CSE-505: Programming Languages

Lecture 7 — Lambda Calculus

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Where we are

- Done: Syntax, semantics, and equivalence
  - For a language with little more than loops and global variables
- Now: Didn't IMP leave some things out?
  - In particular: scope, functions, and data structures
  - (Not to mention threads, I/O, exceptions, strings, ...)

Time for a new model...

Data + Code

Higher-order functions work well for scope *and* data structures

- Scope: not all memory available to all code

```
let x = 1
let add3 y =
   let z = 2 in
   x + y + z
let seven = add3 4
```

- Data: Function closures store data. Example: Association "list"

```
let empty = (fun k -> raise Empty)
let cons k v lst = (fun k' -> if k'=k then v else lst k
let lookup k lst = lst k
```

(Later: Objects do both too)

Adding data structures

Extending IMP with data structures is not too hard:

$$\begin{array}{rcl} e & ::= & c \mid x \mid e + e \mid e * e \mid (e, e) \mid e.1 \mid e.2 \\ v & ::= & c \mid (v, v) \\ H & ::= & \cdot \mid H, x \mapsto v \end{array}$$

$\boxed{H \ ; e \Downarrow v}$ all old rules plus:

$$\frac{H \ ; e_1 \Downarrow v_1 \quad H \ ; e_2 \Downarrow v_2}{H \ ; (e_1, e_2) \Downarrow (v_1, v_2)} \qquad \frac{H \ ; e \Downarrow (v_1, v_2)}{H \ ; e.1 \Downarrow v_1} \qquad \frac{H \ ; e \Downarrow (v_1, v_2)}{H \ ; e.2 \Downarrow v_2}$$

Notice:

- We allow pairs of values, not just pairs of integers
- We now have *stuck* programs (e.g., $c.1$)
  - What would C++ do? Scheme? ML? Java? Perl?
  - Division also causes stuckness

What about functions

But adding functions (or objects) does not work well:

$$
\begin{aligned}
e &::= \ldots \mid \texttt{fun}\ x\ \texttt{->}\ s \\
v &::= \ldots \mid \texttt{fun}\ x\ \texttt{->}\ s \\
s &::= \ldots \mid e(e)
\end{aligned}
$$

$\boxed{H\ ;\ e \Downarrow v}$ $\boxed{H\ ;\ s \rightarrow H'\ ;\ s'}$

Additions:

$$\frac{}{H\ ;\ \texttt{fun}\ x\ \texttt{->}\ s \Downarrow \texttt{fun}\ x\ \texttt{->}\ s} \qquad \frac{H\ ;\ e_1 \Downarrow \texttt{fun}\ x\ \texttt{->}\ s \qquad H\ ;\ e_2 \Downarrow v}{H\ ;\ e_1(e_2) \rightarrow H\ ;\ x := v; s}$$

Does this match "the semantics we want" for function calls?

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$$\begin{array}{rcl} e & ::= & \ldots \mid \texttt{fun}\ x\ \texttt{->}\ s \\ v & ::= & \ldots \mid \texttt{fun}\ x\ \texttt{->}\ s \\ s & ::= & \ldots \mid e(e) \end{array}$$

$$\frac{}{H\ ;\ \texttt{fun}\ x\ \texttt{->}\ s \Downarrow \texttt{fun}\ x\ \texttt{->}\ s} \qquad \frac{H\ ;\ e_1 \Downarrow \texttt{fun}\ x\ \texttt{->}\ s \qquad H\ ;\ e_2 \Downarrow v}{H\ ;\ e_1(e_2) \rightarrow H\ ;\ x := v; s}$$

NO: Consider $\texttt{x} := 1; (\texttt{fun x -> y} := \texttt{x})(2); \texttt{ans} := \texttt{x}$.

Scope matters; variable name does not. That is:

- Local variables should "be local"
- Choice of local-variable names should have only local ramifications

Another try

$$\frac{H\ ;\ e_1 \Downarrow \texttt{fun}\ x\ \texttt{->}\ s \qquad H\ ;\ e_2 \Downarrow v \qquad y\ \text{“fresh”}}{H\ ;\ e_1(e_2) \rightarrow H\ ;\ y := x; x := v; s; x := y}$$

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- "fresh" is not very IMP-like but okay (think malloc)

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- not a good match to how functions are implemented
- yuck: the way we want to think about something as fundamental as a call?

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$$\frac{H\ ;\ e_1 \Downarrow \texttt{fun}\ x\ \texttt{->}\ s \qquad H\ ;\ e_2 \Downarrow v \qquad y \text{ "fresh"}}{H\ ;\ e_1(e_2) \rightarrow H\ ;\ y := x; x := v; s; x := y}$$

- "fresh" is not very IMP-like but okay (think malloc)
- not a good match to how functions are implemented
- yuck: the way we want to think about something as fundamental as a call?
- NO: wrong model for most functional and OO languages
  - (Even wrong for C if s calls another function that accesses the global variable x)

The wrong model

$$\frac{H\ ;\ e_1 \Downarrow \texttt{fun}\ x\ \texttt{->}\ s \qquad H\ ;\ e_2 \Downarrow v \qquad y\ \text{"fresh"}}{H\ ;\ e_1(e_2) \rightarrow H\ ;\ y := x; x := v; s; x := y}$$

$$\begin{array}{l} \texttt{f}_1 := (\texttt{fun x -> f}_2 := (\texttt{fun z -> ans} := \texttt{x} + \texttt{z})); \\ \texttt{f}_1(2); \\ \texttt{x} := 3; \\ \texttt{f}_2(4) \end{array}$$

"Should" set `ans` to 6:

- $\texttt{f}_1(2)$ should assign to $\texttt{f}_2$ a function that adds 2 to its argument and stores result in `ans`

"Actually" sets `ans` to 7:

- $\texttt{f}_2(2)$ assigns to $\texttt{f}_2$ a function that adds *the current value of* `x` to its argument

Punch line

Cannot properly model local scope via a global heap of integers.

- Functions are not syntactic sugar for assignments to globals

So let's build a new model that focuses on this essential concept

- (can add back IMP features later)

Or just borrow a model from Alonzo Church

And drop mutation, conditionals, integers (!), and loops (!)

The Lambda Calculus

The Lambda Calculus:

$$\begin{array}{rcl} e & ::= & \lambda x.\ e \mid x \mid e\ e \\ v & ::= & \lambda x.\ e \end{array}$$

You *apply* a function by *substituting* the argument for the *bound variable*

- (There is an equivalent *environment* definition not unlike heap-copying; see future homework)

Example Substitutions

$$
\begin{array}{rcl}
e & ::= & \lambda x.\ e \mid x \mid e\ e \\
v & ::= & \lambda x.\ e
\end{array}
$$

Substitution is the key operation we were missing:

$$(\lambda x.\ x)(\lambda y.\ y) \to (\lambda y.\ y)$$

$$(\lambda x.\ \lambda y.\ y\ x)(\lambda z.\ z) \to (\lambda y.\ y\ \lambda z.\ z)$$

$$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \to (\lambda x.\ x\ x)(\lambda x.\ x\ x)$$

After substitution, the bound variable is gone, so its "name" was irrelevant. (Good!)

A Programming Language

Given substitution ($e_1[e_2/x] = e_3$), we can give a semantics:

$\boxed{e \rightarrow e'}$

$$\frac{e[v/x] = e'}{(\lambda x.\ e)\ v \rightarrow e'} \qquad \frac{e_1 \rightarrow e_1'}{e_1\ e_2 \rightarrow e_1'\ e_2} \qquad \frac{e_2 \rightarrow e_2'}{v\ e_2 \rightarrow v\ e_2'}$$

A small-step, *call-by-value (CBV)*, left-to-right semantics

- Terminates when the "whole program" is some $\lambda x.\ e$

But (also) gets stuck when there's a *free variable* "at top-level"

- Won't "cheat" like we did with $H(x)$ in IMP because scope is what we are interested in

This is the "heart" of functional languages like OCaml

- But "real" implementations do not substitute; they do something *equivalent*

Roadmap

- Motivation for a new model (done)
- CBV lambda calculus using substitution (done)
- Notes on concrete syntax
- Simple Lambda encodings (it is Turing complete!)
- Other reduction strategies
- Defining substitution

Concrete-Syntax Notes

We (and OCaml) resolve concrete-syntax ambiguities as follows:

1. $\lambda x.\ e_1\ e_2$ is $(\lambda x.\ e_1\ e_2)$, not $(\lambda x.\ e_1)\ e_2$
2. $e_1\ e_2\ e_3$ is $(e_1\ e_2)\ e_3$, not $e_1\ (e_2\ e_3)$
    - Convince yourself application is not associative

More generally:

1. Function bodies extend to an unmatched right parenthesis
   Example: $(\lambda x.\ y(\lambda z.\ z)w)q$
2. Application associates to the left
   Example: $e_1\ e_2\ e_3\ e_4$ is $(((e_1\ e_2)\ e_3)\ e_4)$

- Like in IMP, assume we really have ASTs
  (with non-leaves labeled λ or "application")
- Rules may seem strange at first, but it is the most convenient concrete syntax
    - Based on 70 years experience

Lambda Encodings

Fairly crazy: we left out constants, conditionals, primitives, and data structures

In fact, we are *Turing complete* and can *encode* whatever we need (just like assembly language can)

Motivation for encodings:

- Fun and mind-expanding
- Shows we are not oversimplifying the model ("numbers are syntactic sugar")
- Can show languages are *too expressive* (e.g., unlimited C++ template instantiation)

Encodings are also just "(re)definition via translation"

Encoding booleans

The "Boolean ADT"

- There are two booleans and one conditional expression.
- The conditional takes 3 arguments (e.g., via currying). If the first is one boolean it evaluates to the second. If it is the other boolean it evaluates to the third.

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Here is one of an infinite number of encodings:

"true"	$\lambda x.\ \lambda y.\ x$
"false"	$\lambda x.\ \lambda y.\ y$
"if"	$\lambda b.\ \lambda t.\ \lambda f.\ b\ t\ f$

Example: "if" "true" $v_1\ v_2 \to^* v_1$

Evaluation Order Matters

Careful: With CBV we need to "thunk"...

$$\text{"if" "true"} \ (\lambda x.\ x)\ \underbrace{((\lambda x.\ x\ x)(\lambda x.\ x\ x))}_{\text{an infinite loop}}$$

diverges, but

$$\text{"if" "true"} \ (\lambda x.\ x)\ \underbrace{(\lambda z.\ ((\lambda x.\ x\ x)(\lambda x.\ x\ x))\ z))}_{\text{a value that when called diverges}}$$

does not

Encoding Pairs

The “pair ADT”:

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"mkpair"	$\lambda x.\ \lambda y.\ \lambda z.\ z\ x\ y$
"fst"	$\lambda p.\ p(\lambda x.\ \lambda y.\ x)$
"snd"	$\lambda p.\ p(\lambda x.\ \lambda y.\ y)$

Example:

"snd" ("fst" ("mkpair" ("mkpair" $v_1\ v_2$) v_3)) $\rightarrow^* v_2$

Reusing Lambdas

Is it weird that the encodings of Booleans and pairs both used $\lambda x.\ \lambda y.\ x$ and $\lambda x.\ \lambda y.\ y$ for different purposes?

Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?

Von Neumann: Bits can represent (all) code and data

Church (?): Lambdas can represent (all) code and data

Beware the "Turing tarpit"

Encoding Lists

Rather than start from scratch, notice that booleans and pairs are enough to encode lists:

- Empty list is "mkpair" "false" "false"
- Non-empty list is $\boldsymbol{\lambda h.\ \lambda t.}$ "mkpair" "true" ("mkpair" $\boldsymbol{h}$ $\boldsymbol{t}$)
- Is-empty is ...
- Head is ...
- Tail is ...

Note:

- Not too far from how lists are implemented
- Taking "tail" ("tail" "empty") will produce some lambda
  - Just like, without page-protection hardware, `null->tail->tail` would produce some bit-pattern

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- Write a function that takes an f and calls it in place of recursion
  - Example (in enriched language):

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- The details, especially for CBV, are icky; the point is it is possible and you define "fix" only once
- Not on exam:
  "fix" $\lambda g.\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))(\lambda x.\ g\ (\lambda y.\ x\ x\ y))$

Encoding Arithmetic Over Natural Numbers

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But instead everybody always teaches Church numerals. Why?

- Tradition? Some sense of professional obligation?
- Better reason: You do not need fix: Basic arithmetic is often encodable in languages where all programs terminate
- In any case, we will show some basics “just for fun”

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
...	

- Numbers encoded with two-argument functions
- The "number i" composes the first argument i times, starting with the second argument
  - z stands for "zero" and s for "successor" (think unary)
- The trick is implementing arithmetic by cleverly passing the right arguments for s and z

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"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$

successor: take "a number" and return "a number" that (when called) applies s one more time

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"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$

plus: take two "numbers" and return a "number" that uses one number as the zero argument for the other

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"0"	$\lambda s.\ \lambda z.\ z$
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"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"

times: take two "numbers" m and n and pass to m a function that adds n to its argument (so this will happen m times) and "zero" (where to start the m iterations of addition)

Church Numerals

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"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"
"isZero"	$\lambda n.\ n\ (\lambda x.$ "false") "true"

isZero: an easy one, see how the two arguments will lead to the correct answer

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"
"isZero"	$\lambda n.\ n\ (\lambda x.$ "false") "true"
"predecessor"	(with 0 sticky) the hard one; see Wikipedia
"minus"	similar to times with pred instead of plus
"isEqual"	subtract and test for zero

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Then start type systems

- Later take a break from types to consider first-class continuations and related topics