CSE 521: Design and Analysis of Algorithms **Fall 2023**

Problem Set 5

Deadline: Oct 7th in gradescope

1) a) Let $M \in \mathbb{R}^{n\times n}$ be a (symmetric) PSD matrix. Show that for any $P \in \mathbb{R}^{m\times n}$, $PMP^T \succeq 0$.

   b) Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix that has k positive eigenvalues for some integer $k \geq 1$. Use Cauchy's interlacing theorem (below) to show that for any $P \in \mathbb{R}^{m\times n}$, PMP^T has at most k positive eigenvalue.

   Theorem 5.1 (Cauchy's Interlacing Theorem)**.** *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$. For a vector $v \in \mathbb{R}^n$, let $\beta_n \leq \cdots \leq \beta_1$ be the eigenvalues of $M + vv^T$. Then,*

$$\lambda_n \leq \beta_n \leq \lambda_{n-1} \leq \beta_{n-1} \leq \cdots \leq \lambda_1 \leq \beta_1.$$

2) In this problem you can use the following theorem.

   Theorem 5.2 (Hanson-Wright inequality)**.** *. For $\sigma_1, \ldots, \sigma_n$ independent $\{+1, -1\}$ Radamacher random variable, i.e., $\sigma_i = +1$ w.p. 1/2 and $\sigma = -1$ otherwise. and $A \in \mathbb{R}^{n\times n}$, for all $p \geq 1$*

$$\mathbb{P}\left[|\sigma^T A\sigma - \mathbb{E}\sigma^T A\sigma| > t\right] \leq 2\exp\left(-c\min\left\{\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|}\right\}\right)$$

   where $c > 0$ is a universal constant (independent of n).

   a) Show that for any matrix $A \in \mathbb{R}^{m\times n}$,

$$\|A^T A\|_F^2 \leq \|A\|^2 \|A\|_F^2$$

   b) Show that for a random vector $\sigma \in \mathbb{R}^n$, where σ_i's are independent Radamacher random variables, and for any $A \in \mathbb{R}^{m\times n}$ and $0 < \epsilon < 1$, we have,

$$\mathbb{P}\left[\left|\frac{\|AX\|^2}{\|A\|_F^2} - 1\right| > \epsilon\right] \leq 2\exp\left(-c\frac{\epsilon^2\|A\|_F^2}{\|A\|^2}\right)$$

   c) For an application, let E be a d-dimensional subspace of $\mathbb{R}^n$, and let Π_E be the the projection operator onto E (see lecture 11). Show that for a random Radamacher vector (as above), and $0 < \epsilon < 1$ we have,

$$\mathbb{P}\left[|\frac{d(\sigma, E)^2}{n-d} - 1| > \epsilon\right] \leq 2\exp(-c\epsilon^2(n-d)).$$

   Here, $d(\sigma, E)$ is the Euclidean distance of σ from E.