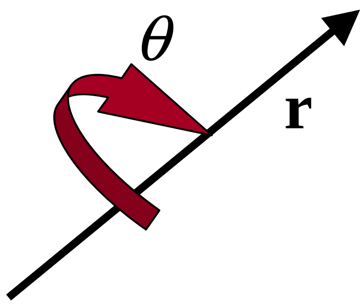
# Lecture 8:
# Quaternions

# Reading

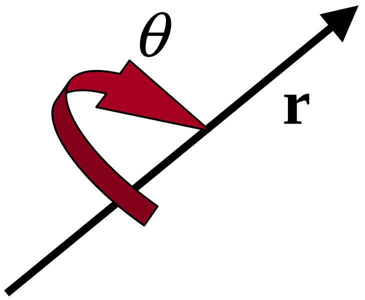
- Hearn & Baker, Section 11-2
- Optional
  - Shoemake, “Animating Rotation with Quaternion Curves”, in SIGGRAPH ’85 proceedings

# Axis-angle Rotation



# Quaternions

$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}$$



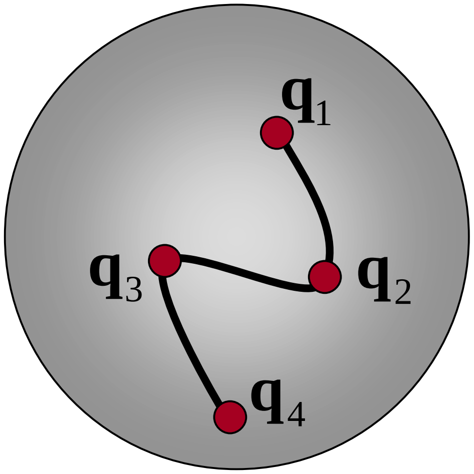

$$\mathbf{q} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{pmatrix}$$

# Unit Quaternions

$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$$|\mathbf{q}| = 1$$

$$\sqrt{x^2 + y^2 + z^2 + w^2} = 1$$



# Quaternion Product

$$\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}\begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{pmatrix}$$

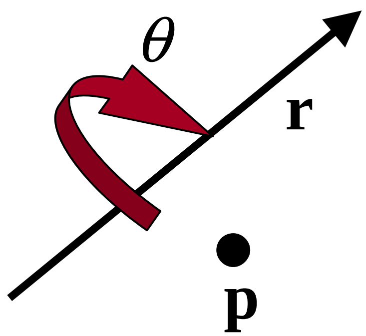
$$\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}\begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix} \neq \begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix}\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}$$

# Quaternion Inverse

$$\mathbf{q}^{-1}\mathbf{q} = 1$$

$$\mathbf{q}^{-1} = \begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix} / |\mathbf{q}| = \begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix} / (w^2 + \mathbf{v} \cdot \mathbf{v})$$

# Quaternion Rotation




$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{p} \end{pmatrix}\begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix}$$

$$= \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}\begin{pmatrix} \mathbf{p}\cdot\mathbf{v} \\ w\mathbf{p}-\mathbf{p}\times\mathbf{v} \end{pmatrix}$$

$$= \begin{pmatrix} w\mathbf{p}\cdot\mathbf{v} - w\mathbf{p}\cdot\mathbf{v} = 0 \\ w(w\mathbf{p}-\mathbf{p}\mathbf{v}) + (\mathbf{p}\cdot\mathbf{v})\mathbf{v} + \mathbf{v}(w\mathbf{p}-\mathbf{p}\times\mathbf{v}) \end{pmatrix}$$

# Matrix Form

$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 \end{pmatrix}$$