# CSE 505: Programming Languages

## Lecture 17 — The Curry-Howard Isomorphism

Zach Tatlock
Autumn 2017

## Language Design

What have we been up to?

- Define a programming language
  - we've been fairly formal
  - pretty close to SML if you squint real hard



## Language Design

What have we been up to?

- Define a programming language
  - we've been fairly formal
  - pretty close to SML if you squint real hard
- Define a type system
  - **outlaw** *bad programs* that "get stuck"
  - sound: no typable programs get stuck
  - incomplete: knocked out some OK programs too, ohwell




## Elsewhere in the Universe (or the other side of campus)

What do logicians do?

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What do logicians do?

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  - tools to precisely state propositions
- Define proof systems
  - tools to figure out which propositions are true

Turns out, we did that too!



## Punchline

We are accidental logicians!



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**The Curry-Howard Isomorphism**

- Proofs : Propositions :: Programs : Types
- proofs are to propositions as programs are to types

## Woah. Back up a second. Logic?!



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Let's trim down our (explicitly typed) simply-typed $\lambda$-calculus to:

$$\begin{array}{rcl} e & ::= & x \mid \lambda x.\ e \mid e\ e \\ & \mid & (e, e) \mid e.1 \mid e.2 \\ & \mid & \mathsf{A}(e) \mid \mathsf{B}(e) \mid \textbf{match } e \textbf{ with A}x.\ e \mid \mathsf{B}x.\ e \end{array}$$

$$\tau \ ::= \ b \mid \tau \to \tau \mid \tau * \tau \mid \tau + \tau$$

- Lambdas, Pairs, and Sums
- Any number of base types $b_1, b_2, \ldots$
- No constants (can add one or more if you want)
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What good is this?!

Well, even sans constants, plenty of terms type-check with $\Gamma = \cdot$



$\lambda x{:}b.\ x$

has type

$$\lambda x{:}b.\ x$$

has type

$$b \rightarrow b$$



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Ohwell, now for a *totally irrelevant* tangent!



## *Totally irrelevant tangent.*



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Not too hard to build a *proof system* to establish tautologyhood.



## Proof System

$$\Gamma \quad ::= \quad \cdot \mid \Gamma, p$$

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$\boxed{\Gamma \vdash p}$

$$\frac{\Gamma \vdash p_1 \qquad \Gamma \vdash p_2}{\Gamma \vdash p_1 \wedge p_2}$$



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$$\frac{p \in \Gamma}{\Gamma \vdash p}$$



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$$\frac{p \in \Gamma}{\Gamma \vdash p} \qquad \frac{\Gamma, p_1 \vdash p_2}{\Gamma \vdash p_1 \supset p_2}$$

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## Wait a second...



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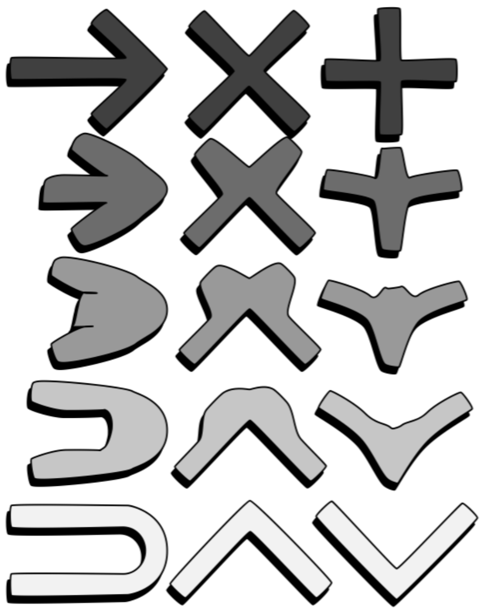



## Wait a second... ZOMG!

That's *exactly* our type system! Just erase terms, change each $\tau$ to a $p$, and translate $\rightarrow$ to $\supset$, $*$ to $\wedge$, $+$ to $\vee$.

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 * \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 * \tau_2}{\Gamma \vdash e.1 : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 * \tau_2}{\Gamma \vdash e.2 : \tau_2}$$

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$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x.\ e : \tau_1 \rightarrow \tau_2} \qquad \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_1}$$

## What does it all mean? The Curry-Howard Isomorphism.

- Given a well-typed closed term, take the typing derivation, erase the terms, and have a propositional-logic proof
- Given a propositional-logic proof, there exists a closed term with that type
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- A term that type-checks is a *proof* — it tells you exactly how to derive the logicical formula corresponding to its type
- Constructive (hold that thought) propositional logic and simply-typed lambda-calculus with pairs and sums are *the same thing*.
  - Computation and logic are *deeply* connected
  - $\boldsymbol{\lambda}$ is no more or less made up than implication
- Revisit our examples under the logical interpretation...



$$\lambda x{:}b.\ x$$

is a proof that

$$b \to b$$



$$\lambda x{:}b_1.\ \lambda f{:}b_1 \to b_2.\ f\ x$$

is a proof that

$$b_1 \to (b_1 \to b_2) \to b_2$$

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$$\lambda x{:}b_1 * b_2.\ \lambda y{:}b_3.\ ((y, x.1), x.2)$$

is a proof that

$$(b_1 * b_2) \rightarrow b_3 \rightarrow ((b_3 * b_1) * b_2)$$

## So what?

Because:

- This is just fascinating (glad I'm not a dog)
- Don't think of logic and computing as distinct fields
- Thinking "the other way" can help you know what's possible/impossible
- Can form the basis for theorem provers
- Type systems should not be *ad hoc* piles of rules!



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So, every typed $\lambda$-calculus is a proof system for some logic...

Is STLC with pairs and sums a *complete* proof system for propositional logic? Almost...



## Classical vs. Constructive

Classical propositional logic has the "law of the excluded middle":

$$\frac{}{\Gamma \vdash p_1 + (p_1 \to p_2)}$$

(Think "$p + \neg p$" – also equivalent to double-negation $\neg\neg p \to p$)



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Can still "branch on possibilities" by making the excluded middle an explicit assumption:

$$((p_1 + (p_1 \to p_2)) * (p_1 \to p_3) * ((p_1 \to p_2) \to p_3)) \to p_3$$



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Theorem: There exist irrational numbers $a$ and $b$ such that $a^b$ is rational.



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Classical Proof:

*Let $x = \sqrt{2}$. Either $x^x$ is rational or it is irrational.*
*If $x^x$ is rational, let $a = b = \sqrt{2}$, done.*
*If $x^x$ is irrational, let $a = x^x$ and $b = x$. Since $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2$, done.*

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Constructive Proof:

*Let $a = \sqrt{2}$, $b = \log_2 9$.*

*Since $\sqrt{2}^{\log_2 9} = 9^{\log_2 \sqrt{2}} = 9^{\log_2 (2^{0.5})} = 9^{0.5} = 3$, done.*

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To prove that something exists, we actually had to produce it. SWEET.

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Our friends Gödel and Gentzen gave us this nice result:

*$P$ is provable in classical logic iff $\neg\neg P$ is provable in constructive logic.*

## Fix

A "non-terminating proof" is no proof at all.

Remember the typing rule for **fix**:

$$\frac{\Gamma \vdash e : \tau \to \tau}{\Gamma \vdash \textbf{fix}\ e : \tau}$$

That let's us prove anything! Example: **fix** $\lambda x{:}b.\ x$ has type $b$

So the "logic" is *inconsistent* (and therefore worthless)

Related: In ML, a value of type 'a never terminates normally (raises an exception, infinite loop, etc.)

```
let rec f x = f x
let z = f 0
```



## Last word on Curry-Howard

It's not just STLC and constructive propositional logic

*Every* logic has a correspondng typed $\lambda$ calculus (and no consistent logic has something as "powerful" as **fix**).



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If you remember one thing: the typing rule for function application is *modus ponens*