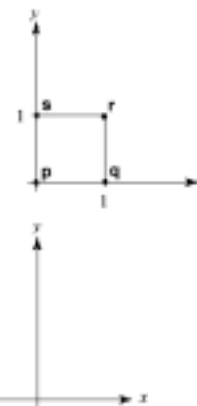
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Effect on unit square

- Let's see how a general 2×2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q & r & s \end{bmatrix} = \begin{bmatrix} p' & q' & r' & s' \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



Effect on unit square, cont.

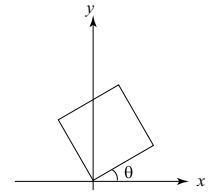
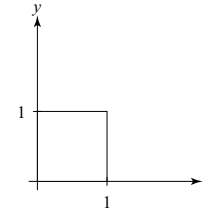
- Observe:
 - » Origin invariant under M
 - » M can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
 - these are the perpendicular basis vectors of the original space
 - » a and d give x - and y -scaling
 - » b and c give x - and y -shearing

Rotation

- From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$



$$M = R(\theta) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Limitations of the 2 x 2 matrix

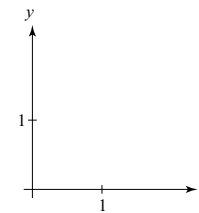
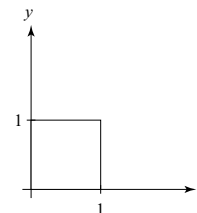
- A 2 x 2 matrix allows
 - » Scaling
 - » Rotation
 - » Reflection
 - » Shearing
- **Q:** What important operation does that leave out?

Homogeneous coordinates

- Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- And then transform with a 3 x 3 matrix:



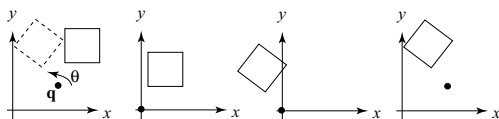
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

- ... gives **translation!**

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, q , about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix:

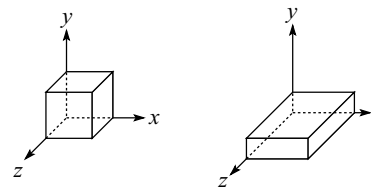


- Translate \mathbf{q} to origin
- Rotate
- Translate back
- Note: Transformation order is important!!

Basic 3-D transformations: scaling

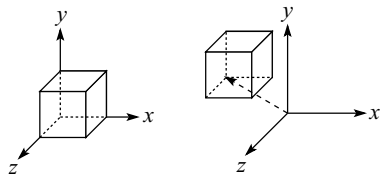
- Some of the 3-D transformations are just like the 2-D ones.
- For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



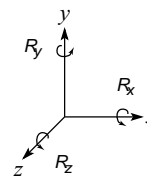
Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation in 3D

- Rotation now has more possibilities in 3D:



Use right hand rule

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

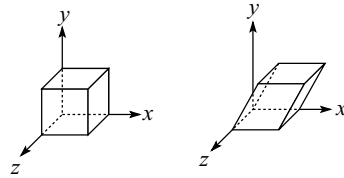
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- How else might you specify a rotation?

Shearing in 3D

- Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

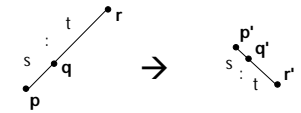


- We'll call this a “shear parallel to the x-z plane” or “shear with respect to the x-z plane.”

Properties of affine transformations

- All of the transformations we've looked at so far are examples of “affine transformations.”
- Here are some useful properties of affine transformations:

- » Lines map to lines
- » Parallel lines remain parallel
- » Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|pq\|}{\|qr\|} = \frac{s}{t} = \frac{\|p'q'\|}{\|q'r'\|}$$

Affine transformations in OpenGL

- OpenGL maintains a “modelview” matrix that holds the current transformation **M**.
- The modelview matrix is applied to points (usually vertices of polygons) before drawing.
- It is modified by commands including:
 - » `glLoadIdentity()` **M** ← **I**
– set **M** to identity
 - » `glTranslatef(tx, ty, tz)` **M** ← **MT**
– translate by (*t_x*, *t_y*, *t_z*)
 - » `glRotatef(θ, x, y, z)` **M** ← **MR**
– rotate by angle *θ* about axis (*x*, *y*, *z*)
 - » `glScalef(sx, sy, sz)` **M** ← **MS**
– scale by (*s_x*, *s_y*, *s_z*)
- Add transformations by *postmultiplication* of modelview matrix.

Summary

- What to take away from this lecture:
 - » All the names in boldface.
 - » How points and transformations are represented.
 - » What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
 - » What homogeneous coordinates are and how they work for affine transformations.
 - » How to concatenate transformations.
 - » The mathematical properties of affine transformations.