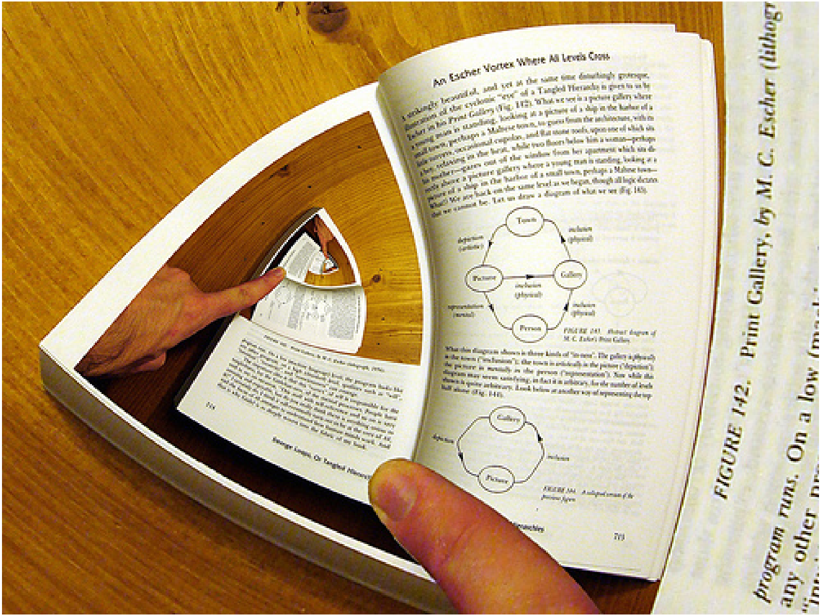
CSE 505: Programming Languages

Lecture 17 — Evaluation Contexts First-Class Continuations Continuation-Passing Style

Zach Tatlock
Autumn 2015

GOTO the past / programs choose their own adventure.



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Our semantics:

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \quad \frac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'} \quad \frac{e \to e'}{\mathsf{A}(e) \to \mathsf{A}(e')} \quad \frac{e \to e'}{\mathsf{B}(e) \to \mathsf{B}(e')}$$

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$$\frac{}{(\lambda x.\ e)\ v \to e[v/x]} \quad \frac{}{(v_1, v_2).1 \to v_1} \quad \frac{}{(v_1, v_2).2 \to v_2}$$

$$\frac{}{\textbf{match A}(v) \textbf{ with A}x.\ e_1 \mid \textbf{B}y.\ e_2 \to e_1[v/x]}$$

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Boring rules to grind sub-expressions down:

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Interesting rules that actually do work:

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We can do better: Separate concerns

Evaluation contexts define where interesting work can happen:

$$\begin{array}{lcl} E & ::= & [\cdot] \mid E\ e \mid v\ E \mid (E, e) \mid (v, E) \mid E.1 \mid E.2 \\ & \mid & \mathsf{A}(E) \mid \mathsf{B}(E) \mid (\textbf{match}\ E\ \textbf{with}\ \mathsf{A}x.\ e_1 \mid \mathsf{B}y.\ e_2) \end{array}$$

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Evaluation with evaluation contexts

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- Given e, find E, e_a, e'_a such that $e = E[e_a]$ and $e_a \xrightarrow{\text{p}} e'_a$

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Many possible eval contexts may match a given e ...

$$([\cdot])[(1, (1, (1, (1, 1))))] = (1, (1, (1, (1, 1))))$$
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Unique Decomposition Theorem: at most one decomposition of e

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Progress Theorem (restated): If e is well-typed, then there is a decomposition or e is a value

Evaluation Contexts: So what?

Small-step semantics (old) and evaluation-context semantics (new) are *very* similar:

- Totally equivalent step sequence
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Evaluation contexts so far just cleanly separate the "find and plug" from the "take that step" by building an explicit E

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$$e \quad ::= \quad \ldots \mid \textbf{letcc}\ x.\ e \mid \textbf{throw}\ e\ e \mid \textbf{cont}\ \boldsymbol{E}$$

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- **cont** E not in source programs: "saved stack (value)"

Examples (exceptions-like)

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$$1 + (\textbf{letcc}\ k.\ (\textbf{throw}\ k\ (\textbf{throw}\ k\ (\textbf{throw}\ k\ 2)))) \to^* 3$$

Another view

If you're confused, think call stacks:

- What if your favorite language had operations for:
  - Store current stack in `x`
  - Replace current stack with stack in `x`

- "Resume the stack's hole" with something different or when mutable state is different
  - Else you are sure to have an infinite loop since you will later resume the stack again

Example ("time travel")

SML/NJ has continuations. This runs and binds 10 to `z`:

```
open SMLofNJ.Cont
val g : int cont option ref = ref NONE
val x = ref true (* avoids infinite loop *)
val y = ref (1 + 2 + (callcc (fn k => ((g := SOME k); 3))))
val z = if !x then (x := false; throw (valOf (!g)) 7) else !y
```

Is this useful?

First-class continuations are a *single* construct sufficient for:

- Exceptions
- Cooperative threads (including coroutines)
  - "yield" captures the continuation (the "how to resume me") and gives it to the scheduler (implemented in the language), which then throws to another thread's "how to resume me"
- Other crazy things
  - Often called the "goto of functional programming" — incredibly powerful, but nonstandard uses are usually inscrutable
  - Key point is that we can "jump back in" unlike boring-old exceptions

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Rather than adding a powerful primitive, we can achieve the same effect via a *whole-program translation* into a sublanguage (source-to-source transformation)

- Every function takes extra arg: *continuation* says what's next

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- *Every expression becomes a continuation-accepting function*
- Will be able to reintroduce **letcc** and **throw** "for free"

CPS examples

Invariant: *every* function takes continuation as extra argument

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let mult' ...
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let rec fact' n k =
  (eq' n 0 (fun b ->
    (if b then
      (k 1)
     else
      (sub' n 1 (fun m ->
        (fact' m (fun p ->
          (mult' n p k))))))))
```

CPS examples

OK, now you convert :

```
let fact n =
  aux n 1

let rec aux n acc =
  if n = 0 then
    acc
  else
    aux (n - 1) (n * acc)
```

The CPS transformation (one way to do it)

A metafunction from expressions to expressions

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Example source language (other features similar):

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- Hence no need for a call-stack: every call is a tail-call
  - Now the *program* is maintaining the evaluation context via a closure that has the next "link" in its environment that has the next "link" in *its* environment, etc.

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You can also manually program in this style (fully or partially)

- Has other uses as a programming idiom too...

A useful advanced programming idiom

- A first-class continuation can "reify session state" in a client-server interaction
  - If the continuation is passed to the client, which returns it later, then the server can be stateless
  - Suggests CPS for web programming
  - Better: tools that do the CPS transformation for you
    - Gives you a "prompt-client" primitive without server-side state
- Because CPS uses only tail calls, it avoids deep call stacks when traversing recursive data structures
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In short, “thinking in terms of CPS” is a powerful technique few programmers have