Readings: K&F 12.1, 12.2, 12.3, 12.4

Approximate Inference I

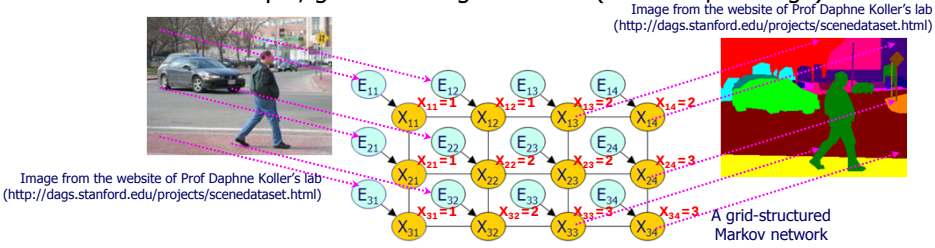
Lecture 15 – May 16, 2011
CSE 515, Statistical Methods, Spring 2011

Instructor: Su-In Lee
University of Washington, Seattle

Inference

- Inference is widely used,
  - Probabilistic queries: Given **E**=**e**, what is P(**X**|**E**=**e**)?
    - For example, given **E**=image what is P(**X**=labels|**E**=image)=?

Image from the website of Prof Daphne Koller's lab (http://dags.stanford.edu/projects/scenedataset.html)



  - Learning parameters with partially observed data
    - EM, gradient ascent, etc

- but exact inference is expensive
  - O(Nn), where $N=\max_i |Val(\mathbf{C}_i)|$
  - Exponential blowup can be in N_i which for factor i can be v^m if factor i has m variables with v values each.

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Approximate Inference Overview

Particle-based approximate inference

- Full particle methods
  - Sampling methods
  - Deterministic particle generation
- Distributional particles
- Applications & Simulation: data association

Today…

Global approximate inference

- Inference as optimization
- Generalized Belief Propagation
- Propagation with approximate messages
- Structured variational approximations

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Particle-based Approximate Inference

General framework

- Generate particles (samples) x[1],…,x[M] from P
- Estimate function f by $E_P(f) \approx \frac{1}{M}\sum_{m=1}^{M} f(x[m])$

- For a function f(**X**)=**1**(**Y**=**y**)
- P(**Y**=**y**) can be written as $E_P[f]$

- How particles are generated?
  - Use forward sampling
  - Use likelihood weighting sampling

Particle-Based Methods Overview

- Full particle methods
  - Sampling methods
    - Forward sampling, Likelihood weighting
    - (Un-normalized/normalized) Importance sampling
    - Markov chain Monte Carlo
  - Deterministic particle generation
- Distributional particles

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Unnormalized Importance Sampling

- For any distribution Q(**X**), since:

$$E_{P(\mathbf{X})}[f(\mathbf{X})] = \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x})P(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbf{X}} Q(\mathbf{x}) f(\mathbf{x}) \frac{P(\mathbf{x})}{Q(\mathbf{x})} = E_{Q(\mathbf{X})}\left[f(\mathbf{X}) \frac{P(\mathbf{X})}{Q(\mathbf{X})} \right]$$

- We can estimate f(x) by generating samples from Q and estimating:

$$E_P(f) \approx \frac{1}{M} \sum_{m=1}^{M} f(x[m]) \frac{P(x[m])}{Q(x[m])}$$

Target distribution

Proposal distribution

  - Can show that estimator variance decreases with more samples M
  - Can show that Q=P is the lowest variance estimator

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Normalized Importance Sampling

- Un-normalized importance sampling assumes known P
  - Usually we know P up to a constant P = P'/α, where $\alpha=\Sigma_x P'(x)$
    - Example: Posterior distribution P(**X**|**E**=**e**) = P(**X**,**E**=**e**) / α where α=P(**E**=**e**)
- We can estimate α by: $E_{Q(\mathbf{X})}\left[\frac{P'(\mathbf{X})}{Q(\mathbf{X})}\right]=\sum_{\mathbf{x}\in\mathbf{X}}Q(\mathbf{x})\frac{P(\mathbf{x})'}{Q(\mathbf{x})}=\sum_{\mathbf{x}\in\mathbf{X}}P'(\mathbf{x})=\alpha$
- Thus

$$\begin{aligned}E_{P(\mathbf{X})}[f(\mathbf{X})] &= \sum_x P(x)f(x)\\ &= \sum_x Q(x)f(x)P(x)/Q(x)\\ &= \frac{1}{\alpha}\sum_x Q(x)f(x)P'(x)/Q(x)\\ &= \frac{1}{\alpha}E_{Q(\mathbf{X})}[f(\mathbf{X})P'(x)/Q(x)]\\ &= \frac{E_{Q(\mathbf{X})}[f(\mathbf{X})P'(x)/Q(x)]}{E_{Q(\mathbf{X})}[P'(x)/Q(x)]}\end{aligned}$$

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Normalized Importance Sampling

$$E_{P(\mathbf{X})}[f(\mathbf{X})]=\frac{E_{Q(\mathbf{X})}[f(\mathbf{X})P'(x)/Q(x)]}{E_{Q(\mathbf{X})}[P'(x)/Q(x)]}$$

- Given M samples from Q, normalized sampling estimates function f by:

$$E_P(f)\approx\frac{\sum_{m=1}^{M}f(x[m])P'(x[m])/Q(x[m])}{\sum_{m=1}^{M}P'(x[m])/Q(x[m])}$$

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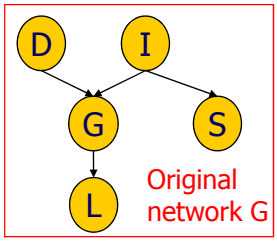
Importance Sampling for BayesNets

- Let's think about how to generate particles to estimate P(**Y**|**e**)
- Define *mutilated network* $G_{E=e}$ as:
  - Nodes X∈E have no parents in $G_{E=e}$
  - Nodes X∈E have CPD that is 1 for X=E[X] and 0 otherwise
  - The parents and CPDs for all other variables are unchanged

E = {S=s^1, G=g^2}



Target distribution P'(**X**,e)

d^0	d^1
0.4	0.6

i^0	i^1
0.7	0.3

g^0	g^1	g^2
0	0	1

s^0	s^1
0	1

G	l^0	l^1
g^0	0.1	0.9
g^1	0.4	0.6
g^2	0.99	0.01

Mutilated network $G_{E=e}$

Proposal distribution Q(**X**,e)=$P_{G_{E=e}}$(**X**,e)

Likelihood Weighting as IS

- Target distribution P'(**X**,**e**)
- Proposal distribution Q(**X**,**e**)=$P_{G_{E=e}}$(**X**,**e**)
- Claim: Likelihood weighting is precisely normalized importance sampling with the above distributions
- Proof sketch:
  - LW estimate: $P(y \mid e) \approx \frac{\sum_{m=1}^{M} w[m]\mathbf{1}\{x[m](Y) = y\}}{\sum_{m=1}^{M} w[m]}$
  - IS estimate: $P(y \mid e) \approx \frac{\sum_{m=1}^{M} P'(x[m]) / Q(x[m])\mathbf{1}\{x[m](Y) = y\}}{\sum_{m=1}^{M} P'(x[m]) / Q(x[m])}$
  - But w[m] in LW precisely comes out to be P'(x[m])/Q(x[m])
    - Since

$$\frac{P'(x[m])}{Q(x[m])} = \frac{\prod_{e_i \in \mathbf{E}} P(e_i \mid Pa(e_i)) \prod_{X_i \notin \mathbf{E}} P(X_i \mid Pa(X_i))}{\prod_{X_i \notin \mathbf{E}} P(X_i \mid Pa(X_i))} = \prod_{e_i \in \mathbf{E}} P(e_i \mid Pa(e_i)) = w[m]$$

Particle-Based Methods Overview

- Full particle methods
  - Sampling methods
    - Forward sampling
    - Importance sampling
    - Markov chain Monte Carlo (MCMC)
  - Deterministic particle generation
- Distributional particles

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Markov Chain Monte Carlo

- Limitations of importance sampling
  - An evidence node affects the sampling only for nodes that are its descendants.
  - The effect on nodes that are non-descendants is accounted for only by the weights w's.
  - What if much of the evidence is at the leaves of the network?
    - We are essentially sampling from the prior distribution P(**X**), which is often very far from the desired posterior P(**X**|**E**=**e**).
    - → Decreases the accuracy of the estimates
- An alternative sampling approach
  - General idea: Define a sampling process that is guaranteed to converge to taking samples from the posterior distribution of interest P(**X**|**E**=**e**)
  - Generate samples from the sampling process
  - Estimate f(X) from the samples

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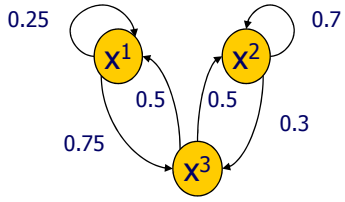
Markov Chains

- A Markov chain consists of
  - A state space Val(**X**)
  - Transition probability T(x→x′) of going from state x to x′
- Distribution over subsequent states is defined as

Distribution over the "next" state | Distribution over the "current" state

$$P^{(t+1)}(X^{(t+1)} = x') = \sum_{x \in Val(X)} P^{(t)}(X^{(t)} = x) T(x \rightarrow x')$$



Simple Markov chain

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Stationary Distribution

Distribution over the "next" state | Distribution over the "current" state

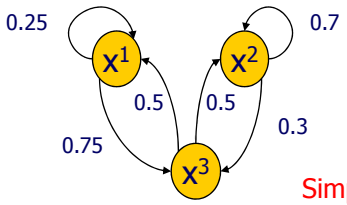
$$P^{(t+1)}(X^{(t+1)} = x') = \sum_{x \in Val(X)} P^{(t)}(X^{(t)} = x) T(x \rightarrow x')$$

- A distribution $\pi(X)$ is a stationary distribution for a Markov chain T if it satisfies

$$\pi(X = x') = \sum_{x \in Val(X)} \pi(X = x) T(x \rightarrow x')$$

$$\begin{aligned}
\pi(x^1) &= 0.25\pi(x^1) + 0.5\pi(x^3) \\
\pi(x^2) &= 0.7\pi(x^2) + 0.5\pi(x^3) \\
\pi(x^3) &= 0.75\pi(x^1) + 0.3\pi(x^2)
\end{aligned}$$

$$\begin{aligned}
\pi(x^1) &= 0.2 \\
\pi(x^2) &= 0.5 \\
\pi(x^3) &= 0.3
\end{aligned}$$



Simple Markov chain

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Markov Chains & Stationary Dist.

- A Markov chain is regular if there is k such that for every x, x′∈Val(X), the probability of getting from x to x′ in exactly k steps is greater than zero
- Theorem: A finite state Markov chain T has a unique stationary distribution if and only if it is regular
- Goal: Define a Markov chain whose stationary distribution is P(**X**|**e**)

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Gibbs Sampling

- States
  - Legal (=consistent with evidence) assignments to variables
- Transition probability
  - $T = T_1 \cdot \ldots \cdot T_k$
  - For each variable X_i, let $\mathbf{X}_{-i}$ be $\mathbf{X}$-{X_i}. Say that $\mathbf{X}_{-i} = \mathbf{x}_{-i}$. $T_i = T((\mathbf{x}_{-i}, x_i) \rightarrow (\mathbf{x}_{-i}, x'_i)) = P(x'_i|\mathbf{x}_{-i})$
- Claim: P(**X**|**e**) is a stationary distribution to the chain

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Gibbs Sampling

A couple slides ago...

- A Markov chain is regular if there is k such that for every x, x′∈Val(X), the probability of getting from x to x′ in exactly k steps is greater than zero
  → ***has a stationary distribution***

- Gibbs-sampling Markov chain is regular if:
  - Bayesian networks: all CPDs are strictly positive
  - Markov networks: all clique potentials are strictly positive

- Gibbs sampling procedure (generating one sample x[m])
  - Set x[m]=x[m-1]
  - For each variable $X_i \in \mathbf{X}$-$\mathbf{E}$
    - Set $\mathbf{x}_{-i}$ = x[m](**X**-X_i)
    - Sample from $P(X_i \mid \mathbf{x}_{-i})$
    - Save sample x[m](X_i) = sampled value
  - Return x[m]

  - Note: $P(X_i \mid \mathbf{x}_{-i})$ is easily computed from Markov blanket

Gibbs Sampling

- How do we evaluate $P(X_i \mid \mathbf{x}_{-i})$?
  - Again, let $\mathbf{x}_{-i} = \{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\}$

- Let's say that P factorizes as

$$P(\mathbf{X}) = \frac{1}{Z}\prod_j \phi_j(\mathbf{D}_j) = \frac{1}{Z}\prod_{j:X_i\in\mathbf{D}_j}\phi_j(\mathbf{D}_j)\prod_{j:X_i\notin\mathbf{D}_j}\phi_j(\mathbf{D}_j)$$

- Let $\mathbf{x}_{j,-i}$ denote the assignment in $\mathbf{x}_{-i}$ to $\mathbf{D}_j$-{X_i},

$$P(x_i' \mid \mathbf{x}_{-i}) = \frac{P(x_i', \mathbf{x}_{-i})}{\sum_{x_i''} P(x_i'', \mathbf{x}_{-i})}$$

$$= \frac{\prod_{j:X_i\in\mathbf{D}_j}\phi_j(x_i', \mathbf{x}_{j,-i})\prod_{j:X_i\notin\mathbf{D}_j}\phi_j(\mathbf{x}_{j,-i})}{\sum_{x_i''}\prod_{j:X_i\in\mathbf{D}_j}\phi_j(x_i'', \mathbf{x}_{j,-i})\prod_{j:X_i\notin\mathbf{D}_j}\phi_j(\mathbf{x}_{j,-i})} = \frac{\prod_{j:X_i\in\mathbf{D}_j}\phi_j(x_i', \mathbf{x}_{j,-i})}{\sum_{x_i''}\prod_{j:X_i\in\mathbf{D}_j}\phi_j(x_i'', \mathbf{x}_{j,-i})}$$

  - The last term uses only the factors involving X_i.

Gibbs Sampling in Practice

- We need to wait until the burn-in time has ended – number of steps until we take samples from the chain
  - We want to wait until the sampling distribution is close to the stationary distribution
  - Hard to provide bounds in general for this 'mixing time'
  - Once the burn-in time ended, all samples are from the stationary distribution
- Note: after the burn-in time, samples are correlated. Consecutive samples from the same trajectory are correlated.
- Since no theoretical guarantees exist, application of Markov chains is somewhat of an art
  - We can evaluate burn-in time by comparing the estimates $\hat{E}_k(f) \approx \frac{1}{M}\sum_{m=1}^{M} f(x^k[m], e)$ from multiple chains 1,…,K

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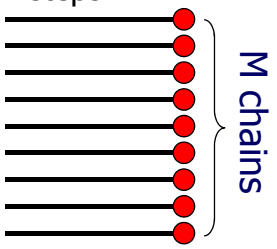
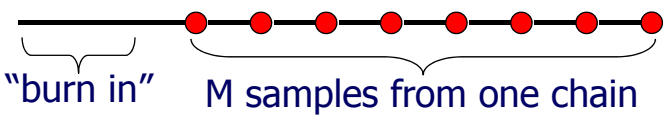
Sampling Strategy

- How do we collect the samples?
- Strategy I:
  - Run the chain M times, each run for N steps
    - Each run starts from a different state
  - Return the last state in each run



- Strategy II:
  - Run one chain for a long time
  - After some "burn in" period, sample points every some fixed number of steps



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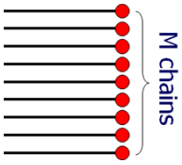
Comparing Strategies

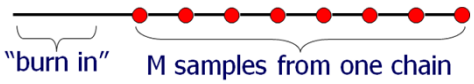
Strategy I:

- Better chance of "covering" the space of points, especially if the chain is slow to reach the stationary distribution
- Have to perform "burn in" steps for each chain



Strategy II:

- Perform "burn in" only once
- Samples might be correlated (although only weakly)



Hybrid strategy:

- Run several chains, and sample few samples from each
- Combines benefits of both strategies

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Particle-Based Methods Overview

- Full particle methods
  - Sampling methods
    - Forward sampling
    - Importance sampling
    - Markov chain Monte Carlo
  - Deterministic particle generation
- Distributional particles

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Deterministic Search Methods

- Idea: if the distribution is dominated by a small set of instances, it suffices to consider only them for approximating the function
  - For instances that we generate x[1],...x[M], estimate is:

$$E_P(f) \approx \sum_{m=1}^{M} f(x[m])P(x[m])$$

  - Note: we can obtain lower and upper bounds by examining the part of the probability mass covered by ΣP(x[m])

$$\sum_{m=1}^{M} f(x[m])P(x[m]) \le E_P(f) \le \left(1 - \sum_{m=1}^{M} (1 - f(x[m]))P(x[m])\right)$$

Sum of probabilities that f is 0.

- Key: how can we enumerate highly probably instantiations?
  - Note: the single most probably instantiation is MPE which itself is an NP-hard problem

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Particle-Based Methods Overview

- Full particle methods
  - Sampling methods
    - Forward sampling
    - Importance sampling
    - Markov chain Monte Carlo
  - Deterministic particle generation
- Distributional particles
  - Until now, we discussed the cases where we are given particles that have full assignment of values on all variables.
  - Can we use a partial assignment to estimate f?

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Distributional Particles

- Idea: use partial assignments to a subset of the network variables, combined with closed form representation of a distribution over the rest
  - X_p – Variables whose assignments are defined by particles
  - X_d – Variables over which we maintain a distribution
  - Distributional particles are a.k.a *Rao-Blackwellized* particles
  - Estimation proceeds as

$$\begin{aligned} E_{P|e}(f) &= \sum_{x_p, x_d} P(x_p, x_d \mid e) f(x_p, x_d) \\ &= \sum_{x_p} P(x_p \mid e) \sum_{x_d} P(x_d \mid x_p, e) f(x_p, x_d) \\ &= \sum_{x_p} P(x_p \mid e) (E_{P(x_d \mid x_p, e)}[f(x_p, x_d)]) \end{aligned}$$

    - We can use any sampling procedure to sample X_p
    - We assume that we can compute the internal expectation efficiently

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Distributional Particles

- Distributional particles define a continuum between
  - For $|X_p|=|X-E|$ we have full particles and thus full sampling
  - For $|X_p|= 0$ we are performing full exact inference
- Distributional Likelihood Weighting
  - Sample over a subset of the variables

$$P(y \mid e) \approx \frac{\sum_{m=1}^{M} w[m] \left(E_{P(X_d \mid x_p[m], e)} \left[\mathbf{1}\{x[m](y) = y\} \right] \right)}{\sum_{m=1}^{M} w[m]}$$

- Distributional Gibbs Sampling
  - Sample only a subset of the variables, transition probability is as before $T((\mathbf{u}_i, x_i) \rightarrow (\mathbf{u}_i, x'_i)) = P(x'_i \mid \mathbf{u}_i)$ but the computation may require inference

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Broader Class of Markov Chains

- Limitations of Gibbs sampling
  - The Gibbs chain uses only very local moves over the state space: moves that chance only one variable at a time.
  - High-probability states will form strong basins of attraction, and the chain will be very unlikely to move away from such a state → the chain will mix very slowly.
  - We want to consider chains that allow a broader range of moves, including much larger steps in the space.
- Solutions
  - Block Gibbs sampling
  - Metropolis-Hastings algorithm

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Block Gibbs Sampling

- There are cases where we can simultaneously sample several variables efficiently
- Block Gibbs algorithm
  - Assume that we can partition the variables $\mathbf{X}$ into several disjoint blocks of variables $\mathbf{X}_1,\ldots,\mathbf{X}_k$, such that we efficiently sample $\mathbf{x}_i$ from $P(\mathbf{X}_i|\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\ldots,\mathbf{x}_k)$
  - Simple modification to Gibbs: Iteratively sample blocks of variables, rather than individual variables
  - It takes much "longer-range" transitions in the state space in a single sample step

Metropolis-Hastings algorithm (1/3)

- Unlike the Gibbs chain, M-H algorithm does not assume that we can generate next-state samples from a particular target distribution
  - It uses the idea of a proposal distribution Q that we have already seen in the case of IS.
- Basic ideas
  - We sample from a different distribution Q and then correct for the resulting error.
  - However, unlike importance sampling, we do not want to keep track of importance weights.
    - They are going to decay exponentially with the number of transitions, leading to a whole slew of problems
  - Instead, we randomly choose whether to accept the proposed transition, with a probability that corrects for the discrepancy between Q and the target distribution P

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Metropolis-Hastings algorithm (2/3)

- Let our proposal distribution T^Q define a transition model over our state space
  - For each state x, T^Q defines a distribution over possible successor states in Val(**X**), from which we select randomly a candidate next state x'
- We can either accept the proposal and transition to the new state x', or reject it and stay at x.
  - For each states x, x', we have an acceptance probability A(x→x'). Then, the actual transition model of the Markov chain is

$$T(x \to x') = T^Q(x \to x')A(x \to x') \qquad x \neq x'$$

$$T(x \to x) = T^Q(x \to x) + \sum_{x \neq x'} T^Q(x \to x')\left(1 - A(x \to x')\right)$$

  - We can show that with the following acceptance probabilities (and the regularity assumption), a chain T has the unique stationary distribution (proof in K&F page 516)

$$A(x \to x') = \min\left[1, \frac{\pi(x')T^Q(x' \to x)}{\pi(x)T^Q(x \to x')}\right]$$

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Metropolis-Hastings algorithm (3/3)

$$A(x \to x') = \min\left[1, \frac{\pi(x')T^Q(x' \to x)}{\pi(x)T^Q(x \to x')}\right]$$

- The M-H algorithm has a particularly natural implementation in the context of graphical models.
  - Each local transition model T_i is defined via an associated proposal distribution $T_i^{Q_i}$, and the acceptance probability for this chain has the form:

$$A(\mathbf{x}_{-i}, x_i \to \mathbf{x}_{-i}, x_i') = \min\left[1, \frac{P(\mathbf{x}_{-i}, x_i')T_i^{Q_i}(\mathbf{x}_{-i}, x_i' \to \mathbf{x}_{-i}, x_i)}{P(\mathbf{x}_{-i}, x_i)T_i^{Q_i}(\mathbf{x}_{-i}, x_i \to \mathbf{x}_{-i}, x_i')}\right]$$

- The proposal distributions are usually fairly simple, so it is easy to compute their ratios.
  - In the case of graphical models, the first ratio can also be computed easily:

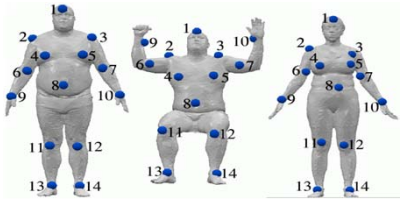
$$\frac{P(\mathbf{x}_{-i}, x_i')}{P(\mathbf{x}_{-i}, x_i)} = \frac{P(x_i' | \mathbf{x}_{-i})}{P(x_i | \mathbf{x}_{-i})}$$

Similarly to Gibbs sampling, $\mathbf{x}_{-i}$ can be reduced to the Markov blanket of X_i

So far, we discussed various sampling strategies. Let me talk about one example of real applications and "show" you how different sampling methods work in a toy model of the application.

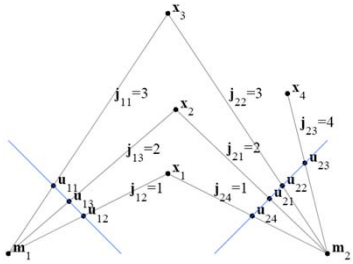
Application & Simulation: Data Association

- Data association problem (aka correspondence problem)
  - Relating sensor measurements to parameters in the model that is being learned.
- For example,
  - We want to find a correspondence between a set of (automatically selected) landmarks on different 3D scans of human bodies.



D. Anguelov, et al. The Correlated Correspondence Algorithm for Unsupervised Registration of Nonrigid Surfaces. NIPS 2004.

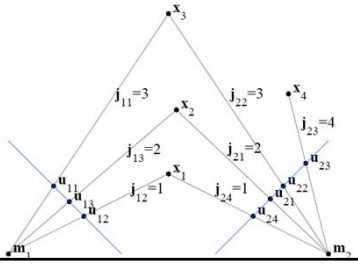
  - Given a set of image measurements **U**={u_{ik} for image i and measurement k}, we want to know which of the 3D objects **X**={x_j, 1≤j≤n} each measurement corresponds to.



F. Dellaert, SM. Seitz, CE. Thorpe and S. Thrun. *EM, MCMC, and Chain Flipping for Structure from Motion with Unknown Correspondence.* Machine Learning 2003.

Application & Simulation: Data Association

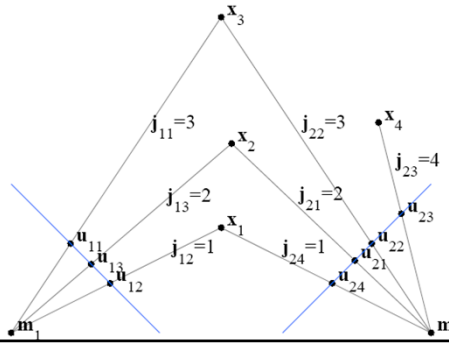
- Data association problem (aka correspondence problem)
  - Relating sensor measurements to parameters in the model that is being learned.
- Example problem
  - Given a set of image measurements **U**={u_{ik} for image i and measurement k}, we want to know which of the 3D objects **X**={x_j, 1≤j≤n} each measurement corresponds to.
  - To model the correspondence between measurements u_{ik} and 3D object x_j, we introduce a correspondence vector **J**.
  - For each measurement u_{ik}, the vector **J** contains an indicator variable j_{ik} indicating that u_{ik} is assigned to the j_{ik}-th object x.



F. Dellaert, SM. Seitz, CE. Thorpe and S. Thrun. *EM, MCMC, and Chain Flipping for Structure from Motion with Unknown Correspondence.* Machine Learning 2003.

Applications: Data Association

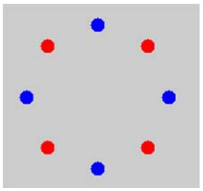
- Maximize $L(\Theta;\mathbf{U}) = \sum_{\mathbf{J}} L(\Theta;\mathbf{U},\mathbf{J})$
  - **J**'s are hidden variables → EM algorithm
  - The E-step requires inference $P(\mathbf{J}|\ \mathbf{U}, \Theta^{(t)})$
  - This problem has been tackled using range of models and a variety of inference methods including MCMC
  - Efficiency is important and we will "see" that how various MCMC-based methods show varying performance



F. Dellaert, SM. Seitz, CE. Thorpe and S. Thrun.
EM, MCMC, and Chain Flipping for Structure from Motion with Unknown Correspondence. Machine Learning 2003.

Toy Model for Data Association

- Blue dots: variables, X_i (i=1,2,3,4)
- Red dots: observations Y_j (j=1,2,3,4)
- Correspondence variable C_i ={1,2,3,4} assigns to X_i one of the observations

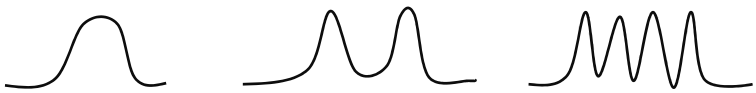


distance

$$\mathbf{P}(\mathbf{X}) \propto \exp(-\sum_{i=1}^{4} \| \frac{x_i - y_{C_i}}{\sigma} \|^2) \quad \text{if every } x_i \text{ has a different value}$$

$$\mathbf{P}(\mathbf{X}) = 0 \quad \text{otherwise}$$

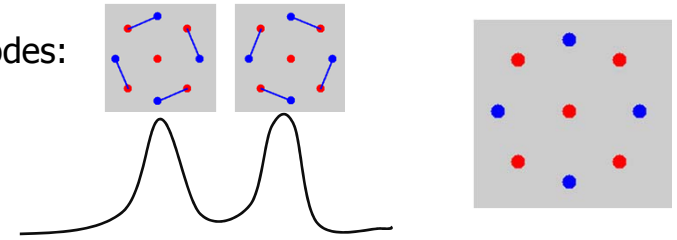
- What does the distribution look like?



(A) (B) (C)

How do we sample from it?

- Add one observation (such that Gibbs would work)
- Two modes:



- Gibbs
  - How does it traverse between the two modes?
- Block Gibbs (block size = 2)
  - How do we sample?
- Metropolis Hasting
  - Take larger steps using a proposal distribution. (We will come to details of this later.)

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Let's "See" How They Work

- Run the following Matlab scripts:

```
VisualMCMC1(10000, 0.1, 0.05);
    % live animation of sampling
    % parameters: num of samples, sigma, pause time
    after each sample
Plot1;
    % the first few lines of Plot1.m contain the parameters
    you may want to play around with
```

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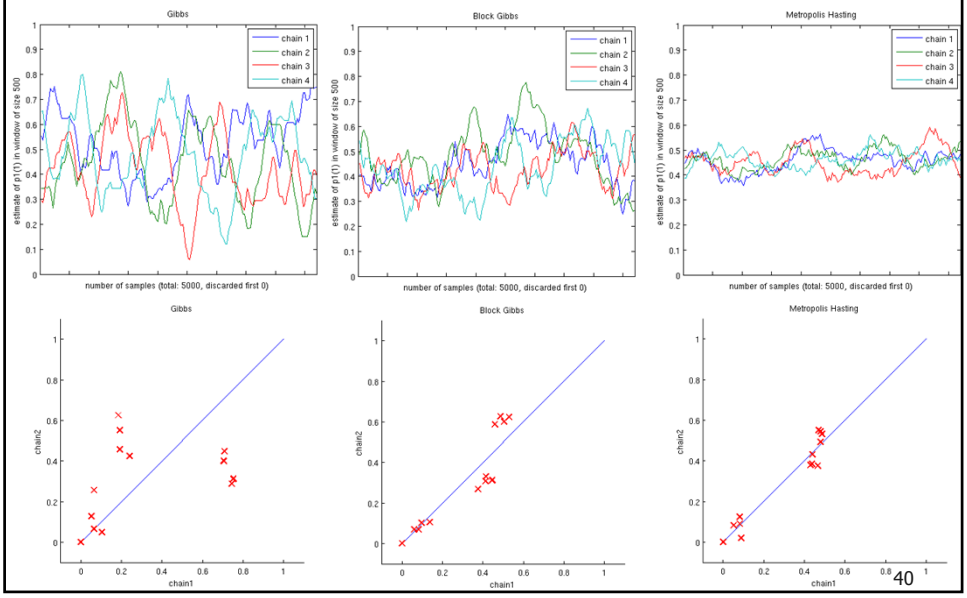
Plots generated by "Plot1"



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